

Preface

Mathematics—it's not just solving for x ; it's also figuring out (wh)y.
—Art Benjamin (1961–)

The study of algebra may be pursued in three very different schools ... The practical person seeks a rule which he may apply, the philological person seeks a formula which he may write, the theoretical person seeks a theorem on which he may meditate. —William Rowan Hamilton (1805–1865)

Mathematicians often extol the beauty of abstract algebra, while students too often find its focus on theory and abstraction opaque and unconnected to their previous education. This text seeks to help students make the transition from a problem-solving, rule-based approach to a theoretical one. The Benjamin quote describes this shift whimsically, while the Hamilton quote gives a more philosophical perspective. Throughout the text I hope to convince students that theoretical algebra retains its practical roots, deepening and extending them through the power of abstraction, as well as possessing the beauty mathematicians prize.

Many texts favor a “groups first” approach conveying as quickly as possible the power of abstraction and key significance of groups throughout mathematics. Unfortunately, too many of these texts provide almost no connection with high school algebra and so some students find these texts intimidating. Other texts favor a “rings first” development to smooth the transition from high school algebra, starting with numbers and number theory. They then expand to polynomials and rings. While students find this approach more connected with their experience, they often get no insight in a single semester why groups or the abstract approach in general are vital for modern mathematics. Few if any texts using either approach make any connection with the long history of algebra leading both to high school algebra and the modern synthesis of abstract algebra. I seek to carve a path connecting the historical and high school understandings of algebra to abstract algebra. However, I don't think that requires postponing the study of groups until nearly the end of the first semester. Instead, I think that elementary examples and properties of both groups and rings should be studied simultaneously to motivate the modern understanding of algebra.

The Prologue tries to provoke leading questions starting from a high school, rule-based perspective. I hope that the historical perspective in Section 1.1 motivates the need for a more theoretical approach. Both of them start to answer what “thinking algebraically” means, an issue needing a response given this book's title. I discuss that briefly here and more fully in the Epilogue. After two years of high school algebra I suspect that students think of algebra as solving equations and manipulating symbols.

A look at history reveals that for most of the four-thousand-year history of mathematics, algebraic thinking centered on solving problems that we can now write in terms of equations. Operating on symbols as though they were numbers, now so characteristic of algebra, developed much more recently, starting in 1591. At approximately the same time, mathematicians expanded what counted as numbers, including negative numbers and complex numbers. Both the expanded sense of “number” and the power of manipulating symbols depend on a willingness to apply the properties of familiar numbers more broadly. Over time people recognized that other things, like polynomials and later vectors and matrices, “work” like numbers. In the nineteenth century a focus on properties—what underlies numbers and other algebraic objects—paid powerful dividends. Algebraic thinking embraced formal reasoning based on properties of symbols. As a result, algebra became both abstract and general—any system satisfying the relevant properties became a legitimate object of study. In turn investigations of properties led algebraists to uncover deeper structure—relationships between algebraic systems. Algebraic thinking employs all these aspects: solving equations, operating on symbols, studying general abstract systems by their formal properties, and investigating structural relationships among systems.

Topics

Section 1.2 provides the key shift towards a focus on properties but does so in the context of familiar algebraic systems. The series of lemmas and corollaries lay out many of the familiar properties of number systems based on properties the systems possess. I envision a class spending enough time on this section for students to present many of the exercises, which include examples and the proofs of most of the lemmas. I purposely didn’t call these results theorems. I want students to think of them as basic tools coming out of their experience with numbers that we can apply to other systems as we encounter them. Many of these basic tools formalize the rules encountered in the Prologue. I hope that this approach will convince students that the focus on properties is not a daunting step up from the more comfortable manipulation of symbols, but rather a reflection on the context for the validity of manipulation.

Section 1.3 introduces what may well be new examples involving symmetry and modular arithmetic, but ones that students generally find relatively concrete. They allow us to revisit many of the properties of the previous section and look for patterns in these finite systems. Both Sections 1.2 and 1.3 contain exercises asking students to find patterns and make conjectures, foreshadowing properties we will later prove. The search for patterns will, I hope, also motivate the shift to abstract algebra in the following chapters. I also try to provide a natural introduction of the number theory ideas so essential to abstract algebra. I purposely do not introduce all the number theory early on because students seem to find it harder when presented up front. Instead I employ a “just in time” approach, so that we introduce and prove number theory results as they are needed for algebra results.

Chapter 2 looks at what I think are the least abstract of the structural ideas. In my experience, students readily understand the idea of when two systems are identical (isomorphism) and later when one system is similar, but not identical, to another (homomorphism). Subgroups and subrings provide ways to explore all systems, and the orders of elements give insights for finite systems. The familiar coordinates of points

and vectors motivate direct products of systems, which greatly expand the range of examples without increasing the difficulty.

Once students have a stock of examples and some experience in proving properties, Chapter 3 makes the transition to a more formal development of groups. Further, the understanding of the integers (mod n) as a ring simplifies a number of the proofs here. There should be enough time in the first semester for students to understand the power of groups to approach many topics. In my experience students find general permutation groups and factor groups more difficult, so I postpone them as long as possible. However, another instructor successfully switched the order. By introducing easier examples and topics first, I intend to develop students' intuition to make these vital topics more understandable. I also choose topics to emphasize the importance of groups in understanding mathematics. I postpone the proof of the fundamental theorem of finite abelian groups (Theorem 3.2.1) to an appendix at the end of the chapter. I think that the theorem is valuable in the first semester of algebra, but not necessarily its proof.

Chapter 4 focuses on topics particular to rings, integral domains, ideals, and fields. The first three sections round out the topics often covered in a first semester abstract algebra course. In those sections I seek to relate ideals and factor rings with the factoring concept so much emphasized in high school algebra. Section 4.4 looks at deeper structural properties of integral domains. Section 4.5 gives a short introduction to Gröbner bases, an important theoretical tool used in a number of recent applications. Section 4.6 briefly considers Boolean models, a developing application of algebra in mathematical biology and other areas.

Chapter 5 starts with a more sophisticated look at linear algebra and an application of it to coding theory. After those two initial sections, the chapter develops material on field extensions to arrive at an introduction to Galois theory and the insolubility of the quintic. Galois' linking of group theory and field theory is one of the most beautiful mathematical topics accessible to undergraduates. It is also an historical culmination of nearly four thousand years of solving equations.

Chapter 6 delves more deeply into theory and applications of group theory. It starts with finite symmetry groups, building on the cyclic and dihedral groups of Section 1.3. The same section also introduces the counting technique often attributed to Burnside or Pólya, although Frobenius first proved it. We then transition to the infinite with frieze groups, wallpaper patterns, and (briefly) crystal patterns. These fit into the more general context of matrix groups, an important family of groups in many applications. These in turn help motivate the more theoretical idea of a semidirect product of groups. We round out the chapter with the Sylow theorems. Together the Sylow theorems and semidirect products enable us to understand some of the richness of finite groups.

Chapter 7 provides a view of some of the other fruitful areas of algebra. For instance lattice theory has applications as well as a rich theory interesting in its own right. It also builds on the lattices of subgroups and subrings students worked with starting in Chapter 2. After that we consider the special case of Boolean algebras so important in logic and computer science. The concept of a semigroup embraces both groups and lattices and so all of the structures studied so far. Finally, a brief taste of universal algebra can give students finishing a year-long course a vision of the perspective possible at a higher level without the severe abstraction of category theory.

Features

Exercises and Projects. Each section has exercises, which are the heart of any mathematics text. Their numbering, like the numbering of theorems, involves three digits: the chapter number, the section number, and the exercise number. Those exercises or parts with a hint or a full or partial answer at the end of the book have a star “★” at their start. In addition each chapter has supplemental exercises at the end denoted x.S.y, where x is the chapter number and y the exercise number. An instructor’s manual provides answers for all the exercises, along with other materials. After the Supplemental Exercises are Projects. Some of the projects, such as Projects 1.P.1 and 3.P.1 in Chapters 1 and 3, seek to motivate topics—in this case, dihedral groups and permutation groups. Most projects, however, involve more in-depth explorations and some are undergraduate research projects appropriate at this stage of a student’s knowledge of algebra.

Examples, Figures, and Tables. Examples are an essential part of the exposition, and this book provides many. Because later sections seldom refer to earlier examples, their single number starts over with each section. Whenever figures and tables can enhance student understanding, I try to include them. They are numbered consecutively within a chapter with two numbers, the first indicating the chapter.

Biographical sketches. I include biographical sketches of mathematicians who made significant contributions to topics in a given section. I think students benefit from understanding some of the background of the ideas they are learning. While there are many important more recent algebraists, for the most part their research is beyond the level of this text.

Prerequisites. Abstract algebra courses need, more than anything else, the nebulous quality of mathematical maturity. Many schools develop the relevant maturity in a linear algebra course and often in an introduction to proofs course. Both of these courses will provide important and sufficient background for this text. The content of linear algebra appears in many exercises, in some examples, and for some motivation. It is essential for Chapter 5. Students will develop their ability to read and formulate proofs, and I try to make early proofs more explicit to model the reasoning. Of course, the exercises also use the skills of high school algebra.

Notation. We indicate the end of a proof with the symbol \square . At the end of an example we place the symbol \diamond . Other notations are explicitly introduced. All symbols are referenced in the index.

Definitions. Definitions are in essence “if and only if” statements, and this text will write them this way. For instance a definition is often of the form “ x is a *blob* if and only if [property 1] and [property 2].” This means if we call something a blob, we affirm that the properties in the definition hold. And conversely, anything that satisfies the properties is a blob. Most mathematical texts and articles use the convention of just saying “if,” assuming that the reader is sophisticated enough to know the meaning. However, everyone agrees that in the statement of theorems, it is vital to distinguish between “if and only if” and “if.” I think pedagogically we should be just as careful with definitions.

Cover Illustration. The stained glass sculpture shown on the cover embeds an octahedron in an icosahedron. Since the time of the ancient Greeks many have admired the symmetrical beauty of these shapes individually. With the advent of group theory mathematicians have studied the symmetries and their structure. The sculpture illustrates concretely that these two polyhedra share some symmetries. In group theory language, their groups share a common subgroup. (See Section 6.1 and the related Supplemental Exercise 6.S.16.) Two former abstract algebra students, Genevieve Ahlstrom and Michael Lah, made this art piece with me. Todd Rosso took the photograph.

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