2

Systems of Linear Equations

Before continuing the study of vector spaces, we want to develop a systematic computational technique for solving certain systems of simultaneous equations. This chapter presents a simple but powerful algorithm for determining all solutions of systems of linear equations. Matrices will provide a convenient notation for conceptualizing the massive arrays of coefficients. Watch for the vector space operations of addition and scalar multiplication lurking in the background at all times.

2.1 Notation and Terminology

You are undoubtedly familiar with methods for solving systems of two simultaneous equations in two unknowns, such as

$$\begin{aligned} 2x + 3y &= 7 \\ 6x - y &= 2 \end{aligned}$$

You may also have considered systems of three or more equations in three or more unknowns, such as

$$\begin{aligned} x + y - z &= 1 \\ 2x + y \phantom{{}+4z} &= 2 \\ x - 2y + 4z &= 0 \end{aligned}$$

The technique we will develop is based on the familiar method of using one equation to eliminate the occurrence of a variable in another of the equations and gradually accumulating values of all the variables.

Mathematical Strategy Session: There are many advantages to proceeding systematically:

We will be able to deal with any number of equations in any number of unknowns.

If there is a solution, the method will find it.

If there is more than one solution, the method will find all of them.

If there are no solutions, the method will let us know.

We will be able to adapt the method to solve related problems.

We will understand how a computer program can assist with the routine arithmetic. ❄

Let us begin with an example of three linear equations in three unknowns:

$$\begin{aligned} x + y - z &= -1 \\ 2x + 4y &= -4 \\ -x + 3y + 2z &= -6 \end{aligned}$$

The term **linear** refers to the fact that the only operations applied to the variables are multiplication by various constants and addition of the resulting terms. The analogy with the two vector space operations gives a hint as to why linear equations are so useful in linear algebra.

We are interested in triples of real numbers corresponding to values for x, y, and z. Let us define a set consisting of those triples of real numbers that make all three equations true when substituted for the three corresponding variables. Taking advantage of our notation for $\mathbb{R}^3$ as the set of all ordered triples of real numbers, we can write

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = -1,\ 2x + 4y = -4,\ \text{and}\ -x + 3y + 2z = -6\}.$$

This set S is called the **solution set** of the system of equations. An element of S is called a **solution** of the system. To check whether a point of $\mathbb{R}^3$ such as $(-4, 1, -2)$ is in S, we plug in -4 for x, 1 for y, and -2 for z, and see whether all three equations hold. For the point $(-4, 1, -2)$, the first two equations hold, but $-(-4) + 3 \cdot 1 + 2 \cdot (-2) = -6$ is false. We conclude that $(-4, 1, -2) \notin S$. Apparently, it is easy to stumble across elements of $\mathbb{R}^3$ that are not in S. We want to know if S contains any points of $\mathbb{R}^3$ at all.

Suppose we simplify the system by replacing the third equation with the sum of the first and the third. That is, consider the new system

$$\begin{aligned} x + y - z &= -1 \\ 2x + 4y &= -4 \\ 4y + z &= -7 \end{aligned}$$

Notice that any solution of the original system will be a solution of the new system. Indeed, the first two equations are exactly the same, and the third holds since it is the result of adding the equal quantities in the first equation to the equal quantities in the third equation of the original system. Also note that any solution of the new system is a solution of the original system. This is because we can recover the third equation of the original system by subtracting the first equation from the new third equation. It follows then that the new system has exactly the same solution set as the old system. The new system has the advantage of being slightly simpler.

To make further progress toward simplification, we can eliminate the variable x in the second equation by adding −2 times the first equation to the second equation. This produces the system

$$\begin{aligned} x + y - z &= -1 \\ 2y + 2z &= -2 \\ 4y + z &= -7 \end{aligned}$$

Notice that we can recover the previous system by adding 2 times the first equation to the second. Thus, we can argue as above that no solutions have been lost and no new solutions have appeared. Hence, we still have exactly the same solution set.

Next we would like to use the y in the second equation to eliminate the occurrences of y in the other equations. First we will multiply the second equation by $\frac{1}{2}$. The system is changed again:

$$\begin{aligned} x + y - z &= -1 \\ y + z &= -1 \\ 4y + z &= -7 \end{aligned}$$

Take a minute to convince yourself that this did not affect the solution set. Now add −4 times the second equation to the third to give the following system (again with the same solution set):

$$\begin{aligned} x + y - z &= -1 \\ y + z &= -1 \\ -3z &= -3 \end{aligned}$$

Multiply the third equation by $-\frac{1}{3}$ to obtain $z = 1$. This gives the following system, where it is clear what value z must have in order for the equations to hold:

$$\begin{aligned} x + y - z &= -1 \\ y + z &= -1 \\ z &= 1 \end{aligned}$$

Subtracting the third equation from the second and then adding the third equation to the first will further simplify things to

$$\begin{aligned} x + y &= 0 \\ y &= -2 \\ z &= 1 \end{aligned}$$

Finally, subtract the second equation from the first to produce the simplest possible system:

$$\begin{aligned} x &= 2 \\ y &= -2 \\ z &= 1 \end{aligned}$$

There is obviously only the one point $(2, -2, 1)$ in the solution set of this system. But we have been careful not to change this solution set as we simplified the system of equations. Hence, $S = \{(2, -2, 1)\}$ is the entire solution set of the original system as well.

Before we consider some other examples to explain more thoroughly this technique for solving systems of equations, there is a simple observation that will save a lot of writing. We can abbreviate the system of equations we started with by recording only the coefficients and the constants on the right side of the equations. Thus, the 3×4 matrix

	coefficients of: x	y	z	constants on the right
coefficients from first equation:	1	1	−1	−1
coefficients from second equation:	2	4	0	−4
coefficients from third equation:	−1	3	2	−6

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & -1 \\ 2 & 4 & 0 & -4 \\ -1 & 3 & 2 & -6 \end{array}\right]$$

will represent the original system of three equations in three unknowns. The vertical line between the third and fourth columns is not part of the matrix; it appears here to indicate that the columns on the left are coefficients for x, y, and z, and the final column contains the constants on the right side of the equations. This matrix is called the **augmented matrix** of the system. The 3×3 matrix of coefficients $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 4 & 0 \\ -1 & 3 & 2 \end{bmatrix}$ has been augmented by the column $\begin{bmatrix} -1 \\ -4 \\ -6 \end{bmatrix}$ of constants. Now, instead of performing operations on the equations to simplify the system, we will just work with the rows of the augmented matrix. For example, the first step in the process described above can be performed by replacing the third row of the original matrix by the sum of the first and third rows. We let R_i denote the ith row of a matrix. This operation can be symbolized by $R_1 + R_3 \to R_3$. We suppress the vertical line in the augmented matrix and write the operation as follows:

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 4 & 0 & -4 \\ -1 & 3 & 2 & -6 \end{bmatrix} \quad R_1 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 4 & 0 & -4 \\ 0 & 4 & 1 & -7 \end{bmatrix}.$$

The entire process can be written

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 4 & 0 & -4 \\ -1 & 3 & 2 & -6 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_2 \to R_2 \\ 1R_1 + R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 1 & -7 \end{bmatrix}$$

$$\tfrac{1}{2}R_2 \to R_2 \quad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 4 & 1 & -7 \end{bmatrix}$$

$$-4R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$$-\tfrac{1}{3}R_3 \to \mathbb{R}_3 \quad \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} -1R_3 + R_2 \to R_2 \\ 1R_3 + R_1 \to R_1 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$-1R_2 + R_1 \to R_1 \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Reinterpreting the last matrix in terms of equations gives the system

$$\begin{array}{ccccc} x & & & = & 2 \\ & y & & = & -2 \\ & & z & = & 1 \end{array}$$

Here is another example of a system of three equations in three unknowns:

$$\begin{array}{rcrcrcr} & - & 3y & + & z & = & 1 \\ x & + & y & - & 2z & = & 2 \\ x & - & 2y & - & z & = & -1 \end{array}$$

The augmented matrix that encodes this system is

$$\begin{bmatrix} 0 & -3 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}.$$

Again we would like to simplify this matrix using the operations as in the previous example. The first step is to eliminate the appearance of x in all but one equation. In the previous example we added appropriate multiples of the first equation to the other equation until the first column of the matrix looked like

$$\begin{bmatrix} 1 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}.$$

The 0 in the upper-left corner of our matrix prevents us from beginning in exactly the same way, but it is obvious how to avoid this difficulty. Let us interchange the first and second rows of the matrix. This corresponds to rewriting the equations in a different order and, of course, does not change the solution set. Thus, the first step can be written

$$\begin{bmatrix} 0 & -3 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 1 & -2 & -1 & -1 \end{bmatrix}.$$

Now we can proceed as in the previous example. We use the first equation to eliminate the x variable from the other equations and then the second equation to eliminate the y variable from the equation below it:

$$\begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 1 & -2 & -1 & -1 \end{bmatrix} \quad -1R_1 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

$$-\tfrac{1}{3}R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

$$3R_2 + R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

Although we could further simplify this matrix, let us stop to look at the third row. It represents the equation $0x + 0y + 0z = -4$. There are no earthly values of x, y, and z that will make 0 equal to -4. We conclude that the solution set S of this last system of equations is empty. Of course, $S = \emptyset$ is also the solution set for the original system of equations. This example illustrates the typical way the simplification procedure leads to the conclusion that there are no solutions to a system of linear equations.

Mathematical Strategy Session: Notice that after the first step in the preceding reduction, we could have subtracted the second row from the third and come to the same conclusion. The savings in time and effort offered by such shortcuts must be weighed against the benefits of having a uniform, systematic procedure that is simple to describe, easy to implement on a computer, and convenient to work with. ❄

Before describing the procedure in detail, let us consider one more example. This example illustrates a system of equations that has an infinite number of solutions:

$$\begin{aligned} - 3y + z &= 1 \\ x + y - 2z &= 2 \\ x - 2y - z &= 3 \end{aligned}$$

This is just a minor modification of the system considered in the previous example; therefore, the reduction process begins as before:

$$\begin{bmatrix} 0 & -3 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & -1 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 1 & -2 & -1 & 3 \end{bmatrix}$$

$$-1R_1 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -3 & 1 & 1 \\ 0 & -3 & 1 & 1 \end{bmatrix}$$

$$-\tfrac{1}{3}R_2 \to R_2 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -3 & 1 & 1 \end{bmatrix}$$

$$3R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-1R_2 + R_1 \to R_1 \quad \begin{bmatrix} 1 & 0 & -\frac{5}{3} & \frac{7}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The third equation has been reduced to $0x + 0y + 0z = 0$. This puts no restriction at all in the solution set. Thus, we need to consider only the equations corresponding to the first two rows of the simplified matrix. They are

$$\begin{aligned} x \quad - \tfrac{5}{3}z &= \tfrac{7}{3} \\ y - \tfrac{1}{3}z &= -\tfrac{1}{3} \end{aligned}$$

At this point it should be clear why there are an infinite number of solutions. If any real number is chosen for any one of the unknowns, there are values of the other two unknowns that will give solutions to the two equations. For example, if we choose $x = 1$, then from the first equation we must have

$$z = -\tfrac{3}{5}(\tfrac{7}{3} - x) = -\tfrac{3}{5}(\tfrac{7}{3} - 1) = (-\tfrac{3}{5})(\tfrac{4}{3}) = -\tfrac{4}{5},$$

and from the second equation

$$y = -\tfrac{1}{3} + \tfrac{1}{3}z = -\tfrac{1}{3} + \tfrac{1}{3}(-\tfrac{4}{5}) = -\tfrac{9}{15} = -\tfrac{3}{5}.$$

Because of the way the two equations are written, however, it is much easier to assign an arbitrary value to z and let that value determine x and y. In fact, let us introduce a new variable r to stand for the specific (but arbitrary) value of z. Then

$$x = \tfrac{5}{3}r + \tfrac{7}{3}$$

and

$$y = \tfrac{1}{3}r - \tfrac{1}{3}.$$

We can write the solution set as follows:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = \tfrac{5}{3}r + \tfrac{7}{3},\ y = \tfrac{1}{3}r - \tfrac{1}{3},\ z = r \text{ for some } r \in \mathbb{R}\}.$$

A slightly neater way to arrive at this solution set is to take advantage of the notation for vector space algebra. The three equations

$$\begin{aligned} x &= \tfrac{5}{3}r + \tfrac{7}{3}, \\ y &= \tfrac{1}{3}r - \tfrac{1}{3}, \\ z &= r \end{aligned}$$

can be written as one equation in terms of 3×1 matrices:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r\begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{7}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}.$$

Since r is allowed to take on all real values, we can even eliminate some fractions by replacing r by $3r$. The vector equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{7}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}.$$

Thus, we can write the solution set

$$S = \{r(5, 1, 3) + (\tfrac{7}{3}, -\tfrac{1}{3}, 0) \mid r \in \mathbb{R}\}.$$

We instantly recognize this set as a line in $\mathbb{R}^3$.

You will find that you can use this example as a pattern whenever a system has an infinite number of solutions.

Mathematical Strategy Session: In the previous example we can read the 3×1 matrices as vectors in $\mathbb{R}^3$ with their coordinates written vertically. This column notation for vectors in $\mathbb{R}^n$ is often convenient, and we will use it freely. At the end of the example we switched back to row notation simply to conserve space in writing the solution set. Although it would not be appropriate to mix notation in a single expression such as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (0, 2),$$

you should feel free to choose the notation that suits your immediate needs. ❄

Exercises

1. For each of the following systems, add a multiple of one equation to another equation to obtain an equivalent system with one unknown in each equation. Write the solution set.

a.[A] $$\begin{aligned} x + 2y &= 7 \\ y &= 2 \end{aligned}$$

b. $$\begin{aligned} x \phantom{{}+6y} &= 2 \\ -2x + 6y &= 5 \end{aligned}$$

c. $$\begin{aligned} x - 3y \phantom{{}+z} &= 2 \\ y &= -1 \\ z &= 5 \end{aligned}$$

d. $$\begin{aligned} x \phantom{{}+y} + 7z &= 0 \\ y &= -3 \\ z &= 2 \end{aligned}$$

2. Find a strategy for adding multiples of one equation to another equation in the following systems to obtain an equivalent system with one unknown in each equation. Write the solution set.

a.[A] $$\begin{aligned} x - 3y - 2z &= 2 \\ y + z &= 1 \\ z &= -2 \end{aligned}$$

b. $$\begin{aligned} x + 5y + 2z &= -2 \\ y + z &= 0 \\ z &= 1 \end{aligned}$$

c. $$\begin{aligned} w + 2x + 3y + 4z &= 5 \\ x + 2y + 3z &= 4 \\ y + 2z &= 3 \\ z &= 2 \end{aligned}$$

d. $$\begin{aligned} w - 3x - 2y \phantom{{}+2z} &= 0 \\ x \phantom{{}-2y} + 2z &= -2 \\ y - 5z &= -3 \\ z &= 2 \end{aligned}$$

e. Which order of eliminating the variables tends to minimize the number of computations?

3.[H] For the following systems of equations, add a multiple of the first equation to each of the other equations to reveal a system that has no solutions.

a. $$\begin{aligned} x + 3y &= 5 \\ 3x + 9y &= 4 \end{aligned}$$

b. $$\begin{aligned} 6x + 9y &= 1 \\ 4x + 6y &= 1 \end{aligned}$$

c. $$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + 3z &= 2 \\ 3x + 4y + 4z &= 0 \end{aligned}$$

d. $$\begin{aligned} 6x + y - 2z &= 0 \\ -3x - y + 4z &= 1 \\ y - 6z &= -1 \end{aligned}$$

e. $$\begin{aligned} 9x + 6y - 3z &= 3 \\ 6x + 4y - 2z &= 4 \end{aligned}$$

f. $$\begin{aligned} x + y \phantom{{}+z} &= 2 \\ y + z &= 2 \\ x \phantom{{}+y} + z &= 2 \\ x + y + z &= 2 \end{aligned}$$

4. The following systems of equations in three unknowns x, y, and z have an infinite number of solutions. Assign an arbitrary variable to one of the unknowns and solve for all the unknowns in terms of the arbitrary variable. Write the solution set in the form of a line in $\mathbb{R}^3$.

a.[A] $\begin{aligned} x + 2y \quad &= 2 \\ z &= 3 \end{aligned}$

b. $\begin{aligned} x \quad - 5z &= 1 \\ y + 2z &= -1 \end{aligned}$

c. $\begin{aligned} y \quad &= 4 \\ z &= 2 \end{aligned}$

d. $\begin{aligned} x - 2y + 4z &= 1 \\ y - 2z &= -1 \end{aligned}$

5. The following systems have an infinite number of solutions. Assign arbitrary variables to two of the unknowns and solve for all the unknowns in terms of these two arbitrary variables. Write the solution set in the form of a plane in Euclidean space.

a.[A] $\begin{aligned} w + 2x \quad + 3z &= 1 \\ y - z &= 2 \end{aligned}$

b. $\begin{aligned} x_1 + 3x_2 - x_3 \quad &= -6 \\ x_4 &= 5 \end{aligned}$

c. $\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 1 \\ x_2 - 2x_3 - x_4 &= 3 \\ 2x_2 - 4x_3 - 2x_4 &= 6 \end{aligned}$

d. $\begin{aligned} x_1 + 3x_2 \quad - x_5 &= 0 \\ x_3 \quad + 2x_5 &= -1 \\ x_4 + x_5 &= 2 \end{aligned}$

6. Give the augmented matrix for each of the following systems of equations.

a. $\begin{aligned} \tfrac{1}{2}x + 3y - 2z &= 3 \\ -x + y - z &= 2 \\ 2x - 2y + 5z &= -1 \end{aligned}$

b. $\begin{aligned} x + 3y &= -4 \\ 2x - y &= 3 \\ 3x + 2y &= -2 \end{aligned}$

c. $\begin{aligned} 5x_1 - 2x_2 + x_3 + 2x_4 &= 3 \\ x_1 \quad - x_3 + x_4 &= 1 \\ 3x_2 - x_3 + 3x_4 &= 5 \end{aligned}$

d. $\begin{aligned} x_1 + 3x_2 \quad - x_5 &= 0 \\ x_3 \quad + 2x_5 &= -1 \\ x_4 + x_5 &= 2 \end{aligned}$

7. Give a system of equations for each of the following augmented matrices.

a. $\begin{bmatrix} 2 & 1 & -2 & 3 \\ 3 & -2 & 1 & -5 \\ 1 & 4 & -2 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 4 & 0 & -\frac{1}{3} & -1 \\ 0 & 0 & -2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -3 & 1 \\ 5 & 0 & 3 \\ 1 & -3 & 4 \\ -2 & 1 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 4 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

8.[S] Geometrically determine the possible solution sets to a system of two equations in two unknowns:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

Sketch pictures to illustrate the possibilities.

9. a. Show that the solution set to the equation $ax + by + cz = d$ is a plane in $\mathbb{R}^3$ as long as one or more of the coefficients a, b, c is nonzero.
   b. Under what conditions on the coefficients does this plane pass through the origin?
   c. What are the possible solution sets if $a = b = c = 0$?

10. Geometrically determine the possible solution sets to a system of two equations in three unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \end{aligned}$$

Sketch pictures to illustrate the possibilities.

11. Geometrically determine the possible solution sets to a system of three equations in three unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Sketch pictures to illustrate the possible ways each of these solution sets can arise.

2.2 Gaussian Elimination

The process we have been using to simplify matrices is known as Gauss-Jordan reduction or, more simply, Gaussian elimination. The method will be described in two parts. First is a description of the three legal row operations. This is followed by the strategy employed in simplifying the matrix.

> **Crossroads:** The article "Gauss-Jordan Reduction: A Brief History" by Steven Althoen and Renate McLaughlin in the February 1987 issue of the *American Mathematical Monthly* gives some interesting background into the origins of this method. ❄

We will find that Gaussian elimination is useful in solving many types of problems that can be formulated in terms of systems of linear equations. In some of these applications we will not be dealing with an augmented matrix that directly represents the system of equations. Thus, it is convenient to describe the process as applied to an arbitrary matrix rather than thinking of the matrix necessarily as the augmented matrix for some system of equations.

2.1 Definition: *The three* **elementary row operations** *are:*

1. *Interchange two rows:* $R_i \leftrightarrow R_j$.
   Interchange the entries in row i with the corresponding entries in another row j. Leave all other rows unchanged.
2. *Multiply a row by a nonzero constant:* $cR_i \to R_i$.
   Replace each entry in row i by the product of that entry with a nonzero real number c. Leave all other rows unchanged.
3. *Add a multiple of one row to another row:* $cR_i + R_j \to R_j$.
   Replace each entry in row j by the sum of that entry and a constant c times the corresponding entry in row i. Leave other rows, including row i, unchanged.

The following definition will be used in describing the strategy for reducing a matrix.

2.2 Definition: *A* **leading entry** *of a matrix is the first nonzero entry of a row of the matrix as you read across the row from left to right. A* **leading 1** *of a matrix is a leading entry that is equal to* 1.

Mathematical Strategy Session: Here is the Gauss-Jordan reduction algorithm for using the three row operations to simplify a matrix. The idea consists of building up a pattern of leading 1s in columns that otherwise contain only 0s. It is important to notice that row operations performed in this process do not disturb the pattern of 1s and 0s obtained in previous steps.

1. Interchange rows so that among all the rows the first row begins with the fewest zeros. Often you will begin with a matrix that has a nonzero entry in the upper-left corner; in such a case no row interchanges need to be done for the first step.
2. Now we want the first row to have a leading 1. To accomplish this, multiply the first row by the reciprocal of its leading entry.
3. In the column below the leading 1 created in the previous step, there may be some nonzero entries. Add the appropriate multiples of the first row to each of the other rows to replace these entries in this column by zeros. At this point all entries below and in any columns to the left of the leading 1 in the first row will be zeros. (Possibly there will be no entries at all to the left of this leading 1.)
4. Next apply steps 1, 2, and 3 to the submatrix formed by ignoring the first row of the matrix. This will produce a leading 1 in the next row with only zero entries below and to the left of it.
5. Continue to repeat this process until you either reach the bottom of the matrix or find that all the remaining rows have only zero entries.

Quick Example: *Apply the first five steps of the Gauss-Jordan reduction algorithm to the matrix* $\begin{bmatrix} 0 & 2 & -6 & 4 \\ -1 & 1 & 1 & 0 \\ 2 & 1 & -13 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 2 & -6 & 4 \\ -1 & 1 & 1 & 0 \\ 2 & 1 & -13 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix} \quad \begin{array}{c}\text{Step 1} \\ R_1 \leftrightarrow R_2\end{array} \quad \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 2 & -6 & 4 \\ 2 & 1 & -13 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{c}\text{Step 2} \\ -1R_1 \to R_1\end{array} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 2 & -6 & 4 \\ 2 & 1 & -13 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{c}\text{Step 3} \\ -2R_1 + R_3 \to R_3 \\ -1R_1 + R_4 \to R_4\end{array} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 2 & -6 & 4 \\ 0 & 3 & -11 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{c}\text{Step 4} \\ \text{(applying steps 1 and 2} \\ \text{to the bottom three rows)} \\ \tfrac{1}{2}R_2 \to R_2\end{array} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 3 & -11 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{c}\text{Step 4, continued} \\ \text{(applying step 3 to the} \\ \text{bottom three rows)} \\ -3R_2 + R_3 \to R_3\end{array} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{c}\text{Step 5} \\ \text{(applying steps 1, 2, and} \\ \text{3 to the bottom two rows)} \\ -\tfrac{1}{2}R_3 \to R_3 \\ -1R_3 + R_4 \to R_4\end{array} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

❄

After these first five steps, the matrix will be in row-echelon form; that is, it will satisfy the conditions of the following definition.

2.3 Definition: *A matrix is in* **row-echelon form** *when:*

1. *Each row that is not entirely zero has a leading* 1.
2. *The leading* 1*s appear in columns farther to the right as you consider the rows of the matrix from top to bottom.*
3. *Any rows of zeros are together at the bottom of the matrix.*

Quick Example: *Verify that the matrix*

$$\begin{bmatrix} 0 & 1 & 5 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row-echelon form.

The first three rows are not entirely zero. Each has a leading 1. The leading 1 in the first row is in column 2, the leading 1 in the second row is in column 4, and the leading 1 in the third row is in column 5. Thus, the leading 1s appear farther to the right as we move down the matrix. Finally, there is one row of zeros at the bottom of the matrix. ❄

Once a matrix has been transformed to have leading 1s in the staircase pattern characteristic of row-echelon form, one further step will produce a matrix that satisfies an additional condition. Our goal is to obtain a matrix that satisfies the four conditions in the following definition.

2.4 Definition: *A matrix is in* **reduced row-echelon form** *when:*

1. *Each row that is not entirely zero has a leading* 1.
2. *The leading* 1*s appear in columns farther to the right as you consider the rows of the matrix from top to bottom.*
3. *Any rows of zeros are together at the bottom of the matrix.*
4. *Only zeros appear in the portions of the columns above the leading* 1*s.*

Mathematical Strategy Session: After the five steps given previously have been applied to put a matrix in row-echelon form, here is the final step that will transform the matrix to reduced row-echelon form. It involves using each leading 1 in turn to clear out the nonzero entries in the portion of the column above it. By starting at the bottom, you will be able to create zeros that will reduce the amount of arithmetic later.

6. Beginning with the bottommost row that contains a leading 1 and working upward, add appropriate multiples of this row to the rows above it to introduce zeros in the column above the leading 1. ❄

Quick Example: *Apply the sixth step in the Gauss-Jordan reduction algorithm to put the matrix*

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{c} \text{Step 6} \\ \text{(using the leading 1} \\ \text{in the third row)} \\ 3R_3 + R_2 \to R_2 \\ 1R_3 + R_1 \to R_1 \end{array} \quad \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{Step 6, continued} \\ \text{(using the leading 1} \\ \text{in the second row)} \\ 1R_2 + R_1 \to R_1 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

❄

As you practice putting a few matrices in reduced row-echelon form, keep in mind the relation between the row operations you are performing and the idea of eliminating unknowns in a system of linear equations. This will help you keep track of the steps to be performed.

It should become clear that the Gauss-Jordan reduction algorithm is an efficient, mechanical, straightforward way of transforming any matrix to reduced row-echelon form. The most annoying feature of the process is the necessity of dealing with fractions. Most of the examples and exercises in this text have been rigged to minimize the amount of complicated arithmetic. You should feel free to take advantage of any available calculator or computer facilities that enable you to concentrate on the reduction algorithm rather than the arithmetic.

Crossroads: As you apply the Gauss-Jordan reduction algorithm, you may occasionally notice a shortcut that will save a few steps or make the arithmetic easier. For example, you may be able to reduce the number of computations involving fractions if you delay changing a leading entry to a leading 1 until you have cleared out the nonzero entries below the leading entry. Of course, these different sequences of row operations will not change the solution set if the matrix comes from a system of equations (no row operation ever changes the solution set). Yet what if the two sequences of operations result in two different row-echelon forms? It is reassuring to find that the resulting reduced row-echelon forms will be the same. The proof that there is only one possible result when a given matrix is put in reduced row-echelon form requires some ideas that will be developed soon. The following theorem is for background information; it will not be used in this text. A one-paragraph proof of this result can be found in the article "The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof" by Thomas Yuster in the March 1984 issue of *Mathematics Magazine*. (You might also enjoy the cartoon and riddle that immediately follow the article.) ❄

2.5 Theorem: *Suppose a sequence of elementary row operations transforms a matrix M into a matrix M' in reduced row-echelon form. Suppose another sequence transforms M into a matrix M'', also in reduced row-echelon form. Then $M' = M''$.*

Exercises

1.[H] How many columns of the matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ contain leading 1s? How many rows of this matrix contain leading 1s?

2. For each type of row operation, show that there is a row operation that will undo it. That is, if M is transformed into M' by a certain row operation, determine a row operation that can be applied to M' to yield M.

3. If two row operations are applied in succession to transform the matrix M into the matrix M', describe the row operations that will transform M' back to M. What if n row operations are used?

4. What is the next step in applying the Gauss-Jordan reduction algorithm to each of the following matrices?

a. $\begin{bmatrix} 1 & 5 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & -5 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \\ 0 & 1 \\ 4 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -1 & -\frac{4}{7} & 0 & 6 \\ 0 & 1 & \frac{4}{7} & 0 & -6 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{9}{8} \end{bmatrix}$ d. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

5.[A] Put the following matrices in reduced row-echelon form.

a. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 6 \\ -1 & 2 & -2 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 1 & -2 & -5 \\ 1 & 1 & -1 & -3 \\ 3 & 2 & -2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & -1 & -7 & 0 \\ 2 & -3 & -7 & -7 \end{bmatrix}$ d. $\begin{bmatrix} 1 & -2 & 2 & 11 \\ -1 & 2 & 3 & -1 \\ -2 & 4 & 0 & -14 \end{bmatrix}$

6. Put the following matrices in reduced row-echelon form.

a. $\begin{bmatrix} 2 & -4 \\ -3 & 6 \\ 1 & 2 \\ -2 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 2 & 0 \\ -2 & 0 & -6 & -1 \\ 4 & -1 & 10 & 2 \\ 1 & 0 & 3 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 4 & 1 & 1 & 1 \\ 0 & -1 & -4 & 1 & -1 & 3 \\ 0 & 1 & 4 & 0 & 1 & -1 \\ 0 & 2 & 8 & 3 & 2 & 4 \end{bmatrix}$

7.[S] Regard two row-reduced matrices as having the same echelon pattern if the leading 1s occur in the same positions. If we let an asterisk act as a wild card to denote any number, the four echelon patterns for a 2×2 matrix are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

a. Write down the possible echelon patterns for a 2×3 matrix.

b. How many patterns are possible for 3×5 matrices?

8.[H] Give an example to show that the row-echelon form of a matrix is not unique.

9.[H] a. Show that the system

$$\begin{aligned} ax + by &= r \\ cx + dy &= s \end{aligned}$$

has a unique solution if $ad - bc \neq 0$.

b. Show that the system does not have a unique solution if $ad - bc = 0$. What are the possibilities in this case?

10. One measure of the efficiency of a numerical algorithm is a count of the number of the arithmetic operations required to carry out the algorithm. For the Gauss-Jordan reduction algorithm, we can simplify the count in several ways. First, ignore row interchanges. Second, when multiplying a row by a constant, count one operation for each entry to the right of the leading entry (that is, ignore zeros to the left of the leading entry as well as the leading entry itself). Third, when adding a multiple of one row to another row, count one operation for each entry to the right of the leading entry in the first row (that is, combine the multiplication and addition into one operation and ignore zeros to the left of the leading entry as well as the leading entry itself). Finally, assume the leading entries always have to be changed to leading 1s and that they occur in successive columns staring on the left of the matrix.

a.[H] How many operations are required to put a 3×4 matrix into row-echelon form?

b. How many operations are required to put a 4×5 matrix into row-echelon form?

c.[H] How many operations are required to put an $m \times (m+1)$ matrix into row-echelon form?

d. How many additional operations are required to take a 3×4 matrix in row-echelon form and to put it in reduced row-echelon form?

e. How many additional operations are required to put a 4×5 matrix in row-echelon form and to put it in reduced row-echelon form?

f. How many additional operations are required to put an $m \times (m + 1)$ matrix in row-echelon form and to put it in reduced row-echelon form?

g.[H] For $n > m$, extend your results to determine the number of operations required to put an $m \times n$ matrix in row-echelon form.

h. For $n > m$, extend your results to determine the number of additional operations required to take an $m \times n$ matrix in row-echelon form and to put it in reduced row-echelon form.

2.3 Solving Linear Systems

Since the applications of Gaussian elimination tend to be related to solving systems of linear equations, let us return to that problem and formulate a standard way to solve such systems and to write the solution set S.

The first step is to form the augmented matrix of the system. Each row of this matrix corresponds to one of the equations. The rightmost column contains the constants that appear on the right side of the equations. Each of the other columns corresponds to one of the unknowns; the entries are the coefficients of that unknown. The next step is to apply the Gauss-Jordan reduction process to put the matrix in reduced row-echelon form.

We know from examples in Section 2.1 that there are three possibilities. If the rightmost column contains a leading 1, the row it is in corresponds to the equation $0 = 1$. Thus, the system has no solution, and we can describe the solution set S by writing

$$S = \emptyset.$$

If every column but the rightmost contains a leading 1, then the system has a unique solution. The coordinates of this single vector can be read in the rightmost column of the reduced matrix. We can write the solution set as

$$S = \{(b_1, b_2, \ldots, b_n)\}.$$

Finally, suppose there are columns other than the rightmost that do not contain leading 1s of any row. The corresponding unknowns are called **free variables**, and we can assign arbitrary values to them. We will typically use parameters such as r, s, and t to denote the real numbers assigned to these variables. The **leading variables**, those unknowns corresponding to columns with leading 1s, will be determined by the values assigned to the free variables. For example, suppose the reduced matrix of a system turns out to be

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & 0 & 5 & -1 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that the second and fifth columns do not contain a leading 1 of any row. We assign the values r and s to the corresponding free variables x_2 and x_5 and allow r and s independently to take on all real values. Then the first row corresponds to the equation

$$x_1 = -r + 2s + 4.$$

The second and third rows give equations for the other two leading variables x_3 and x_5 in terms of the parameters r and s. Altogether, we have one equation for each variable:

$$\begin{aligned} x_1 &= -r + 2s + 4, \\ x_2 &= r, \\ x_3 &= \quad - 5s - 1, \\ x_4 &= \quad - 3s, \\ x_5 &= \quad s. \end{aligned}$$

Using the notation of column vectors, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ -5 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, the set of all solutions is a plane in $\mathbb{R}^5$. We can switch back to row notation and write the solution set as

$$S = \{r(-1, 1, 0, 0, 0) + s(2, 0, -5, -3, 1) + (4, 0, -1, 0, 0) \mid r, s \in \mathbb{R}\}.$$

Now that we have seen how to apply the Gauss-Jordan reduction technique to solve any system of linear equations, let us concentrate on the special situation where the constants on the right side of the equations are all equal to zero. Such a system is called a **homogeneous** system. That is, a homogeneous system of m equations in n unknowns will be of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

The first fact you should notice about a homogeneous system is that the solution set will never be empty. Indeed, $\mathbf{0} = (0, 0, \ldots, 0)$ will always be a solution. This is known as the **trivial solution**. For a homogeneous system, the question is whether the solution set S contains any nontrivial solutions (solutions other than $\mathbf{0}$).

Next, notice that the augmented matrix of a homogeneous system will have a column of zeros as its rightmost column. Furthermore, the three row operations used in the Gauss-Jordan reduction process will always preserve this column of zeros. Hence, when we apply the reduction process to a homogeneous system, we will frequently omit the augmentation column. We will apply the standard reduction process to the coefficient matrix with the understanding that the constants on the right side of the equations are all equal to zero.

Quick Example: *Solve the homogeneous system*

$$\begin{array}{ccccccc} x & - & 2y & - & z & = & 0 \\ 2x & - & y & + & 2z & = & 0 \\ x & + & y & + & 3z & = & 0 \end{array}$$

We begin by reducing the coefficient matrix:

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \quad \begin{matrix} -2R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{bmatrix}$$

$$\tfrac{1}{3}R_2 \to R_2 \quad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & \frac{4}{3} \\ 0 & 3 & 4 \end{bmatrix}$$

$$-3R_2 + R_3 \to R_3 \quad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$2R_2 + R_1 \to R_1 \quad \begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

This final coefficient matrix corresponds to the system

$$\begin{aligned} x \quad + \tfrac{5}{3}z &= 0 \\ y + \tfrac{4}{3}z &= 0 \\ 0 &= 0 \end{aligned}$$

If we write our choice of an arbitrary value for the free variable z as $3r$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

as the general form of a solution. Thus, the solution set is a line in $\mathbb{R}^3$. In the standard form for a line, the solution set looks like

$$S = \{r(-5, -4, 3) \mid r \in \mathbb{R}\}.$$

❄

Crossroads: Notice that the solution set of the linear system in the previous example contains not just the trivial solution, but an infinite number of other points. The solution set is a line in $\mathbb{R}^3$ through the origin. In particular (see Exercise 23 of Section 1.8), this set is a subspace of $\mathbb{R}^3$. In Chapter 5 we will develop the tools to make it easy to show that the solution set of any homogeneous system of linear equations in n unknowns is a subspace of $\mathbb{R}^n$. In fact, by the time you reach Exercise b. of Section 5.1, this result will be a straightforward application of the Subspace Theorem. ❄

You may be familiar with the general result that a homogeneous system with more unknowns than equations will always have a nontrivial solution. This result will play a key role in our development of the concept of the dimension of a vector space.

2.6 Fundamental Theorem of Homogeneous Systems: *A homogeneous system of m linear equations in n unknowns with $n > m$ has at least one nontrivial solution.*

Proof: The coefficient matrix of such a system will be an $m \times n$ matrix. In particular, it will have more columns than rows. Thus, when we apply the Gauss-Jordan reduction process to this matrix, the number of leading 1s—which cannot exceed m, the number of rows—will be less than n, the number of columns. It follows that at least one of the columns will not have a leading 1. The unknown corresponding to such a column can be assigned an arbitrary real number. In particular, any nonzero value for this free variable will lead to a nontrivial solution of the system of equations. ❄

Exercises

Solve the systems of linear equations. For each system, write the solution as a set with zero or one point, or as a line, plane, or higher-dimensional analog in $\mathbb{R}^n$. Once you have mastered the steps in the Gauss-Jordan reduction algorithm, you are encouraged to take advantage of any available computer facilities for reducing the matrices associated with these systems of equations.

1. a.
$$\begin{aligned} x + y - 2z &= 1\\ 3x + 2y + z &= -5\\ -x + y + z &= -11 \end{aligned}$$
b.
$$\begin{aligned} 3x - 5y + 2z &= -8\\ x + y &= -3\\ 2y - 4z &= -2 \end{aligned}$$
c.
$$\begin{aligned} x + 2y + z &= 1\\ x - y + 2z &= 2\\ x + 8y - z &= -1 \end{aligned}$$
d.
$$\begin{aligned} x + 2y + z &= 1\\ x - y + 2z &= 2\\ x + 8y - z &= -2 \end{aligned}$$

2. a.
$$\begin{aligned} 2x - 4y + z &= 1\\ 4x + 2y - z &= 1 \end{aligned}$$
b.
$$\begin{aligned} 3x + 6y + z &= 0\\ x + 2y + z &= 2 \end{aligned}$$

3. a.
$$\begin{aligned} 4x - 6y + \tfrac{2}{3}z &= 2\\ 6x - 9y + z &= 3 \end{aligned}$$
b.
$$\begin{aligned} 4x - 6y + \tfrac{2}{3}z &= 2\\ 6x - 9y + z &= 2 \end{aligned}$$

4. a.
$$\begin{aligned} w + 3x + y - z &= 0\\ w + 3x + 2y - z &= -2\\ 2w + x + y + z &= 0\\ -w + 2x - y + z &= 1 \end{aligned}$$
b.
$$\begin{aligned} x_1 + x_2 + x_3 &= 3\\ -x_1 + x_2 + x_4 &= 0\\ x_1 - x_2 + x_3 + x_4 &= -4 \end{aligned}$$

c.
$$\begin{array}{rcrcrcrcr} & & & & x_3 & + & x_4 & = & 1 \\ 2x_1 & + & x_2 & + & 2x_3 & & & = & 0 \\ & & x_2 & & & - & 4x_4 & = & -4 \end{array}$$

d.
$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & = & 6 \\ x_1 & - & x_2 & + & x_3 & - & x_4 & + & x_5 & - & x_6 & = & -6 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & - & x_5 & - & x_6 & = & -2 \\ & & x_2 & - & x_3 & + & x_4 & - & x_5 & + & x_6 & = & 8 \end{array}$$

5. a.
$$\begin{array}{rcrcrcr} x & + & 5y & + & 3z & = & -7 \\ -x & - & 5y & - & 2z & = & 4 \\ 2x & + & 10y & + & 5z & = & -11 \end{array}$$

b.
$$\begin{array}{rcrcrcrcr} w & & & + & y & - & 2z & = & 1 \\ 2w & + & x & & & & & = & 4 \\ w & + & x & - & y & + & z & = & 3 \end{array}$$

c.
$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & + & 2x_3 & - & 3x_4 & - & 2x_5 & = & 5 \\ & & & & 2x_3 & + & x_4 & + & x_5 & = & 1 \\ -4x_1 & - & 2x_2 & & & + & 7x_4 & + & 8x_5 & = & -9 \end{array}$$

6. a.
$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 7 \\ 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 3 \\ x_1 & - & 3x_2 & + & 2x_3 & & & + & 2x_5 & = & -4 \end{array}$$

b.
$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 3 \\ x_1 & - & 3x_2 & + & 2x_3 & & & + & 2x_5 & = & -4 \end{array}$$

c.
$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 7 \\ 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 3 \\ x_1 & - & 3x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & -4 \end{array}$$

d.
$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & - & 2x_5 & = & 0 \\ 2x_1 & - & x_2 & + & x_3 & + & x_4 & & & = & 0 \\ x_1 & - & 3x_2 & + & 2x_3 & & & + & 2x_5 & = & 0 \end{array}$$

7. a.
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

b.
$$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + 4x_2 + x_3 &= 0 \end{aligned}$$

c.
$$\begin{aligned} 12x - 3y - 2z &= 0 \\ x - y - z &= 0 \\ 6x + 3y + 4z &= 0 \end{aligned}$$

d.
$$\begin{aligned} x + y + z &= 0 \\ x - y + 2z &= 0 \\ 2x + 3z &= 0 \\ x - 3y + 3z &= 0 \\ x + 3y &= 0 \end{aligned}$$

8.[A] a.
$$\begin{aligned} x + y - z &= 1 \\ 2x + 4y &= 4 \\ -x + 3y + 2z &= 6 \end{aligned}$$

b.
$$\begin{aligned} x + y - z &= 3 \\ 2x + 4y &= 12 \\ -x + 3y + 2z &= 18 \end{aligned}$$

c.
$$\begin{aligned} x + y - z &= -2 \\ 2x + 4y &= -8 \\ -x + 3y + 2z &= -12 \end{aligned}$$

d.
$$\begin{aligned} x + y - z &= 0 \\ 2x + 4y &= 0 \\ -x + 3y + 2z &= 0 \end{aligned}$$

e. Compare your solutions of these systems with the solution of the example near the beginning of Section 2.1. Make a conjecture as to the solution of

$$\begin{aligned} x + y - z &= \tfrac{1}{2} \\ 2x + 4y &= 2 \\ -x + 3y + 2z &= 3 \end{aligned}$$

f. Formulate a general principle about what seems to be the relation between the constants on the right side of the equations and the solution.

9. Suppose (x_1, y_1, z_1) and (x_2, y_2, z_2) are solutions of the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Prove that $(x_1, y_1, z_1) - (x_2, y_2, z_2)$ is a solution of the homogeneous system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

10. Suppose (x_0, y_0, z_0) is a solution of the homogeneous system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

and (x_1, y_1, z_1) is a solution of the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Prove that $(x_0, y_0, z_0) + (x_1, y_1, z_1)$ is a solution of the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

11.[S] a. Find a solution (x_1, y_1, z_1) of the system of equations

$$\begin{aligned} x - 2y + z &= 8 \\ x + 2y - 2z &= 8 \\ x + y - z &= 7 \end{aligned}$$

b. Find a solution (x_2, y_2, z_2) of the system of equations

$$\begin{aligned} x - 2y + z &= 1 \\ x + 2y - 2z &= 0 \\ x + y - z &= 1 \end{aligned}$$

c. Show that $(x_1, y_1, z_1) + (x_2, y_2, z_2)$ is a solution of the system of equations

$$\begin{aligned} x - 2y + z &= 9 \\ x + 2y - 2z &= 8 \\ x + y - z &= 8 \end{aligned}$$

d. Show that $(x_1, y_1, z_1) - (x_2, y_2, z_2)$ is a solution of the system of equations

$$\begin{aligned} x - 2y + z &= 7 \\ x + 2y - 2z &= 8 \\ x + y - z &= 6 \end{aligned}$$

e. Show that $10(x_1, y_1, z_1) + 4(x_2, y_2, z_2)$ is a solution of the system of equations

$$\begin{aligned} x - 2y + z &= 84 \\ x + 2y - 2z &= 80 \\ x + y - z &= 74 \end{aligned}$$

f. Make and verify a conjecture about solutions of the systems of equations

$$\begin{aligned} x - 2y + z &= 804 \\ x + 2y - 2z &= 800 \\ x + y - z &= 704 \end{aligned} \quad \text{and} \quad \begin{aligned} x - 2y + z &= 161 \\ x + 2y - 2z &= 160 \\ x + y - z &= 141 \end{aligned}$$

g. Make and verify a conjecture about solutions of the systems of equations

$$\begin{aligned} x - 2y + z &= 8a + b \\ x + 2y - 2z &= 8a \\ x + y - z &= 7a + b \end{aligned}$$

for arbitrary real numbers a and b.

12. a. Find a solution (x_1, y_1, z_1) of the system of equations
$$\begin{aligned} x - y - z &= 1 \\ -2x + y + z &= 0 \\ -x + 2y + z &= 0 \end{aligned}$$
b. Find a solution (x_2, y_2, z_2) of the system of equations
$$\begin{aligned} x - y - z &= 0 \\ -2x + y + z &= 1 \\ -x + 2y + z &= 0 \end{aligned}$$
c. Find a solution (x_3, y_3, z_3) of the system of equations
$$\begin{aligned} x - y - z &= 0 \\ -2x + y + z &= 0 \\ -x + 2y + z &= 1 \end{aligned}$$
d. Show that $2(x_1, y_1, z_1) - 3(x_2, y_2, z_2) + (x_3, y_3, z_3)$ is a solution of the system
$$\begin{aligned} x - y - z &= 2 \\ -2x + y + z &= -3 \\ -x + 2y + z &= 1 \end{aligned}$$
e. Show that $a(x_1, y_1, z_1) + b(x_2, y_2, z_2) + c(x_3, y_3, z_3)$ is a solution of the system
$$\begin{aligned} x - y - z &= a \\ -2x + y + z &= b \\ -x + 2y + z &= c \end{aligned}$$

13. Prove that if a homogeneous system of linear equations has a nontrivial solution, then it has an infinite number of solutions.

14. Prove that if a system of linear equations has two solutions, then it has an infinite number of solutions.

2.4 Applications

This section is devoted to sampling the variety of applications of systems of linear equations that arise in problems in mathematics and other quantitative disciplines.

In Section 1.9 we saw how systems of linear equations are intimately related to the study of lines and planes in Euclidean spaces. We will frequently encounter important results in linear algebra that we will be able to reformulate in terms of systems of linear equations.

You have probably also used systems of linear equations in other mathematics courses, perhaps in the integration technique involving partial fractions decompositions or in the method of undetermined coefficients for solving differential equations. You may be surprised to see how naturally linear systems arise in other areas of the physical, biological, and social sciences.

Let us begin with an example of curve fitting. The general problem is to find a mathematical expression for a curve that satisfies certain specified conditions. Mathematicians have studied the problem of curve fitting as a basic technique for approximating a complicated function by a simpler function. You may recall, for example, that

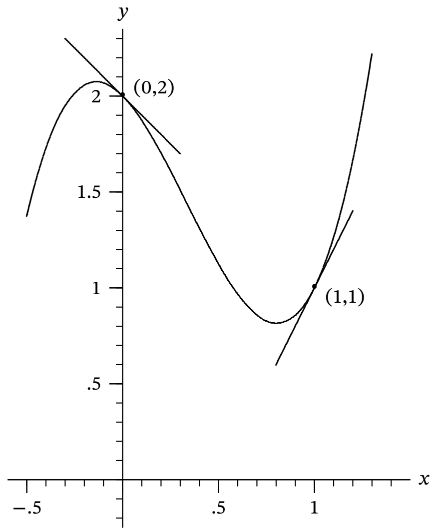

Figure 2.7. Polynomial p satisfying $p(0) = 2$, $p(1) = 1$, $p'(0) = -1$, and $p'(1) = 2$.

Simpson's rule is based on integrating quadratic polynomials that approximate a more complicated function. The development of computer graphics has further increased the interest in finding formulas for curves that pass through predetermined points in predetermined directions.

Quick Example: *Determine a polynomial p whose graph passes through* $(0, 2)$ *with a slope of* -1 *and through* $(1, 1)$ *with a slope of* 2.

It seems likely that the four coefficients of a cubic polynomial defined by $p(x) = ax^3 + bx^2 + cx + d$ will provide sufficient flexibility to meet the four requirements. For the graph of p to pass through $(0, 2)$ and $(1, 1)$, we must have $p(0) = 2$ and $p(1) = 1$. To meet the conditions on the slope, p must also satisfy $p'(0) = -1$ and $p'(1) = 2$. An instant after we compute the derivative $p'(x) = 3ax^2 + 2bx + c$, we can write down four linear equations in four unknowns:

$$\begin{aligned} d &= 2\\ a + b + c + d &= 1\\ c &= -1\\ 3a + 2b + c &= 2 \end{aligned}$$

The single solution $(a, b, c, d) = (3, -3, -1, 2)$ gives the cubic polynomial defined by $p(x) = 3x^3 - 3x^2 - x + 2$ that satisfies the four conditions. See Figure 2.7. ❄

The field of twentieth-century mathematics known as linear programming deals with linear systems involving both equalities and inequalities. Problems of allocating limited resources frequently lead to linear programming models. The following example illustrates such a situation in which most of the linear constraints are equalities.

Quick Example: *A manufacturer makes three kinds of plastics A, B, and C. Each kilogram of A produces* 10 *grams of particulate matter discharged into the air and* 30 *liters of liquid waste discharged into the river. Each kilogram of B produces* 20 *grams of particulate matter and* 50 *liters of liquid waste. Each kilogram of C produces* 20 *grams of particulate matter and* 110 *liters of liquid waste. The Environmental Protection Agency limits the company to* 2550 *grams of particulate matter per day and* 7950 *liters of liquid waste discharge per day. Determine the production levels of the three plastics that will result in emission levels that reach the maximum values for both particulate matter and liquid waste.*

We begin by introducing variables x, y, and z to represent the number of kilograms of the three kinds of plastics that can be made. The constraints on the particulate and liquid waste levels translate into the two equations

$$\begin{aligned} 10x + 20y + 20z &= 2550,\\ 30x + 50y + 110z &= 7950. \end{aligned}$$

Our standard techniques yield the solution set

$$\{r(-12, 5, 1) + (315, -30, 0) \mid r \in \mathbb{R}\}.$$

Keeping in touch with reality, we need to avoid specifying negative production levels for any of the three plastics. The condition $x = -12r + 315 \geq 0$ results in the restriction $r \leq \frac{315}{12} = 26.25$. The condition $y = 5r - 30 \geq 0$ results in the restriction $r \geq 6$. This also ensures that $z = r$ will be safely positive. Thus, the set of possible values for x, y, and z is the line segment

$$\{r(-12, 5, 1) + (315, -30, 0) \mid 6 \leq r \leq 26.25\}.$$ ❄

The final example in this section deals with the problem of determining the distribution of material in a system at equilibrium. Although it is stated as a physiological problem, simple reformulations result in problems as diverse as transfer of energy in thermodynamics and migration patterns of insect colonies in entomology.

Quick Example: *A patient is injected with* 1 *gram of a long-lasting drug. At any time thereafter, a portion of the drug is in use by the body, a portion is stored in the liver, and the remainder is in circulation throughout the body. Each day, suppose that of the amount in the circulatory system on the previous day,* 20% *is stored in the liver,* 30% *goes into use, and the rest remains in circulation. Suppose that* 20% *of the amount in the liver and* 10% *of the amount in use on the previous day are released into the bloodstream, with no direct transfer between the liver and the sites of utilization.*

Find the amount of the drug in the various locations when the distribution reaches equilibrium.

Let x denote the number of grams of the drug in the circulatory system, let y denote the amount stored in the liver, and let z denote the amount in use by the body. The condition can be concisely formulated as the system

$$\begin{aligned} x + y + z &= 1 \\ 0.5x + 0.2y + 0.1z &= x \\ 0.2x + 0.8y &= y \\ 0.3x + 0.9z &= z \end{aligned}$$

After rewriting the system with all the variables on the left of the equal signs and performing Gaussian elimination, we determine a unique solution $(x, y, z) = (0.2, 0.2, 0.6)$ for the system. Under the assumption that the drug does reach an equilibrium distribution, we expect to find 20% in the circulatory system, 20% stored in the liver, and 60% in use. ❄

Exercises

1. Find the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ whose graph passes through the four points $(-1, -3)$, $(0, -4)$, $(1, -7)$, $(2, 6)$.
2. Find the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ whose graph passes through $(-1, 1)$ with a horizontal tangent line and through $(1, 0)$ with a slope of 1.
3. Find an equation of a parabola that coincides with the graph of the sine function at $x = 0$, $x = \pi/2$, and $x = \pi$.
4.[A] a. Find the polynomial of minimal degree whose graph coincides with the graph of the exponential function at $x = 0$, $x = 1$, and $x = 2$.
   b. What if we additionally require the slopes of the two graphs to agree at $x = 0$ and $x = 2$?
5. Find values of a, b, and c so that the circle with equation
$$x^2 + y^2 + ax + by + c = 0$$
will pass through the three points $(1, 1)$, $(-2, 0)$, and $(0, 1)$.
6.[A] Find values of a, b, and c so that the function defined by $f(x) = a + b\sin x + c\cos x$ satisfies $f(0) = 1$, $f(\frac{\pi}{2}) = 1$, and $\int_0^{\pi/2} f(x)\,dx = 1$.
7. A city has 700 teachers, 80 firefighters, 160 police officers, and 290 clerical workers. Within each of these four groups, all employees are to receive a salary increase of the same number of dollars. A total of \$500,000 is available for raises for these four groups of employees. The raise for a firefighter is to be \$100 larger than the raise for a police officer. A teacher must get a raise equal to the average raises of a firefighter and a police officer. What are the possible raises?
8.[A] Suppose a certain virus exists in forms A, B, and C. Half of the offspring of type A will also be of type A, with the rest equally divided between types B and C. Of the offspring of type B, 5% will mutate to type A, and the rest will remain of type B.

Of the offspring of type C, 10% will mutate to type A, 20% will mutate to type B, and the other 70% will be of type C. What will be the distribution of the three types when the population of the virus reaches equilibrium?

9.[A] [*Partial fractions decomposition*] Set up and solve a system of linear equations in the unknowns a and b so that

$$\frac{1}{x^2+4x+3} = \frac{a}{x+1} + \frac{b}{x+3}.$$

10. [*Partial fractions decomposition*] Set up and solve a system of linear equations in the unknowns a, b, c, and d so that

$$\frac{11x^3-x+2}{(x^2-1)(x^2+2x)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{x+2} + \frac{d}{x}.$$

11. [*Partial fractions decomposition*] Set up and solve a system of linear equations in the unknowns a, b, c, and d so that

$$\frac{(x+1)^2}{(x^2+1)(x-1)} = \frac{ax+b}{x^2+1} + \frac{c}{x-1}.$$

12.[A] [*Method of undetermined coefficients*] Set up and solve a system of linear equations in the unknowns a, b, and c to determine the values of the coefficients so that $y = ax^2 + bx + c$ is a solution of the differential equation $y'' + y = x^2$.

13. [*Method of undetermined coefficients*] Find values of the coefficients a and b so that $y = a\sin 2x + b\cos 2x$ is a solution of the differential equation $y'' + y' + y = \sin 2x$.

14.[A] [*Method of undetermined coefficients*] Find values of the coefficients a, b, and c so that $y = ae^x + bxe^x + cx^2e^x$ is a solution of the differential equation $y' + y = e^x + xe^x + x^2e^x$.

15.[S] Set up a system of linear questions in the unknowns a, b, c, and d so that the chemical equation

$$a\,C_3H_8 + b\,O_2 \longrightarrow c\,CO_2 + d\,H_2O$$

is balanced. That is, find the number of each of the four types of molecules so that the number of atoms of each element in the reactants will equal the number of atoms of that element in the products of this chemical reaction. Solve the system and choose values of the free variable to give the smallest positive-integer solution.

16.[A] Set up a system of linear questions to balance the chemical equation

$$NH_3 + O_2 \longrightarrow NO + H_2O.$$

Solve the system and write the balanced equation with the smallest positive-integer solution.

17.[A] Set up a system of linear questions to balance the chemical equation

$$Mg(OH)_2 + H_3PO_4 \longrightarrow Mg_3(PO_4)_2 + H_2O.$$

Solve the system and write the balanced equation with the smallest positive-integer solution.

18. [*Motivation for basis of a subspace*]
   a. Find four cubic polynomials q, r, s, and t with the following properties:
$$\begin{array}{llll} q(0)=1, & q'(0)=0, & q(1)=0, & q'(1)=0, \\ r(0)=0, & r'(0)=1, & r(1)=0, & r'(1)=0, \\ s(0)=0, & s'(0)=0, & s(1)=1, & s'(1)=0, \\ t(0)=0, & t'(0)=0, & t(1)=0, & t'(1)=1. \end{array}$$
   b. Show that $2q - r + s + 2t$ satisfies the four conditions of the example at the beginning of this section. Verify that $2q - r + s + 2t$ agrees with the polynomial p determined in that example.
   c. Show that $5q + 3r - 2s + 8t$ is a cubic polynomial that passes through $(0, 5)$ with slope 3 and through $(1, -2)$ with slope 8.
   d. Generalize these results to show how to write a sum of scalar multiples of q, r, s, and t that yields a cubic polynomial with arbitrary value a and slope b at 0 and arbitrary value c and slope d at 1.

Project: Numerical Methods

The Gauss-Jordan reduction algorithm is a tool of amazing versatility. We will see it play a crucial role in the development of certain theoretical results, and it will repeatedly appear as a standard technique in working with concrete examples. Outside of textbook exercises, the reduction of matrices commonly relies on the speed and accuracy of a digital computer. However, since computers typically represent numbers with a limited number of significant figures, a certain amount of error is inherent in such computations. The project in this section concerns some numerical difficulties that arise as small initial errors are propagated through a sequence of computations.

1. Determine the solution to the system
$$\begin{array}{rcrcl} (1/a)x & + & y & = & 1 \\ -x & + & y & = & 0 \end{array}$$
in terms of the value of the parameter a. What happens to the system and the solution as a increases to infinity?

2. Investigate the effect of error introduced when calculations are limited in precision. Take $a = 1000$ in the preceding system and apply the Gauss-Jordan reduction algorithm, rounding after each step to three significant figures. How does the approximate solution compare with the true solution rounded to three figures? Trace through the steps of the reduction process to locate the causes of this trouble.

Interchange the rows of the augmented matrix and repeat this investigation. The leading entry that is used to create zeros elsewhere in its column is known as the **pivot entry**. The process of interchanging rows prior to creating these zeros is known as **pivoting**.

What do you suggest should be the pivot entry for $\begin{bmatrix} 100 & 100a & 100a \\ -1 & 1 & 0 \end{bmatrix}$ if a is large? This example shows that the pivot entry should be chosen by looking at its size relative to the other entries in its row. Suggest some formulas for measuring the size of a row of a matrix. Which of these is simplest from the standpoint of implementation on a computer? Devise a strategy for choosing pivot entries to reduce the effect of round-off errors.

3. Test your pivoting strategy. Create some systems of equations for which you know the exact solutions. Be sure to use some systems where the coefficients are not all of the same order of magnitude. Take advantage of any computer facilities available to you in carrying out these tests. Is the accuracy of your results comparable to the precision used in the intermediate steps? How do your results compare with those obtained when pivoting is used only to avoid division by zero?

4. You may be interested in reading the article "Why Should We Pivot in Gaussian Elimination?" by Edward Rozema in the January 1988 issue of the *College Mathematics Journal*. This article discusses pivoting from the point of view outlined here. It also discusses some ill-conditioned systems where pivoting may lead to increased errors.

5. Techniques for dealing with linear systems are important topics in numerical analysis. You may wish to investigate other aspects of the Gauss-Jordan reduction process (total pivoting or LU-decompositions, for example), iterative methods for repeatedly reducing the error in an approximate solution, or the condition number associated with a matrix that leads to an estimate of the size of the error in an approximate solution. Most textbooks on numerical analysis contain a wealth of information on these topics.

Project: Further Applications of Linear Systems

There is an inexhaustible supply of applications of linear systems. Whenever a relation exists between two sets of variables, the simplest model to try is one in which the dependent variables are sums of constants times the independent variables. When we want to determine the values of the constants or to reverse the roles of dependent and independent variables, we must solve a system of linear equations. Here are brief descriptions of three broad topics where linear systems are crucial. You may wish to explore the mathematical background involved in the topic, or you may prefer to investigate a specific application to a model of particular interest to you.

1. Linear programming is a technique for allocating limited resources under linear constraints. The constraints typically involve a system of inequalities. The set of points that satisfy the inequalities is called the **feasible region**. The **objective function** measures the success of the allocation program. Like the constraints, the objective function involves a sum of constants times the variables. One of the basic results of linear programming is that the extreme values of an objective function occur at the vertices of the feasible region.

You may want to take a typical linear programming problem and try a brute-force approach to solving it. Find all vertices determined by subsets of the constraints, determine which are in the feasible region, and choose one that optimizes the objective function. The simplex method is a very successful technique for reducing the amount of trial and error involved in solving a linear programming problem. Some references to linear programming, the simplex method, and related results are:

Frederick Hillier and Gerald Lieberman, *Introduction to Operations Research*, 11th ed., McGraw-Hill, 2021.

Margaret Lial, Raymond Greenwell, and Nathan Ritchey, *Finite Mathematics and Calculus with Applications*, 10th ed., Pearson, 2016.

Wayne Winston, *Operations Research: Applications and Algorithms*, 4th ed., Thomson Brooks/Cole, 2004.

2. Data gathered in the physical world are seldom perfectly consistent. The discrepancies are usually attributed to inaccurate measurements and other sources of error. Regression analysis attempts to estimate the values of parameters in a general model so that it best fits the observed data. The most common measure of how well a model fits a set of data is the sum of squares of the deviation at each data point. If we differentiate this quantity with respect to each of the parameters in the model, the minimum value will occur at a point where these derivatives are all zero. This gives a system of equations. For a simple model, the system is often linear.

Consult any statistics textbook for examples of regression analysis. A good starting point is:

Robert Hogg, Elliot Tanis, and Dale Zimmerman, *Probability and Statistical Inference*, 9th ed., Pearson, 2015.

You are likely to find a discussion of the distributions of the parameters as well as formulas for estimating their most likely values. You may want to try the technique in more complicated models or for other measures of goodness of fit. If you discover a new probability distribution, you can name it after yourself.

3. If a soap film or cell membrane is supported by a rigid frame, the surface quickly relaxes to an equilibrium shape. The position of each point will be determined by the positions of points immediately surrounding it. Imagine a grid of points superimposed on the surface. The position of each of the grid points can be approximated by some kind of average of the positions of neighboring points on the grid. This relaxation method can be adapted to many other situations where the value of a quantity is determined as an average of values of the quantity at neighboring points. Temperature distribution on a conductive plate with fixed boundary conditions is a classic case; the distribution of pollution in a trapped air mass is a three-dimensional version.

Create a relaxation model for some phenomenon of interest to you. Define a grid over a region, assign values to certain grid points (perhaps along the boundary of the region), and decide how you want to approximate the value at each grid point in terms of the values of neighboring points. You should obtain a system of linear equations. There will be one equation and one unknown for each grid point whose value is not predetermined.

The size of the system of equations increases quite rapidly as finer grids are used to attain more accuracy. You might want to solve a large system with an iterative technique: initialize all variables with some reasonable values, and apply your averaging formula to compute new values of the variables one at a time while making repeated passes through the grid. Try to speed up the convergence of this iterative process.

Think of a good strategy for making the initial assignment of values. Try different averaging formulas. Extrapolate the change in a value during one iteration to anticipate what its change might be in the next iteration.

Summary: Chapter 2

Chapter 2 presents the Gauss-Jordan reduction algorithm as a tool for solving systems of linear equations. Three types of elementary row operations allow us to simplify the coefficient matrix of a linear system of equations. Then we can easily determine the solution set of the system.

Computations

The three elementary row operations

Recognize leading entries of a matrix

Gauss-Jordan reduction algorithm
- systematic use of the three row operations
- transform a matrix to row-echelon form
- transform a matrix to reduced row-echelon form

Systems of linear equations
- code a system as an augmented matrix
- perform the Gauss-Jordan reduction algorithm
- read off the solution set from the reduced matrix
  - three possibilities: $\emptyset$, singleton, infinite set of solutions
- geometric interpretation of infinite solution sets
  - lines, planes, higher-dimensional analogs in $\mathbb{R}^n$

Theory

Advantages of a systematic and effective solution technique

Homogeneous systems of linear equations
- the trivial solution always exists
- ignore augmentation column
- possible nontrivial solutions
- Fundamental Theorem of Homogeneous Systems

Applications

Curve fitting and the method of undetermined coefficients

Linear constraints

Equilibrium distributions

Review Exercises: Chapter 2

1.[S] a. The second type of elementary row operation allows us to multiply a row of a matrix by a nonzero constant. Why is it important that the constant be nonzero?

b. The third type of elementary row operation allows us to add a multiple of one row to another row of a matrix. What happens if the multiplier is zero? Why is it not important to prevent this from happening?

2. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   a. How many rows of A have leading 1s?

   b. How many columns of A have leading 1s?

   c. How many leading 1s does A have?

3. Illustrate the steps in the Gauss-Jordan reduction algorithm in putting the matrix

$$\begin{bmatrix} 0 & 2 & 4 & 0 & 6 \\ 1 & -1 & 2 & 1 & 1 \\ 2 & -2 & 4 & 2 & 2 \\ -2 & 3 & -2 & 1 & 4 \end{bmatrix}$$

in reduced row-echelon form.

4.[S] Suppose a system of nine linear equations in seven variables has an augmented matrix that, when put in reduced row-echelon form, has four leading 1s.

   a. Describe the solution set if one of the leading 1s is in the rightmost column.

   b. Describe the solution set if none of the leading 1s is in the rightmost column.

5. Suppose a system of m equations in n variables has an augmented matrix that, when put in reduced row-echelon form, has r leading 1s.

   a. Describe the solution set if one of the leading 1s is in the rightmost column.

   b. Describe the solution set if none of the leading 1s is in the rightmost column.

6. For each of the following systems of linear equations, give the augmented matrix corresponding to the system. Put the matrix in reduced row-echelon form. Write the solution set of the system as a set with zero or one point, or as a line, plane, or higher-dimensional analog.

   a.
$$\begin{aligned} -x_1 + x_2 - 2x_3 \phantom{{}+ 3x_4} &= -2 \\ x_1 - x_2 + x_3 + x_4 &= 1 \\ 2x_1 - 2x_2 + x_3 + 3x_4 &= 1 \end{aligned}$$

   b.
$$\begin{aligned} x_2 + 2x_3 + x_4 &= -3 \\ 2x_1 + 4x_3 - 6x_4 &= 6 \\ 5x_1 + 3x_2 + 16x_3 - 12x_4 &= 6 \end{aligned}$$

7. In applying the Gauss-Jordan reduction algorithm to transform each of the following matrices to reduced row-echelon form, what elementary row operation would be applied first? Write down the matrix resulting from this operation.

a. $\begin{bmatrix} 4 & 2 & 0 & -8 \\ 1 & 3 & 2 & 2 \\ 3 & 0 & 7 & -1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 3 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 5 & 2 \\ -3 & 7 & 2 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 0 & 4 & 0 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

8. Which of the following matrices are in row-echelon form? Which are in reduced row-echelon form?

a. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 3 & 0 & 4 & 0 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

9.[A] Solve the following systems of linear equations. Write the solution set as a set with zero or one element, or in the standard form of a line or plane.

a.
$$\begin{aligned} -w + 2x + y + z &= -4 \\ w + x - y + z &= 3 \\ 2w - 3x + 2z &= 3 \\ x + 4y - z &= 7 \end{aligned}$$

b.
$$\begin{aligned} -w + 2x + y + z &= -4 \\ w + x - y + z &= 3 \\ 2w - 3x + 2z &= 3 \\ 3x + 2z &= -1 \end{aligned}$$

c.
$$\begin{aligned} -w + 2x + y + z &= -4 \\ w + x - y + z &= 3 \\ 2w - 3x + 2z &= 3 \\ 3x + 2z &= 1 \end{aligned}$$

10. Solve the following homogeneous system of linear equations:

$$\begin{aligned} -x_1 - x_2 + 2x_3 - 3x_4 &= 0 \\ 2x_1 + 2x_2 - x_3 + 9x_4 &= 0 \\ x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

11. a. Suppose the solution set of a system of linear equations is known to be the empty set. Explain why the system cannot be homogeneous.

b. Suppose **0** is known to be a solution of a system of linear equations. Explain why the system must be homogeneous.

12.[A] Consider a system of five linear equations in six unknowns.

a. Is it possible for the solution set to be ∅?
b. Must the solution set be ∅?
c. Is it possible for the solution set to consist of a single point?
d. Must the solution set consist of a single point?
e. Is it possible for the solution set to be a line?
f. Must the solution set be a line?
g. Is it possible for the solution set to be a plane?
h. Must the solution set be a plane?

13. Consider a system of six linear equations in five unknowns.

a. Is it possible for the solution set to be ∅?
b. Must the solution set be ∅?
c. Is it possible for the solution set to consist of a single point?
d. Must the solution set consist of a single point?
e. Is it possible for the solution set to be a line?
f. Must the solution set be a line?
g. Is it possible for the solution set to be a plane?
h. Must the solution set be a plane?

14. Determine the coefficients of the polynomial $p \in \mathbb{P}_n$ that satisfies $p(0) = c_0$, $p'(0) = c_1, \ldots, p^{(n)}(0) = c_n$ for constants $c_0, c_1, \ldots, c_n$. Show that there is only one polynomial in $\mathbb{P}_n$ that satisfies these conditions. Where have you run across such polynomials in your calculus course?

15. Find a cubic polynomial $p(x)$ that satisfies $p(0) = -1$, $p'(0) = 5$, $p(1) = 3$, $p'(1) = 1$.

16.[A] Suppose a country has three political parties: Left, Center, and Right. Every election one-fifth of the Leftists and one-fourth of the Rightists switch to the Center party, and one-third of the Centrists move to the extreme groups, equally divided between the Left and Right. If the number of members in each of the three parties remains constant from one year to the next, determine the fractions of the politically active citizens that are members of each of the three parties.

17. Find values of the coefficients a and b so that $y = a \sin 2x + b \cos 2x$ is a solution to the differential equation $y'' + 3y' - y = \sin 2x$.