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Introduction

Number Theory and Mathematical Thinking

One of the great steps in the development of a mathematician is becoming an independent thinker. Every mathematician can look back and see a time when mathematics was mostly a matter of learning techniques or formulas. Later, the challenge was to learn some proofs. But at some point, the successful mathematics student becomes a more independent mathematician. Formulating ideas into definitions, examples, theorems, and conjectures becomes part of daily life.

This textbook has two equally significant goals. One goal is to help you develop independent mathematical thinking skills. The second is to help you understand some of the fundamental ideas of number theory.

You will develop skills of formulating and proving theorems. Mathematics is a participatory sport. Just as a person learning to play tennis would expect to play tennis, people seeking to learn to think like a mathematician should expect to do those things that mathematicians do. Also, in analogy to learning a sport, making mistakes and then making adjustments are clear parts of the experience.

Number theory is an excellent subject for learning the ways of mathematical thought. Every college student is familiar with basic properties of numbers, and yet the study of those familiar numbers leads us into waters of extreme depth. Many simple observations about small, whole numbers can be collected, formulated, and proved. Other simple observations about small, whole numbers can be formulated into conjectures of amazing richness. Many simple-sounding questions remain unanswered after literally thousands of years of thought. Other questions have recently been settled after being unsolved for hundreds of years.

Throughout this book, we will continue to emphasize the dual goals of developing mathematical thinking skills and developing an understanding of number theory. The two goals are inextricably entwined throughout and seeking to disentangle the two would be to miss the essential strategy of this two-pronged approach.

The mathematical thinking skills developed here include being able to

- look at examples and formulate definitions and questions or conjectures;
- prove theorems using various strategies;
- determine the correctness of a mathematical argument independently without having to ask an authority.

Clearly these thinking skills are applicable across all mathematical topics and outside mathematics as well.

Note on the approach and organization

Each chapter contains definitions, examples, exercises, questions, and statements of theorems. Definitions are generally preceded by examples and discussion that make that definition a natural consequence of the experience of the examples and the line of thinking presented. We want you to see the development of mathematics as a natural exploration of a realm of thought. Never should mathematics seem to be a mysterious collection of definitions, theorems, and proofs that arise from the void and must be memorized for a test.

Theorem statements arise as crystallized observations. Proofs are clear reasons that the statements are true.

Each chapter concludes with some selective historical remarks on the chapter's content. This is meant to place the ideas on an historical timeline. It is fascinating to see threads begin in antiquity and continue into the 21st century with no clear end in sight.

Chapters one through four present concepts that are used in all the future chapters. Chapter five on cryptography does not contain material that is required for the future chapters. Chapters six, seven, and eight are sequentially dependent. Chapters nine and ten are independent and can be read any time after chapter four. In a semester course, the authors generally treat chapters one through five, using the further chapters for future work and independent study projects.

Number theory contains within it some of the most fascinating insights in mathematics. We hope you will enjoy your exploration of this intriguing domain.

Methods of thought

Methods of thought, proof, and analysis are not facts to be learned once and set aside. They become useful tools as they appear recurrently in different contexts and as you begin to incorporate them into your habits of approaching the unknown.

While looking at numbers and finding patterns among them, it will be natural to develop an understanding of various ways to give convincing arguments. These different styles of proofs will become familiar and logically sound. We do not present these methods of proof in the abstract, but instead you will develop them as naturally occurring methods of stating logically correct reasons for the truth of statements.

Some methods of thought, proof, and analysis are:

- Finding patterns and formulating conjectures.
- Making precise definitions.
- Making precise statements.
- Using basic logic.
- Forming negations, contrapositives, and converses of statements.
- Understanding examples.
- Relating examples to the general case.
- Generalizing from examples.
- Measuring complexity.
- Looking for elementary building blocks.
- Following consequences of assumptions.
- Methods of proof:
  - induction,
  - contradiction,
  - reducing complexity,
  - taking reasoning that works in a special case and making it general.

By the end of the course these abilities and techniques will be natural strategies for you to use in your mathematical investigations and beyond.

We hope you enjoy your inquiry into number theory.

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