Preface

If you love calculus, then you will love this text, because it answers two natural questions:

- How might a real-valued function, such as one typically studied in calculus, be extended to a complex-valued domain?
- What are the extended function's properties of limits, derivatives, and integrals, and how might they appear in real-world applications?

The answers describe complex analysis as a natural augmentation of the calculus on real functions. The powerful bridge that connects the two studies is the Extension Theorem. It says any real analytic function (one represented as a power series with a positive radius of convergence) extends into a region of the complex plane as an analytic, infinitely differentiable complex function. The Identity Theorem in Chapter 1 proves the extension is unique once the real domain is extended out to any complex domain, no matter how small. Taylor series and the Root Test are at its core. Placing the Extension Theorem at the beginning of the text allows calculus, a favorite course of most mathematics, physics, and engineering majors, to flow uninterrupted as an enjoyable introduction to complex functions. Instead of a new, foreign study, complex analysis becomes a continuation of what happens with real functions. The fun of calculus continues.

Real and complex functions live in the same universe. But the study of complex functions reveals broader vistas. For example, complex algebraic operations such as addition and multiplication extend those for real numbers. Real analysis sits inside this study and can be, but is not necessarily, a prerequisite for the course. This book thereby offers curricular flexibility. Depending upon the course's placement in the curriculum, teachers can direct students at different levels of theoretical engagement, from gainfully digesting every detail to merely skimming important proofs, just as they can when teaching calculus with a text that offers Cauchy's delta-epsilon description of limits or a proof of the product rule. In this way, students can enjoy complex analysis alongside or as a natural supplement to multivariate calculus, or after a proofs course (or even as a proofs course—cf. [**168**]), or before, during, or after real analysis. The calculus delight of limits, derivatives, and integrals continues, but with additional investigations into the exquisite theory and properties of complex functions. Some limitations of real function theory (such as the fact that not every real-valued differentiable function is twice differentiable over an interval) become rectified in the extension to a complex domain. Functions in the complex setting take on a richness, a gracefulness of mathematical properties—a maturity or growth out of what happens in general for real functions. As a diplomatically considerate person would not stress differences between a teenager and a fully mature adult but instead would celebrate the adult's maturity, this text focuses on the graceful beauty out of which the theory of complex functions blossoms from real-valued seeds.

The book celebrates complex theory while enjoying examples and calculations. Theory and calculations enhance one another, especially as examples firm up an understanding of nuances. A broadening of calculus into the world of complex functions relies on precise and well defined theoretical ideas, about which the book cares deeply. Its expansion of calculus also reflects what motivates the development of complex function theory from a historical standpoint. Much of the important nineteenth century work of Cauchy, Weierstrass, and Riemann is motivated by the desire to extend the analysis involved in real variable calculus: Cauchy via the integral; Weierstrass via infinite series; and Riemann via the geometry of the functional graphs. At the same time not everyone reading this book has experience with mathematical proofs, and an ideal text presents material with minimal prerequisites to broaden access. The book explains theorems and proofs in a descriptive, understandable, and mathematically rigorous way without losing broad undergraduate accessibility and interest. Exercises that grapple with theory do so with a hope to guide readers into discovery and engage them to think in productive and correct ways about complex analysis.

Of course not every complex function is analytic nor is every analytic complex function an extension of a function real-valued on a real domain.[1] But analytic complex functions are beautifully introduced by the Extension Theorem, form an ideal starting place for a thorough introduction to complex variables, and set forth important ideas. Plus, material on analytic functions feels familiar. For example, the complex derivative's definition symbolically matches what students learn and love in calculus. The majority of functions arising as examples and useful in engineering and science are analytic (or real parts of analytic functions). Analytic functions as a starting place also immediately prompts further questions, such as, "What are examples of functions not always analytic on a domain, and what can we say about them?" This text also discusses such questions—the standard ones in a complex analysis course. And yet the book goes further, successfully leaping into introductions of topics regularly left to courses at the graduate level or not discussed at all. It is the first complex variables text that allows for a further reach in a single course because it starts by "jumping right in," talking about analytic functions right away. Extra topics are easier to teach in one term. Professors can show students more. Readers can learn more.

Use for a One-term or Two-term Course. The text offers options for its use. Though this situation is rare, all six chapters are manageable in a single semester's course for students at the highest level of engagement and with a significant theoretical background. If nothing else, this claim may surprise. Using other texts and in the standard, commonly understood delivery of complex analysis, it is surely impossible to cover in a single course most of this book's advanced topics in its last three chapters, and sometimes not in a full-year's course. But this text has an advantage. Its introduction of analytic functions at the start, along with the power of the Extension Theorem to provide students intuition from calculus, quickly gets to material such as harmonic functions, streamlines and equipotentials, and other nice applications and abstract topics faculty members often struggle to find time to include. Plus, teachers

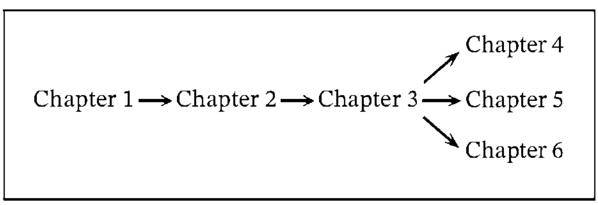
[1] A simple example is $f(z) = iz$.

can be sure the text will not run out of material for a semester course. It is better for a chef to serve any meal with leftovers than to leave diners hungry. This book serves a delightful six course dinner menu.

Another clear option is for a two-term course. In that design, each term could comfortably have students work through three chapters. Explorations of every concept could occur in detail, with the possibility of using a strong student-engagement pedagogy with much self-discovery that could come by reading the text carefully, with a focus on both theory and calculations, working through explorations, and walking through a large majority of the text's homework exercises.

The text recognizes the fact that complex analysis teachers have different goals for the course. It provides proofs of theorems, understanding that many teachers may wish to minimize the theory and skip over proofs such as delta-epsilon limit proofs to emphasize calculations and applications, just as a standard calculus text provides these proofs even though not all calculus students engage in them. The text understands that readers will have a large variability in backgrounds (everything from multivariate calculus to real analysis). The text occasionally uses footnotes to help bridge the gap. Readers with an advanced background can enjoy the exposition without breaks in the discussion caused by unnecessary review, not needing to remind themselves of concepts reviewed in the footnotes. Readers with fewer prerequisites under their belt can feel comfortable with the text's reference to such a concept, using each corresponding footnote to bring them up to speed. The text is set up in a clever way, so that readers can refer whenever necessary to footnotes placed at just the right spots. The footnote usage allows any reader to enjoy the advanced discussion. Prerequisites are minimized. A surprisingly wide audience becomes properly engaged.

Again, full coverage is uncommon, though anyone can continue reading after a course concludes. A more typical one-term use is with the first three chapters and then a choice of



topics in at least one additional chapter. Interdependence between the last three chapters is minimal and manageable for a course that wants to skip either Chapters 4 or 5. The accompanying figure shows how such choices can be outlined in the simplest of manners. Other ways are possible. For example, a curriculum designed for engineers would add portions of Chapter 4 and Section 5.1, while one interested in showing how complex analysis contributed toward major discoveries would choose Chapter 5, and one offering a taste of operator theory or linear algebra using complex scalars might choose Chapter 6. Chapters 4, 5, and 6 are essentially independent of each other. They include advanced theory that undergraduates can both understand and get excited about. Each one shows the power of complex functions in applications. (For example, Section 5.1 introduces Laplace and Fourier transforms.) They scan a surprisingly wide vista of mathematical subfields. (Section 5.2 introduces complex number theory, independent of Section 5.1.) They introduce amazing facts. (Section 5.3 shows how to define the Riemann zeta function and its connection to the distribution of primes.) They look at graduate topics, including the geometry of analytic function spaces to study functional analysis and operator theory. (Section 5.4 describes the Weierstrass factorization as a generalization of the Fundamental Theorem of Algebra, and it is independent of the other sections in Chapter 5.) A popular option in a single term is to work through the first three chapters, have a class collectively learn about Section 4.1 and 4.2's conformal maps and potential theory, and then to assign individual learning projects in each student's topic of choice. For example, an engineering student might elect to write a paper on Section 5.1's Fourier transform, and a student headed for graduate school might choose to describe material in Chapter 6, including what makes a Hilbert space operator bounded.

For the first time in an undergraduate complex book, advanced students can see how complex functions provide a playing field for operator theory and linear algebra. Chapter 6 successfully introduces infinite-dimensional function spaces in some detail. This material turns out to be manageable especially for students who have taken linear algebra and want to see more. Students learn that collections of analytic functions on a given domain are closed with respect to arithmetic, algebra, composition, and other analytic processes such as differentiation, with respect to limits of so-called Cauchy sequences in appropriate definitions of norm.[2] Analytic functions thereby form vector spaces of infinite dimension. The structure of infinite-dimensional vector spaces constitutes the main research framework for mathematical investigations in function theory and operator theory today. For example, the Hilbert space of square-integrable analytic functions $H^2(\mathbb{D})$ is easy to define, since it can be thought of in terms of the Maclaurin power series associated with each boundary function on the unit circle.[3] Lebesgue measurability of functions in $H^2(\mathbb{D})$ is automatic and need not be discussed. What's more, although the integral is extraordinarily important in various investigations (as it lays out functional properties), the Lebesgue integral is not needed to define the $H^2(\mathbb{D})$ norm or inner product. Chapter 6 expresses them in terms of power series coefficients.[4]

The topic of function spaces leads to the book's exciting denouement. At an undergraduate and quite accessible level, the study of analytic functions gives the reader a direct pathway into the most current research questions about linear operators on infinite-dimensional vector spaces. Chapter 6 offers this icing on the cake. It introduces the modern ideas of the functional calculus acquirable for the first time in an undergraduate text. Functional calculus forms a differential and integral calculus for *operator-valued* functions of a complex variable. (An "operator" is a bounded linear map from one complete normed vector space to another.) The book shows, via an accessible description of the Riesz projection, how the Jordan form of a matrix extends to compact operators on infinite-dimensional Hilbert spaces and Banach spaces. In that sense, the text hints at how important complex analysis is to a full understanding of linear algebra, how the functional calculus extends the ideas of linear algebra and is

[2]Being "closed" means that the operation on two analytic functions produces an analytic function. For example, the sum of two analytic functions is analytic.

[3]This fact also shows the equivalence of $H^2(\mathbb{D})$ with ℓ^2.

[4]For an undergraduate introductory real study of Banach and Hilbert spaces without need for a real analysis prerequisite, see [**98**], which describes the Lebesgue integral and can partner with this book to give undergraduates a powerful ability to research function theory. It can be read independently, before or after this text.

based on complex analysis, and how the course is linked not only to calculus as a continuation of that study, but also to linear algebra and to an abundance of mathematical study and research beyond. It's time that undergraduate majors see in broad accessible ways advanced theoretical analysis topics they can enjoy. This text shows them many of these beautiful views.

Unique Features. The text is meant to be exceptionally readable, so that students and teachers find it clear and student friendly, while preserving the important exactness and rigor of its beautiful mathematics. *Important Concepts* at the end of each section optionally offer a check, questioning the reader on important ideas and providing a way to confirm their understanding. They could also be useful for a writing-intensive version of the course. *Explorations* are embedded exercises within the text, providing student engagement with the material in real time as they read. Abbreviated but instructional answers to every exploration are provided at the end of the section to give a quick check for students as they proceed. Approximately 1000 impressively rich exercises offer professors and readers a choice of a thorough variety of problems to assign and work. *Engaged Learning Modules* are short, often technology-driven units that teachers can treat either as in-class material or can assign for student self-discovery. Though *Mathematica* along with its simplified web-accessed version *Wolfram Alpha* is the CAS technology choice this text uses, other systems certainly could substitute. (Student projects to make such substitutions could be a part of an engaged assignment.) *Historical Notes* appear at the end of each chapter, making the discovery of complex function theory come alive for those readers who might enjoy and benefit from it. The notes connect some aspects of the chapter's discussion with the mathematicians who produced them.

Complex geometry is important. The text is the first to offer a new, powerful geometric framework to view complex function 4-D graphs via *Mathematica* outputs. Section 1.3 describes how to visualize them. These are *not* domain only or range only graphs, neither are they plots of real parts or imaginary parts only. Instead, they are the full 4-D graph, with the complex domain in the horizontal (x-y plane) and the complex range given in polar coordinates, with the modulus on the vertical axis and the polar angle cleverly denoted in color. The cover shows six. For any of us, of course it takes time to study and interpret any individual 4-D graph, since our brains are geared toward 3-D. But these graphs are powerful. They illustrate fully the function's geometry in a single picture, just as a real function in a calculus course can be fully realized via its graph in the Cartesian plane. The complex 4-D graph can reveal properties. For example, the 4-D graph of a linear function shows its geometric interpretation as rotation, scaling, and translation. The book also describes traditional ways to view a function's geometry, such as a map from a type of domain set to a corresponding range set in two separate 2-D frames. But it presents the 4-D graphs in a way that illustrates all such individual set mappings in a single view. The imagery is so new at the time of this book's publication that one might expect it to produce new interpretations, linking purely analytic functional properties to a function's graph. One hope for the text is that it sets in motion a broad appreciation of this new geometric visualization.

The table of contents is bold. After finishing the book, a reader can:

- *Understand topics presented in any undergraduate complex analysis course.* Besides analytic functions, a few highlights are Cauchy's Theorem, Laurent series, use of complex integration to evaluate real-valued integrals, conformal mappings, and solving the Dirichlet problem.

- *Feel invited to study several topics typically presented at the graduate level.* These can be omitted if desired. A few examples are the Poisson integral formula, the Riemann hypothesis, Hilbert and Banach spaces, and bounded operators.
- *Have fun thinking about a few representative modern research topics—these are open questions—that actively interest mathematicians today.* The text briefly introduces students to the modern study of function theory, just enough to whet their appetite. The last section lists ten interesting research questions. These give the reader a feel for modern research. They include a few of the most challenging open questions, as well as ones that invite the reader into manageable investigations right away.

As a result, the text engages a fullest possible study of complex analysis in a manageable outline, one that can fulfill in a spectacular way a course's study in a single academic term.