Preface

> "The only way to learn mathematics is to do mathematics."
> Quote attributed to Paul Halmos

For the student

Embracing Halmos's quote above, this textbook provides you (the student) with opportunities to *do* mathematics: to perform experiments and grapple with problems; to formulate, test, and revise conjectures; to develop theories that bring coherence to observed results; and to express understanding using precise mathematical language. In essence, you get to experience mathematics as a mathematician does.

As its title suggests, this textbook is a friendly introduction to abstract algebra, a study of algebraic structures and relationships. I hope you'll not only read the textbook, but *actively engage* with it. As you work through the examples or exercises, you should seek patterns, make bold conjectures, and try proving them. Create your own examples to further your understanding. Ask your own questions. Have fun!

For the instructor

This textbook is intended to be used in a first-semester course in abstract algebra. Students should have completed Calculus I and II, not because they need the calculus content, but to acquire enough mathematical experience and maturity to handle the abstraction and proof writing that are part of the textbook. Familiarity with matrices is helpful, but not required. (Chapter 7 contains a brief introduction to matrices and covers matrix concepts that are needed in the textbook.) Prior proof-writing experience is *not* expected. In fact, an underlying goal of the textbook is to guide students toward writing clear and precise mathematical proofs. To support this goal, the textbook includes "Proof know-how," frequent, context-specific, short tips on proof writing.

Paul Halmos once said, "A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept." Following this advice, almost all concepts in this textbook are introduced through concrete examples. Ideas are foreshadowed, revisited, and developed over time. For instance, in an exercise about modular arithmetic (in Chapter 4), students compute the order of units modulo p and make conjectures about these orders. Eight chapters later (in Chapter 12), the notion of the order of a group element is formally defined. By then, students have seen enough examples so that the concept feels familiar.

Abstract algebra often acts as a "gateway" to upper-level mathematics courses and to a successful completion of a mathematics major. But it can seem impenetrable due to its (seemingly) theoretic nature. By taking a more concrete approach to the subject and by allowing students to develop their own understanding of the material, this textbook makes abstract algebra more accessible to more students.

Below, we highlight the pedagogical features of this textbook.

"Under the hood" perspective. This textbook provides students access to the "under the hood" work that mathematicians do. Rather than starting with a general theorem or definition (i.e., a finished product), the textbook lets students in on how that finished product is developed. In Chapter 13, for instance, we consider the cyclic group $\langle g \rangle$ generated by a group element g of order 12. We compute the product $g^9 \cdot g^7 = g^{9+7} = g^4$ in $\langle g \rangle$ and notice how this is just like the sum $9 + 7 = 4$ in $\mathbb{Z}_{12}$. Indeed, multiplication in $\langle g \rangle$ *feels like* addition in $\mathbb{Z}_{12}$. In Chapter 16, we build on this observation to motivate the definition of a *group isomorphism*. By providing access to the mathematical thinking that goes into the finished product, this textbook helps students make sense of the concepts in abstract algebra.

"How did you come up with that?" This is a question that I often get from abstract algebra students, especially when it comes to proof writing. Students can typically follow a proof that is presented to them, but they struggle with deriving the key steps on their own. This textbook addresses this issue through in-depth analyses of the proofs. In the proof of Theorem 19.14, for instance, there is a tricky step of coming up with an element $h = a^{-1}g$. Here's an excerpt from the "Proof know-how" following the proof:

> Coming up with the element $h = a^{-1}g$ employed the familiar "working backwards" technique. Our goal was to show that $g = ah$ for some $h \in H$, so we solved this equation for h by left-multiplying each side by a^{-1}, which yielded $h = a^{-1}g$. As before, this process of solving for h is scratch work and does *not* belong in the proof. Instead, the focus of the argument is showing that $g = ah$ for $h = a^{-1}g$.

Students, even mathematics majors, often have a (false) impression of the subject, that mathematicians produce new ideas out of thin air by writing down a theorem and effortlessly proving it. This textbook teaches students not only the content, but also the skills and know-how to do mathematics and create new ideas on their own.

Experience before formality. Providing students with *concrete experiences* is at the heart of this textbook, accomplished through example-driven exposition where theoretical concepts are introduced through examples. This approach makes the content more accessible to more students.

A key feature of this textbook is the set of exercises at the end of each chapter, in which students work on examples that lead to or reveal certain patterns. After working on such exercises, students are often asked to make a conjecture and/or prove a generalization. I tell my students that the role of the proof is *not* to convince, but to understand and explain. In other words, they shouldn't try to prove any theorem that they don't already believe is true. And this belief typically comes from concrete experiences that lead to the statement of the theorem. This is another instance of how, through this textbook, students experience mathematics as a mathematician does. Again, such an experience has the effect of making the subject more accessible to more students.

Accessible but still rigorous content. In this textbook, certain choices were made with the aim of making the content as accessible as possible, while still maintaining the mathematical rigor. For instance, we define a polynomial $f(x)$ to be *factorable* when $f(x) = p(x) \cdot q(x)$ with $\deg p(x), \deg q(x) < \deg f(x)$. (In other words, $f(x)$ is a product of "smaller" polynomials.) Otherwise, we say that $f(x)$ is *unfactorable*. This treatment is a bit unorthodox in a couple of ways. First, we use the terms *factorable* and *unfactorable*, rather than the more commonly used *reducible* and *irreducible*. Second, a more typical approach is to define *irreducible* polynomials as satisfying the following property: If $f(x) = p(x) \cdot q(x)$, then $\deg p(x) = 0$ or $\deg q(x) = 0$; and otherwise, $f(x)$ is said to be *reducible*. In this textbook, this property of irreducible (or unfactorable) polynomials is proved in Theorem 30.8.

These decisions about polynomials were made because the approach we take more closely resembles students' prior experiences with polynomials. In fact, many of the topics that we cover (e.g., polynomials, integers, the commutative law, putting on socks and shoes, to name a few) are already familiar to students. Whenever possible, we build on students' existing knowledge to make the content more accessible.

Note about rings

In this textbook, a ring will contain the multiplicative identity element *by definition*. (For instance, we do *not* consider the set $2\mathbb{Z}$ of even integers to be a ring.) We do so for two reasons. First, we wanted our definition of a ring to closely mimic what we observe in the ring of integers $\mathbb{Z}$. Second, every relevant example of a ring that we examine contains the multiplicative identity.

Road map

There are 37 chapters in the textbook. Depending on your students' background and/or the structure of your algebra course, here are some suggested road maps:

- Chapters 1 through 25 cover group theory. If students have had prior experience with proof writing, you may choose to omit Chapters 1 and 2. Similarly, students may have already studied the notion of divisibility (of integers), particularly if they have taken an introductory number theory course. In such a case, Chapter 3 may be skipped as well. I recommend *not* skipping Chapter 4 on modular arithmetic, however, since many group-theoretic concepts are introduced in that chapter.

- Chapter 7 contains a brief introduction to matrices and covers matrix concepts that are needed in the textbook. If your students have had a linear algebra course, then Chapter 7 may be omitted or assigned to the students to read on their own as a refresher.

- Chapters 26 through 37 cover ring theory, with an emphasis on polynomial rings. It is possible to end the course with Chapter 35, where we complete the proof of Theorem 35.1 (i.e., $F[x]/\langle g(x)\rangle$ is a field if and only if $g(x)$ is unfactorable).

- Chapters 36 and 37 guide the students to prove Theorem 35.1 again using the notion of maximal ideals. It is a lovely proof, so I recommend its inclusion in your course, if time permits.

Acknowledgments

This textbook is based on the materials that I developed for Math 252, a first-semester abstract algebra course at St. Olaf College. I want to thank all of my Math 252 students (past, present, and future) for their hard work, enthusiasm, and patience, as I put them through the experience of learning mathematics by *doing* mathematics.

My mathematical mentors, David Rohrlich, Glenn Stevens, and Al Cuoco, taught me how to think mathematically and what it means to experience mathematics as a mathematician does. I hope their impact can be seen on every page of this textbook.

I am grateful to Steve Kennedy from the Mathematical Association of America for seeing the potential in the first draft of this textbook and for his continual guidance and encouragement. I also want to thank the MAA editors Will Green, for his thorough review of an early manuscript, and Suzanne Larson, for her support throughout the writing and production stages. Moreover, I appreciate the meticulous work by Arlene O'Sean and Chris Thivierge from the American Mathematical Society to make this textbook a reality.

My kids, Elizabeth and Anita, lent their names to the recurring characters in this textbook. They take full credit for the insightful comments that the characters make but bear no responsibility for their errors.

As always, I am most indebted to my wife, Sarah, the real algebraist of the family. When I showed her my Math 252 course notes, she immediately said, "Send it to Steve Kennedy." She knew that I could write an algebra textbook long before I believed it myself.