Preface

Abstract mathematics is a mixture of rigor and intuition, and to develop either of these, we need repeated exposure and keen practice. As students progress through high school and even early college, they are typically made to spend years working exclusively with algorithmic mathematics ("differentiate this," "move this term over there," "plug that value into it,") before they are allowed to see any abstraction. It is therefore natural for students to have trouble in their first course with abstraction. They may not have seen the language of advanced mathematics. They may not yet be familiar with basic concepts. To them, even the simplest abstract ideas with which the subject works may feel overwhelming.

To help students make the change to abstract mathematics from a more algorithmic approach, many universities offer transition courses designed to provide some warm-up to the students. This book was written for one such course at California State University Northridge.

In the competition between developing the language and tools for writing proofs—statements, logical expressions, sets, functions, induction—on the one hand and developing mathematical intuition on the other, the balance in this book tips slightly towards the intuition side of the equation. I feel that many tools are best developed in context. So, while this book definitely has whole chapters on statements and on sets and functions and on induction, complete with exercises, the bulk of the book focuses on mathematical ideas and on developing creativity. Thus, the book has chapters on combinatorics (counting, the pigeonhole principle), on elementary number theory (divisibility, primes, the Unique Prime Factorization theorem and consequences), on analysis (convergence, continuity, the completeness of the real number system and consequences), on graph theory (the Königsberg Bridge problem, Eulerian trails and circuits), and on elementary group theory (permutations, cyclic groups, matrix groups). Each of these chapters is liberally augmented with exercises, parenthetical remarks, comments on mathematical culture and tips on how to study mathematics.

I feel that a strictly logical presentation of topics, Bourbaki style with all terms defined and every assertion carefully proved, can be confusing to a student at this stage of development—contradictory as this view might be in a course designed to expose students to rigorous mathematics! I believe learning is non-linear: sometimes we need to work with concepts for a long time before we can truly understand them, and it is often pedagogically sound to let students see concepts early even if all the background i's have not been dotted nor all the background t's crossed. So, where necessary, I have not hesitated in doing a bit of "hand-waving" to get to the heart of a subject without getting bogged down in technicalities. For instance, in the first chapter ("Introduction") I introduce students informally to the rings $\mathbb{Z}/n\mathbb{Z}$, representing the elements by just the remainders $0, \ldots, n-1$, and ask them to make computations with these remainders and even prove various assertions, without formally indicating why basic algebraic rules like associativity and distributivity of multiplication over addition hold. I believe that when material is presented in this manner, students can start to work with such objects, and as they do so, these objects will appear less and less intimidating, and learners will develop confidence and build intuition. (Of course, associativity and distributivity are considered later, in Chapter 9, after equivalence relations are introduced.)

The prerequisites for this book are simply high school level mathematics (and naturally, a strong desire to learn!). Various chapters of the book can be used to teach a course on mathematical reasoning in a liberal arts program. (And indeed, I have taught precisely such a course at Krea University from this book.) Just as easily, this book could be used by a bright high school student wanting to learn mathematics beyond the traditional school curriculum. While I invoke complex numbers in various examples, the usage of these numbers in these examples is not critical, and one can simply substitute complex numbers with real numbers without any measurable loss. Calculus is not a prerequisite at all, but since many students are likely to have seen it, I make allusions on occasion to examples or ideas from calculus. Once again, these are not critical, and a student who has not seen calculus can simply skip the relevant material and remain unaffected.

Chapters 1 through 9 deal with the basic bricks from which mathematics is constructed: statements, sets and their cardinalities, functions, and equivalence relations, along with elementary tools: principle of induction, the pigeonhole principle, and simple counting techniques. This is followed by an introductory study of four areas of mathematics: elementary number theory involving divisibility and unique prime factorization in the integers (Chapter 10), beginning analysis covering convergence of sequences and continuity of functions and limits (Chapter 11), the least upper bound principle and various consequences, including the monotone convergence theorem, the Bolzano-Weierstrass theorem, and tests for convergence of series (Chapter 12), beginning group theory covering examples of the dihedral, symmetric, and cyclic groups, and also subgroups, leading up to Lagrange's theorem for finite groups (Chapter 13), and finally, beginning graph theory built from the Königsberg Bridge Problem, leading up to a proof of necessary and sufficient conditions for a graph to have an Eulerian circuit (Chapter 14).

Several courses can be fashioned from this book. Really, this is a matter of instructor taste, but here are some suggestions. For a semester-long course intended for students who need to have their foundations built from scratch, a leisurely study of Chapters 1 through 9, with just a toe-dip, if there is time at all, into any of the remaining chapters on unique factorization (Chapter 10), or sequences (Chapter 11), or groups (Chapter 13) or graphs (Chapter 14) will be apt. For a semester-long course intended for students with greater mathematical preparation, to whom the language and culture of mathematics will come a bit more naturally, a relatively fast study of Chapters 1 through 9, followed by a detailed study of elementary number theory (Chapter 10), followed perhaps by the two analysis chapters (Chapters 11 and 12), or perhaps by a combination of the group theory and graph theory chapters (Chapters 13 and 14), would be more suitable. For a course on mathematical reasoning in a general education program, one could first focus on the introductory chapter (Chapter 1), on the pigeonhole principle (Chapter 2), on counting in finite sets (Chapter 4), and induction (Chapter 7), so that students will be exposed to some fun problem solving, and after a brief study of the chapters on statements (Chapter 3) and sets and functions (Chapter 5), move either to the chapter on the Königsberg Bridge Problem (Chapter 14) or to the proof of the unique prime factorization theorem (Chapter 10). As for bright high school students, such students are likely to be reading this book as a self-study project. Once they have absorbed the basic material on statements (Chapter 3) and sets and functions (Chapter 5), they can pick and choose as they please: it will all be useful and hopefully all be fun and beautiful!

I wish to thank Steve Kennedy at AMS Publications, who shepherded this book through the publication process, as well as my editor Suzanne Larson and her team of reviewers. The suggestions they all made were detailed and invaluable. Of course, the usual caveats apply: only I am to blame for mistakes that remain. I also wish to take this opportunity to thank Loretta Bartolini, the gentle and encouraging editor at Springer-Verlag where too this book was accepted. She and her reviewers also helped influence this book, and I am grateful to have been considered by them.

I wish to thank my colleagues in the Mathematics department at California State University Northridge who used this book to teach the transition course there. Katherine Stevenson provided me with many suggestions as she tested this book in her course. Sungjin Kim was an enthusiastic adopter, who also provided me with comments. Jerry Rosen was a firm supporter, teaching out of the earliest version of the book.

I am grateful to the National Science Foundation for their generous research grant CCF 1318260, under whose broad support the core of this book was written.

B. Sethuraman
California State University Northridge
Krea University, India