Chapter 1

Basic Concepts

1.1 Introductory Remarks

Writing a book on problem solving is a bit foolish; it is also foolish to write a book on swimming or piano playing. For you cannot learn any of these skills by reading about them. You must *do* them. In fact you must *immerse yourself* in them. Just as developing your torso would entail pumping iron in a rigorous and planned fashion, so developing your problem solving skills will involve a regimen of practice and hard work.

Nevertheless, learning to solve problems can be fun and rewarding. It is a process of developing and extending your mental powers, and it will equip you with a body of techniques that will be useful in other parts of your studies and of your life.

This book is organized around certain types of problem solving techniques, and around certain mental processes that are involved in problem solving. Each concept is illustrated with a number of sample problems; these examples will help to train you in the techniques of problem solving. You should work through each example with care, for the specific examples are much more important than the philosophical remarks that precede them.

Some examples may take a fair amount of time for you to master; but it is important to do so. If you are using this book in a class, then be sure to talk to your instructor, and to your classmates, about the problems you are studying and the techniques you are learning. *Learn to ask questions.* Part of the learning process is to learn to formulate precise statements and precise questions. Another related part is to learn to *communicate* processes of reasoning and analysis. This is strenuous mental exercise. Do it alone, but also do it in a group. Throw the ball back and forth and run around the track (mentally).

Another critical part of this book, this course, and your education in general, is to *learn to read*. By this we do not mean the attainment of literacy. If you are reading these words then you have that problem solved. Instead we mean to read a problem, or an analytical passage, or a solution, and to get to the bottom of it, to completely understand it, and finally to internalize it.

Those ideas and techniques that you internalize are those that finally belong to you, that you can use in practice in your life, that are at your finger tips and have become your own personal tools. This book is designed to familiarize you with the process of internalization and to make it part of your working mental regimen.

Like most books, this one is written in linear order. That is to say, the ideas on any page may utilize, or at least make reference to, the ideas on the preceding pages. But it is no crime to flip ahead in the book, to look at problems that are coming up, or to dip into the book as your interest dictates.

In the next section we will begin to solve problems. At first, our main interest will be to learn to defeat "mental inertia." Even for an experienced and talented problem solver, a tempting way to react to a tough, meaty problem is to look off into space, scratch the old head, and say "Oh, gosh, I don't know. What's for lunch?" Your goal should instead be to train to become a problem fighting machine. You see a new problem and you say "I've seen something like that before. Let's try this ... Let's draw a figure ... Let's reformulate it as follows ... Let's try an example ..." Study this book and you will learn to think in this fashion, not just in math class but in any situation.

1.2 A First Problem

We begin with the analysis of a simple problem. It stands alone, in that one cannot imagine relating it to another problem, or another technique. Its solution requires no specialized knowledge or experience.

PROBLEM 1.2.1 *Determine how many zeros end the number 100!*

Solution: Recall that

$$100! = 100 \cdot 99 \cdot 98 \cdots 3 \cdot 2 \cdot 1.$$

Adding a zero to the end of a product occurs precisely when we multiply by 10. Thus multiplication by any number ending in 1,3,7,9 cannot possibly add a zero to the product (since none of these numbers divides 10). In fact the prime factorization of 10 is $10 = 5 \cdot 2$. We endeavor to solve this problem by counting the factors of 5 in 100!

In the numbers 1–10, only the numbers 5 and 10 have factors of 5. The number 5 may be paired with 2 to yield 10 and the number 10 does not need to be paired. The two resulting factors of 10 contribute two zeros to the full product that forms the factorial.

In the numbers 11–20, only the numbers 15 and 20 have factors of 5. Reasoning as in the last paragraph, we count two additional zeros.

The numbers between 21 and 30 are a bit different. As before, 25 and 30 are the only numbers having a factor of 5, but 25 has *two* factors of 5. Thus

$$22 \times 24 \times 25 = 11 \times 12 \times (2 \times 5) \times (2 \times 5)$$

and this will contribute $10 \times 10 = 100$, or two zeros. Thus the range 21–30 contributes a total of three zeros.

The ranges 31–40 is a simple one, like the first two ranges we considered. It contributes two zeros.

The range 41–50 is special because 45 contributes one factor of 5 but 50 contributes two factors. Thus this range of numbers contributes three zeros (as did the range 21–30).

The range 51–60 and the range 61–70 are like the first two. There are no multiple factors of 5, and each range contributes two zeros.

The range 71–80 is special because 75 contributes two factors of 5 and 80 contributes one factor of 5. The total contribution is three zeros.

The range 81–90 contributes two factors of 5 in the usual fashion, and thus adds two zeros.

The range 91–100 contains 95 and 100. The first of these contributes one factor of five and the second contributes two. Thus three zeros are added.

Taking all of our analyses into account, we have six ranges of numbers that each contribute two zeros and four ranges that each contribute three zeros. This gives a total of 24 zeros that will appear at the end of 100! $\square$

This example already exhibits several important features of successful problem solving:

- We identified the essential feature on which the problem hinges (that a trailing zero comes from multiplication by 10).
- We began by analyzing a special case (i.e., the product $10 \cdot 9 \cdot 8 \cdots 3 \cdot 2 \cdot 1$).
- We determined how to pass from the special case to the full problem.

It is not always true that examining a special case, or a smaller case, will lead to a solution of the problem at hand. But it will get you started. This will be one of our many devices for attacking a problem.

Looking back at our solution of the first problem, we see that we could have been more clever. The numbers from 1 to 100 contain $100 \div 5 = 20$ multiples of 5. Four of these multiples of 5 are in fact multiples of 25, hence contribute two 5's. That gives a total of 24 factors of 5 in 100! Pairing each of these with an even number gives a factor of ten, and hence a zero. We conclude that there are 24 zeros at the end of 100!

Here is another example of specialization:

PROBLEM 1.2.2 *A math class has 12 students. At the beginning of each class hour, each student shakes hands with each of the other students. How many handshakes take place?*

Solution: We begin with a special case and build up to the case of 12 students.

Suppose that there are just 2 students. Then only one handshake is possible.

Now suppose that a new student walks in the door. He/she must shake hands with each of the students that is already in the room. So that makes two more handshakes. The total number of handshakes is $1 + 2 = 3$.

If a fourth student walks in the door, then he/she must shake hands with each of the students already in the room. The total number of handshakes is then $1 + 2 + 3 = 6$.

The pattern is now clear: the addition of a fifth student would result in $1 + 2 + 3 + 4$ handshakes. When we get up to twelve students, we will have required

$$1 + 2 + 3 + \cdots + 9 + 10 + 11 = 66$$

handshakes.

That solves the problem. $\square$

Many times the solution or analysis of one problem will suggest others. Here is another problem that Problem 1.2.2 suggests:

PROBLEM 1.2.3 *Assume that k is a positive integer. What is the sum of the integers*

$$S = 1 + 2 + 3 + \cdots + (k - 1) + k \ ?$$

Before we present a solution, we conduct some preliminary discussion of this problem.

We think of S as function: set

$$S(k) = 1 + 2 + 3 + \cdots + (k - 1) + k.$$

What sort of function might this be? If a function $f(k)$ increases by a fixed amount, say 3, each time that k is increased by 1, then f must be a linear function. Indeed, f must have the form $f(k) = 3k + b$.

Likewise, if the function g increases by a linear function of k each time that k is increased by 1, then we might suspect that g is quadratic. (For those who know calculus, think of the concept of derivative: the derivative of a quadratic function is linear.) For instance, if $g(k) = k^2$ then $g(k+1) - g(k) = 2k + 1$, and that difference is linear.

These considerations motivate our attack on the present problem:

Solution: A useful method for analyzing a sum is to rewrite each term so that some cancellations are introduced. Notice that

$$\begin{array}{rcl} 2^2 - 1^2 & = & 3 = 2 \cdot 1 + 1 \\ 3^2 - 2^2 & = & 5 = 2 \cdot 2 + 1 \\ 4^2 - 3^2 & = & 7 = 2 \cdot 3 + 1 \\ & \cdots & \\ k^2 - (k-1)^2 & = & 2 \cdot (k-1) + 1 \\ (k+1)^2 - k^2 & = & 2 \cdot k + 1 \end{array}$$

Now we add the columns:

$$\begin{array}{l} [2^2 - 1^2] + [3^2 - 2^2] + [4^2 - 3^2] + \cdots + [(k+1)^2 - k^2] \\ \quad = \ [2 \cdot 1 + 1] + [2 \cdot 2 + 1] + [2 \cdot 3 + 1] + \cdots + [2 \cdot k + 1]. \end{array}$$

The left hand side "telescopes" (that is, all but the first and last terms cancel) and the right side may be factored. The result is

$$(k+1)^2 - 1^2 = 2[1 + 2 + 3 + \cdots + k] + [\underbrace{1 + 1 + 1 + \cdots + 1}_{k \text{ times}}]$$

or

$$k^2 + 2k = 2 \cdot S + k.$$

Recall here that S is that sum that we wish to calculate.

Solving for S, we find that

$$S = \frac{k^2 + k}{2}. \qquad \square$$

The formula derived in the last problem is often attributed to Carl Friedrich Gauss (1777–1855), although there is evidence that it was known much earlier.

CHALLENGE PROBLEM 1.2.4 *Imitate the method used in the last problem to find a formula for the sum*

$$1^2 + 2^2 + 3^2 + \cdots + k^2$$

when k is a positive integer.

The solution of our last problem sheds some light on the preceding "handshake" problem. For if a class contains k students and the class period begins with everyone shaking everyone else's hand, then our solution of Problem 1.2.2 shows that the total number of handshakes that occurs is $1 + 2 + 3 + \cdots + (k - 1)$. Now Problem 1.2.3 teaches us that this last sum equals $[(k - 1)^2 + (k - 1)]/2 = [k^2 - k]/2$.

Here is another question that one might ask about the handshake problem:

PROBLEM 1.2.5 *Refer again to the situation in Problem 1.2.2, but suppose now that there are k students in the class. If k is even then will the number of handshakes that takes place be even or odd? If k is odd then will the number of handshakes that takes place be even or odd?*

Solution: If there are 2 students (an even value for k) then the total number of handshakes is 1, an odd number. If we add one student, that adds two handshakes: the number of students is 3 (odd) and the number of handshakes is 3 (odd). If we add yet another student then there are 3 additional handshakes. Thus the total number of handshakes is 6 (even) while the total number of students is 4 (even).

In fact if we draw up a chart then a pattern begins to emerge:

# students	# handshakes	parity of handshakes
0	0	even
1	0	even
2	1	odd
3	3	odd
4	6	even
5	10	even
6	15	odd
7	21	odd
8	28	even
9	36	even
10	45	odd
11	55	odd
12	66	even
13	78	even

As is sometimes done in mathematics, we include the somewhat silly cases of 0 students and 1 student for completeness, and to simplify the discussion that follows.

We see that the first two numbers for handshakes are even, then there are two odd, then there are two even, and so forth.

The pattern repeats itself in increments of *four*. Now notice that the number in the "# handshakes" column in the row for $(k+1)$ students is obtained by adding the two numbers in the row for k students. We know that the number of handshakes for k students is $(k^2-k)/2$. Verify this formula for yourself by plugging the numbers $1, 2, 3, 4, 5, 6, 7, 8$ into the expression. What you are in fact checking is that (for ℓ a non-negative integer)

(i) In rows $4\ell, 4\ell+1$, the number of handshakes is even;
(ii) In rows $4\ell+2, 4\ell+3$, the number of handshakes is odd.

That completes our analysis of the parity of the number of handshakes of k students. $\square$

Notice that our analysis of Problem 1.2.5 did not conform to the theme of this section: it was not solved by first considering a simple

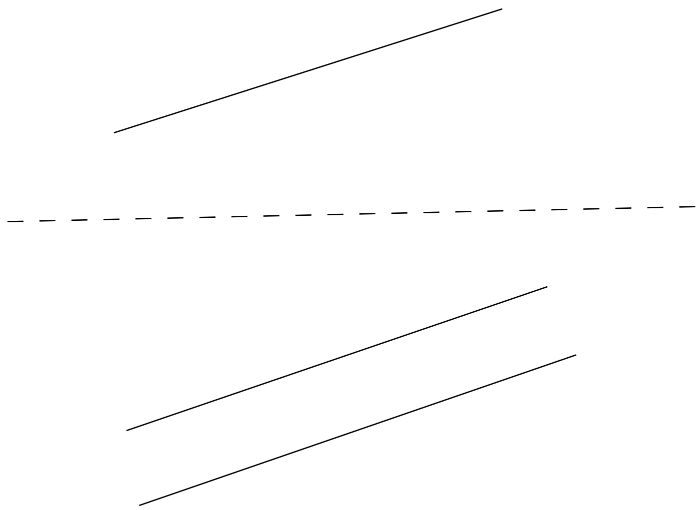


Figure 1

case. Instead, we explored Problem 1.2.5 as something of a side-track—because it was suggested by the result of Problem 1.2.2.

We now turn to a final example of the technique of specialization:

PROBLEM 1.2.6 *What is the greatest number of regions into which three straight lines (of infinite extent) can divide the plane?*

Solution: We begin with the simpler question: "What is the greatest number of regions into which one line can divide the plane?" Of course there is nothing to discuss, for one line will always separate the plane into two regions.

Next we look at two lines. Refer to Figure 1 (top). If the two lines coincide, then the plane is still divided into just two regions. If, instead, the two lines are distinct but parallel (Figure 1, bottom) then the plane is separated into three separate regions.

We think of the two cases just described as degenerate or atypical for the following reason: if you drop two straws onto a floor, then the probability that they will land on top of each other, or land in a configuration so that they are parallel, is zero. Rather, with probability one, the straws will land so that they are skew (or non-parallel). We refer to this last situation as "general position" for the two straws.

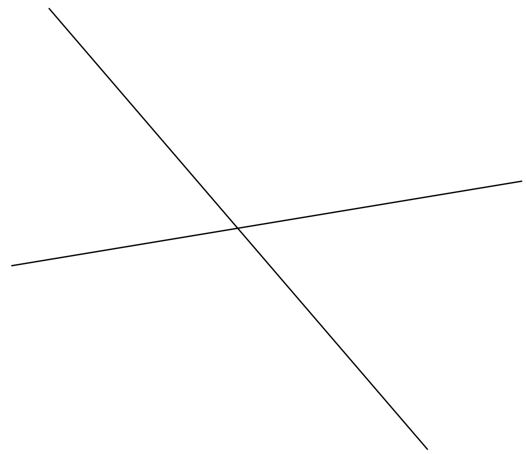


Figure 2

Now suppose that our two lines are in general position. This situation is illustrated in Figure 2. Then the plane is separated into four regions.

Finally we pass to three lines. If all three lines coincide, then we are in the situation for one line. If two of the lines coincide, then we are in the situation for two lines. So suppose that the three lines are distinct.

If the three lines are parallel, then the plane is separated into four regions (Figure 3). If two are parallel, and the third is skew to them, then the plane is separated into six regions (Figure 4). Now suppose that no two of the three lines are parallel.

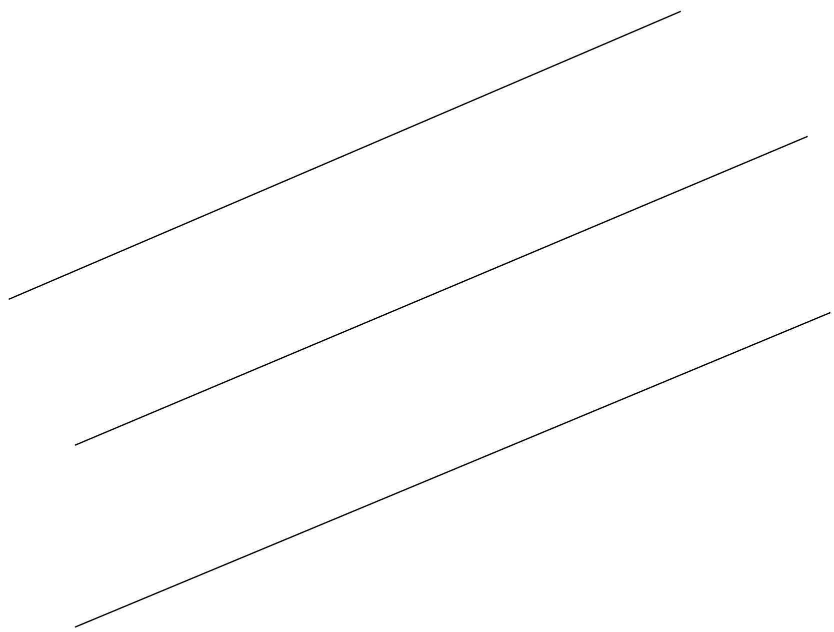


Figure 3

If the three lines pass through a single point, then (Figure 5) the plane is separated into six regions. If the three lines do not pass through a single point, and no two of them are parallel (this is the general position situation, that is, the one that occurs with probability one—see Figure 6) then the plane is separated into seven regions. Thus seven is the maximal number of regions into which three lines can divide the plane. □

In the last problem we certainly used the method of specializing to simple cases to begin to get a grasp of how the problem works. But we also used the *method of exhaustion*. We used the notions of parallelism and intersection—what we ended up calling "general position"—to lay out all the possible configurations for the lines. You may find it an instructive exercise to generalize the last problem to four lines or five lines.

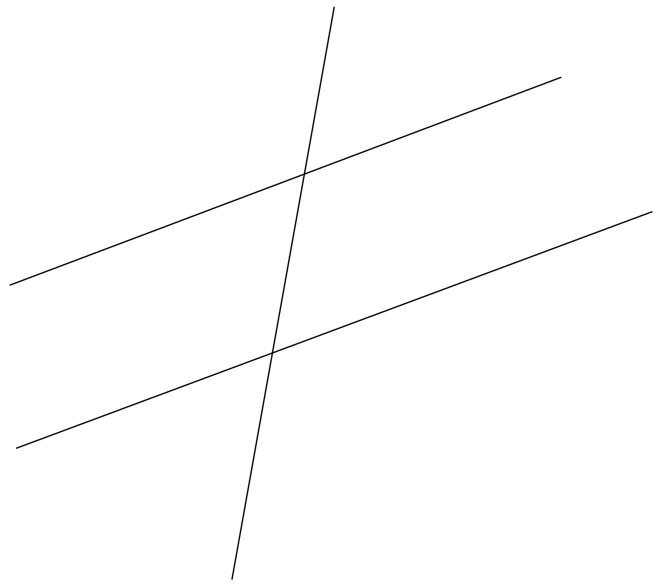


Figure 4

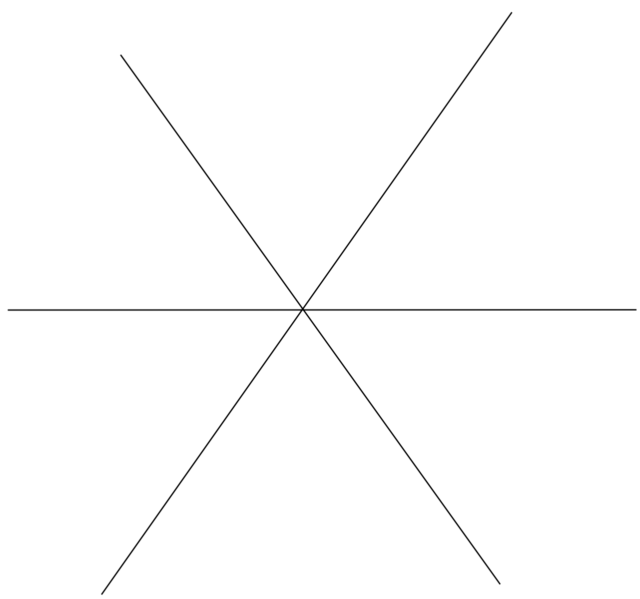


Figure 5

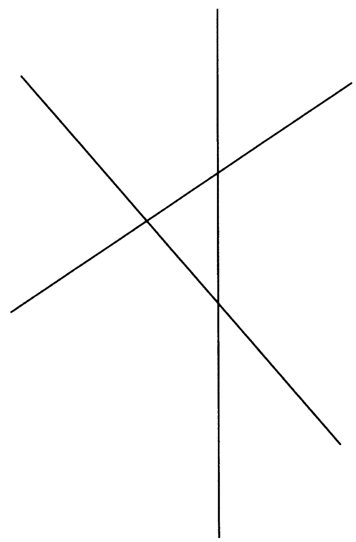


Figure 6

We wrap up the section with the second of several CHALLENGE PROBLEMS. These are problems—for you to do—that are closely related to those in the text. You should attempt to do them right away, right after you have read the cognate problems and solutions that have been presented in the text. Sometimes the CHALLENGE PROBLEMS will be routine, involving no essentially new ideas. Other times they will exceed the usual level of difficulty, or trickiness, of the typical examples in the text or the exercises. They are included to encourage you to stretch your abilities and imagination. Particularly, you should use them as an excuse to talk to others in an effort to generate ideas.

CHALLENGE PROBLEM 1.2.7 *What is the greatest number of regions into which five planes can divide three dimensional space?*

In fact the answer is 26 regions, and this is rather difficult to visualize. The remarkable fact is that the actual solution is not very difficult; what is difficult is harnessing your three dimensional geometric powers of visualization so that you can come upon the solution. John Sununu, a member of President George Bush's White House Staff, was able to solve this problem as a young man. We shall discuss its complete solution later in the book.

1.3 How to Count

Already in Problem 1.2.1 we got a taste of a "counting problem." Counting problems come up in many guises: How many different five card poker hands are there? How many different ways can you roll 8 with two dice? How many different ways can you make up $1 with nickels, dimes, and quarters?

The essence of counting technique is to have an organizational strategy. We begin with the most elementary counting question.

PROBLEM 1.3.1 *We are given k objects $\{a_1, \ldots, a_k\}$. How many different ordered pairs may be made up from those k objects?*

Solution: There are k possible choices (namely $a_1, a_2, \ldots, a_k$) for the first element of the ordered pair. Having chosen an object for the first element, how many choices remain for the second element? The answer is that there are $(k-1)$ of the original $\{a_1, \ldots, a_k\}$ remaining.

So if we chose a_1 for the first element then we may choose any of $a_2, a_3, \ldots, a_k$ for the second element—that's $(k-1)$ choices. If we chose a_2 for the first element then we may choose any of $a_1, a_3, a_4, \ldots, a_k$ for the second element—that's $(k-1)$ choices. And so forth.

In summary, there are k choices for the first element of the ordered pair. For each of those choices, there are $(k-1)$ choices for the second element of the ordered pair. The total number of possible ordered pairs, chosen from among $\{a_1, a_2, \ldots, a_k\}$, is then $k \cdot (k-1)$. $\square$

We can use the counting strategy from this last problem to come up with a basic fact about "permutations," or orderings, of finite sets:

PROBLEM 1.3.2 *We are given k objects $\{a_1, \ldots, a_k\}$. In how many different orders can we arrange these objects?*

Solution: Suppose that we have k positions into which to put the objects (Figure 7). There are k different objects (namely $a_1, a_2, \ldots, a_k$) that we may put in the first position.

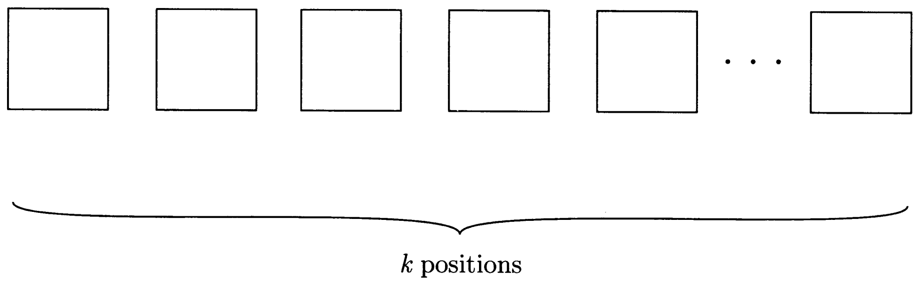


Figure 7

Having placed an object into the first position, there remain $(k-1)$ different objects to put into the second position. Thus, reasoning as in the last problem, there are $k \cdot (k-1)$ choices of pairs of objects to put into the first two positions.

Having chosen two objects to put into the first two positions, we see that there remain $(k-2)$ objects to put into the third position. Thus there are $k \cdot (k-1) \cdot (k-2)$ choices of objects to put into the first three positions.

We may continue to reason in this fashion. We find that there are $k \cdot (k-1) \cdot (k-2) \cdot (k-3)$ choices for the first four positions, $k \cdot (k-1) \cdot (k-2) \cdot (k-3) \cdot (k-4)$ choices for the first five positions, and so forth.

In the end, there are

$$k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1 = k!$$

possible different orderings of the k objects $\{a_1, a_2, \ldots, a_k\}$. $\square$

A successful attack on many counting problem relies on knowledge of the "choose function." We now turn to that notion:

PROBLEM 1.3.3 *We are given k objects $\{a_1, a_2, \ldots, a_k\}$. Suppose that m is a positive integer that is less than or equal to k. In how many ways can we choose m objects from among the original k?*

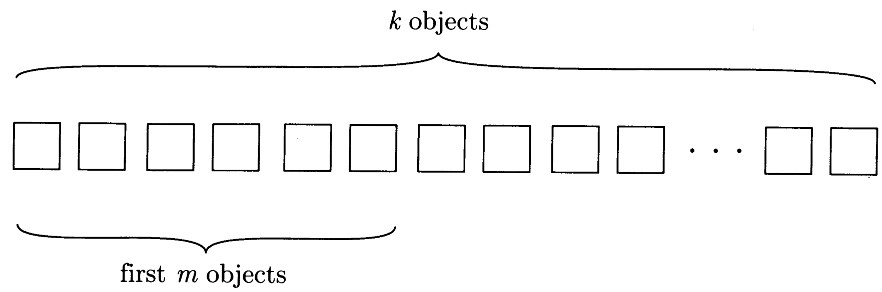


Figure 8

An example of this last type of problem is to ask how many different 5 card poker hands may be chosen from a deck of 52. Or how many different 11 person football teams may be chosen from a group of 25 people. What is interesting about the question, and differentiates it from the preceding problems, is that *we are not interested in ordering.* That is, the poker hand

$$A\spadesuit\ ,\ K\heartsuit\ ,\ J\spadesuit\ ,\ 9\clubsuit\ ,\ 7\diamondsuit$$

is just the same as the poker hand

$$K\heartsuit\ ,\ 7\diamondsuit\ ,\ 9\clubsuit\ ,\ A\spadesuit\ ,\ J\spadesuit.$$

Now we turn to the solution of our problem:

Solution: We need a system for selecting m objects from among the original k. Suppose we proceed as follows: We select an ordering for the entire set of k objects and choose as our subcollection of m objects the first m in the ordering. See Figure 8. Since there are $k!$ different ways to order the k objects (see the last problem), this suggests that there are $k!$ different subcollections of m objects.

Of course there must be something wrong with this reasoning, since the answer $k!$ does not seem to depend on m. What is wrong is this: We are counting different orderings of those first m objects as different (see Figure 9). We do not wish to do this, so we divide out by the

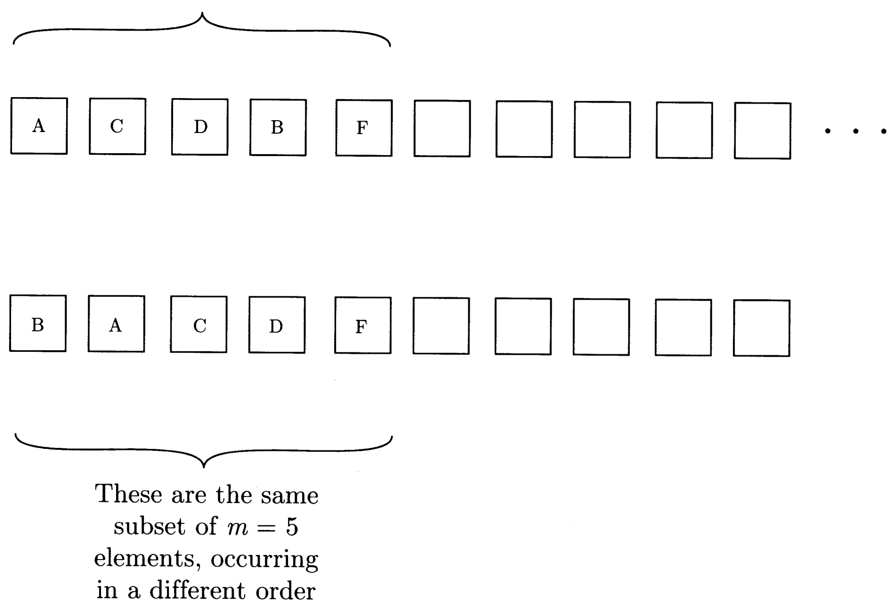


Figure 9

number of possible orderings of these m objects—namely we divide out by $m!$ Likewise, we are counting different orderings of the last $k - m$ objects as different. So we must divide out by the number of possible orderings of those objects—namely we divide out by $(k-m)!$. Now our counting scheme is accurate, and enumerates different subcollections of m objects chosen from among a total of k objects.

We have discovered that the number of different subcollections of m objects chosen from among a total of k objects is

$$\frac{k!}{m! \cdot (k-m)!}.$$

Note once again that our strategy in deriving the formula was this: we took the number of possible orderings of the totality of k objects, and we thought in terms of selecting the first m from among any such ordering. But we must divide out by the different possible orderings of those first m objects. And we must divide out by the different possible orderings of the remaining $k - m$ objects. $\square$

The quantity

$$\frac{k!}{m! \cdot (k-m)!}$$

is used universally in counting arguments, and is commonly referred to as "k choose m." It is written $\binom{k}{m}$. Thus we have

$$\binom{k}{m} = \frac{k!}{m! \cdot (k-m)!}.$$

PROBLEM 1.3.4 *How many different 5 card poker hands may be had from a deck of 52 cards?*

Solution: With the ideas that we have developed, this is an easy problem. For the number of such poker hands is just "52 choose 5:"

$$\# \text{ of poker hands} = \binom{52}{5} = \frac{52!}{5!47!} = 2,598,960. \qquad \square$$

PROBLEM 1.3.5 *How many pairs of bridge hands may be dealt from a deck of 52 cards?*

Solution: Recall that bridge is played by two teams of two people. Each person is dealt 13 cards. Consider one of the teams.

The first team member is dealt 13 cards from among the total of 52. The number of possible hands that this person could be dealt is

$$C_1 = \binom{52}{13} = \frac{52!}{13! \cdot 39!}.$$

The second team member is also dealt 13 cards, chosen from among the remaining 39 cards. [Notice here that in an actual game we do not first give 13 cards to the first team member and then give 13 cards to the second team member. But the order in which the cards are distributed is irrelevant here. The only point is that the first team member is given 13 cards at random and the second team member is given 13 *other* cards at random.] Thus the number of possibilities for the second team member are

$$C_2 = \binom{39}{13} = \frac{39!}{13! \cdot 26!}.$$

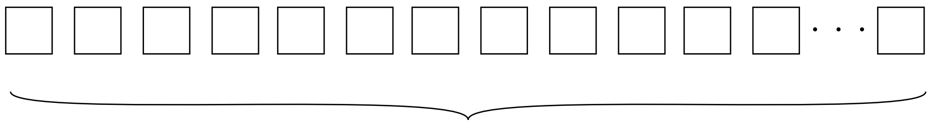


Figure 10

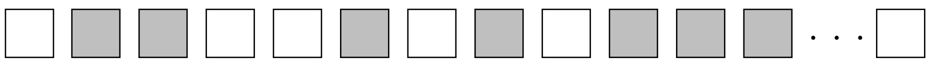


Figure 11

The total number of *pairs of hands* for the two team members is

$$C_1 \cdot C_2 = \binom{52}{13} \cdot \binom{39}{13} = \frac{52!}{13!\,39!} \cdot \frac{39!}{13!\,26!} \approx 5.1578 \times 10^{21}. \qquad \square$$

PROBLEM 1.3.6 *Assume that k and m are positive integers. How many different monomials of degree m are there in $\mathbb{R}^k$?*

First we clarify what we are asking here. The space $\mathbb{R}^k$ consists of elements of the form $(x_1, x_2, \ldots, x_k)$, where $x_1, x_2, \ldots, x_k$ are real numbers. A *monomial* is an expression like $(x_1)^2 \cdot (x_3)^3 \cdot (x_5)^6$. In other words, it consists of *some* powers of *some* of the variables multiplied together. We say that this particular monomial has *degree* 11 because the sum total of all the powers that occur is $2 + 3 + 6 = 11$. The problem asks for the number of all possible monomials of a given degree m in $\mathbb{R}^k$.

Solution: We now learn a new counting device. Consider $m + k - 1$ boxes, as shown in Figure 10. We will shade any $(k-1)$ of them. What remains is some clusters of unshaded boxes—the total number of these remaining boxes is m. Refer to Figure 11.

Between the leftmost edge of all the boxes and the first shaded box is a group of unshaded boxes. Say that there are m_1 of them. Note

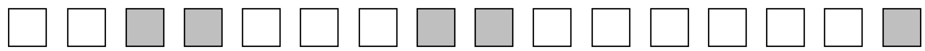


Figure 12

that $0 \leq m_1 \leq m$. Next, to the right of the first shaded box and to the left of the second shaded box there is a group of unshaded boxes, say m_2 of them. Continue in this fashion.

We see that the shading of $k-1$ boxes gives rise to non-negative integers $m_1, m_2, \ldots, m_k$ such that $m_1+m_2+\cdots+m_k = m$. In turn, this k-tuple of numbers corresponds to the monomial $(x_1)^{m_1} \cdot (x_2)^{m_2} \cdots (x_k)^{m_k}$.

The process runs in reverse. Any monomial $(x_1)^{m_1} \cdot (x_2)^{m_2} \cdots (x_k)^{m_k}$ has a corresponding k-tuple $(m_1, m_2, \ldots, m_k)$, and this k-tuple in turn corresponds to a shading of $k-1$ boxes from among $m+k-1$ boxes. In fact Figure 12 illustrates the shading of boxes that corresponds to $(x_1)^2 \cdot (x_3)^3 \cdot (x_5)^6$ in $\mathbb{R}^6$.

Thus counting the monomials of degree m in $\mathbb{R}^k$ corresponds exactly to counting the different ways to shade (or to choose) $k-1$ boxes chosen from among $m+k-1$ boxes. *Note that there are no redundancies or ambiguities.* Thus the number of monomials that we wish to count is

$$\binom{m+k-1}{k-1} = \frac{(m+k-1)!}{(k-1)! \cdot m!} \qquad \square$$

1.4 The Use of Induction

Mathematical Induction is one of the most powerful techniques in all of mathematics. Before we begin to study it, we should draw a distinction between the use of "induction" in other fields of human endeavor and the use of "Mathematical Induction" in mathematics.

Most scientific subjects rely heavily on induction. A chemist or physicist or biologist examines a certain number of cases of some phenomenon and then attempts to *induct* from this data some general rule or dictum. The process of passing from the data to the rule can take many forms. It is not defined in advance. And the main test of whether the process has been valid is further experimentation and gathering of data.

"Mathematical Induction" is limited in scope, and proceeds in a more rigid fashion. Here is the scheme for mathematical induction. Suppose that we have a statement $P(k)$ for each positive integer k. For example, the statement could be "$k^2 - 2k + 1 \geq 0$." Or it could be "The number $2k + 4$ can be written as the sum of two odd primes." The method of Mathematical Induction is used to proved $P(k)$ for all k in the following fashion:

(i) First verify $P(1)$.
(ii) Verify that $P(j) \Rightarrow P(j+1)$ for every $j \in \{1, 2, 3, \ldots\}$.

Assuming that these two statements have been verified, we notice the following. Using (i) and the special instance $j = 1$ of (ii) gives us $P(1)$ and $P(1) \Rightarrow P(2)$. From this we may conclude $P(2)$. Now the special instance $j = 2$ of (ii) gives us $P(2) \Rightarrow P(3)$. From this, and the statement $P(2)$ obtained in the preceding step, we may conclude $P(3)$. Continuing in this fashion, we see that $P(k)$ is true for every positive integer k.

The discussion that we have just presented to explain why the method of Mathematical Induction is a valid method of reasoning should be taken as intuitive. A rigorous treatment of the method is intimately bound up with set theory and the construction of the natural numbers; we cannot go into the details here. We refer the reader to [KRA1] and [SUP] for more detailed discussions.

Now we turn to some examples where induction is useful.

PROBLEM 1.4.1 *Verify the formula*

$$1 + 2 + 3 + \cdots + (k-1) + k = \frac{k + k^2}{2}.$$

Solution: We solved this problem by a different method in Section 1.2. The technique that we used at the time may have seemed like an unmotivated trick. Once you get used to induction, by studying examples like the one we are doing now, it will seem like a standard device for problems like this one.

When using the method of induction (we now use this language, instead of the more formalistic Mathematical Induction), it is important to proceed systematically.

First, what is the statement $P(k)$ that is to be verified? It is

$$1 + 2 + 3 + \cdots + (k-1) + k = \frac{k + k^2}{2}.$$

To verify $P(1)$, we note that

$$1 = \frac{1 + 1^2}{2}.$$

The most interesting, and subtle, part of the method of induction is part (ii). We *assume* that $P(j)$ is known to hold. Thus, in the present problem, we are assuming that

$$1 + 2 + 3 + \cdots + j = \frac{j + j^2}{2}. \qquad (*)$$

From this we wish to derive the corresponding statement for $(j+1)$.

To this end, we add $(j+1)$ to both sides of $(*)$. We obtain

$$1 + 2 + 3 + \cdots + j + (j+1) = \frac{j + j^2}{2} + (j+1).$$

Simplifying, we have

$$1 + 2 + 3 + \cdots + (j+1) = \frac{j + j^2 + 2(j+1)}{2}$$

or

$$1 + 2 + 3 + \cdots + (j+1) = \frac{(j+1) + (j+1)^2}{2}.$$

This last is precisely the statement $P(j+1)$.

Notice that, assuming the validity of $P(j)$, we have derived $P(j+1)$. That is precisely part (ii) of the method of induction.

The verification is complete. According to the method of induction, once we have verified Steps (i) and (ii), then we can be sure that $P(k)$ holds for all k. Thus we have solved the problem. $\square$

Sometimes it is convenient to begin an induction at a point other than $j = 1$. In the next problem we begin at $j = 0$.

PROBLEM 1.4.2 *Suppose that S is a set with k elements. Show that S has precisely 2^k subsets.*

Solution: We will use the method of induction. First recall that a set A is said to be a subset of a set B if each element of A is also an element of B. In particular, $\emptyset \subset A$, where $\emptyset$ is the "empty set" (or the set with no elements). Also $A \subset A$.

Now our inductive statement $P(k)$ is this: "If a set S has k elements then S has 2^k subsets."

As already noted, we begin our induction at 0, rather than 1. For step (i), notice that if $S = \{\} = \emptyset$ has no elements then the only subset of S is S itself. Thus S has $1 = 2^0$ subsets. Thus we have verified $P(0)$.

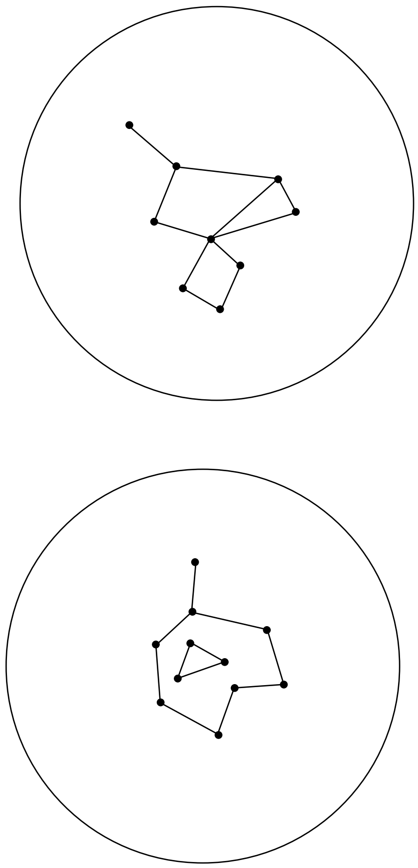
For step (ii), we assume that $P(j)$ is valid. This means that any set with j elements has 2^j subsets. Now let $S = \{s_1, s_2, \ldots, s_j, s_{j+1}\}$ be a set with $(j+1)$ elements. Set $S' = \{s_1, s_2, \ldots, s_j\}$. Notice that the set S' has j elements. By hypothesis, S' has a total of 2^j subsets. Now we count the subsets of S itself.

Certainly any subset of S' is also a subset of S. That accounts for 2^j subsets of S. Also, if A is any subset of S' then $A \cup \{s_{j+1}\}$ is a subset of S. That accounts for another 2^j subsets of S. We thus have identified a total of $2^j + 2^j = 2^{j+1}$ subsets of the set S. Notice that we have in fact accounted for *all* the subsets of S, since any subset of S either contains s_{j+1} or it does not. Therefore we have derived $P(j+1)$ from $P(j)$. That is part (ii) of the method of induction.

The verification is complete. □

PROBLEM 1.4.3 *Suppose that we have an admissible graph on the unit sphere in three dimensional space. Here, by "admissible graph" we mean a connected configuration of arcs. Two arcs may be joined only at their endpoints. The endpoints of the arcs in the graph are called vertices. The arcs are called edges. An edge is that portion of an arc that lies between two vertices. A* face *is any two dimensional region, without holes, that is bordered by edges and vertices. Figure 13 illustrates an admissible graph and a non-admissible graph.*

This problem asks you to verify Euler's formula *for an admissible graph. We let V be the number of vertices, E the number of edges,*



admissible

non-admissible

Figure 13

and F the number of faces. Then Euler's formula is

$$V - E + F = 2.$$

Solution: We begin with some special cases, just to be sure that we understand what is going on.

The simplest graph that is admissible, according to our definitions, consists of a single vertex and nothing else (Figure 14). The complement of that single vertex, in the sphere, is a valid face. Thus $V = 1$, $E = 0$, and $F = 1$. Then

$$V - E + F = 1 - 0 + 1 = 2$$

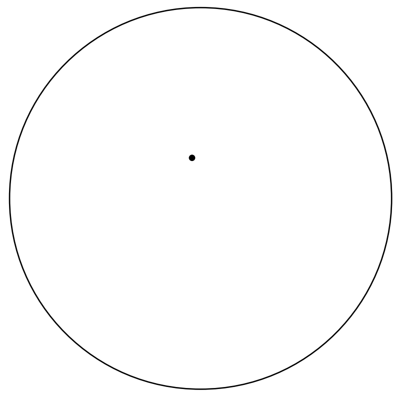


Figure 14

and we see that Euler's formula is valid.

The next most complex graph has one edge, with a vertex on each end, and nothing else. The complement (in the sphere) of this edge with its endpoints is a single valid face. See Figure 15. Thus, in this case, $V = 2$, $E = 1$, and $F = 1$. We see that

$$V - E + F = 2 - 1 + 1 = 2.$$

Thus Euler's formula is valid in this case as well.

Now we let $P(k)$ be the statement "Euler's formula is valid for any admissible graph with k edges." We shall use induction to prove this statement for every k.

The statement $P(1)$ has already been verified. That is part (i) of the method of induction.

For part (ii), we assume that Euler's formula is valid for any admissible graph having j edges. Now let $\mathcal{G}$ be a graph having $(j+1)$ edges. There is some edge that can be removed from $\mathcal{G}$ so that the remaining graph $\mathcal{G}'$ is still admissible (exercise—for example, an edge that separates two different faces will do). Say that V', E', and F' denote the numbers of vertices, edges, and faces for the graph $\mathcal{G}'$. Now consider what the corresponding numbers V, E, F for the graph $\mathcal{G}$ might be.

The graph $\mathcal{G}$ is obtained from $\mathcal{G}'$ (we are reversing the construction that produced $\mathcal{G}'$) by adding an edge. If the edge is added by attaching

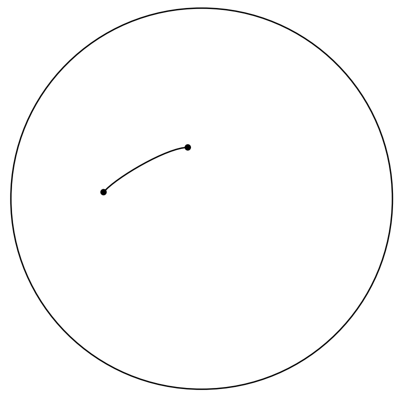


Figure 15

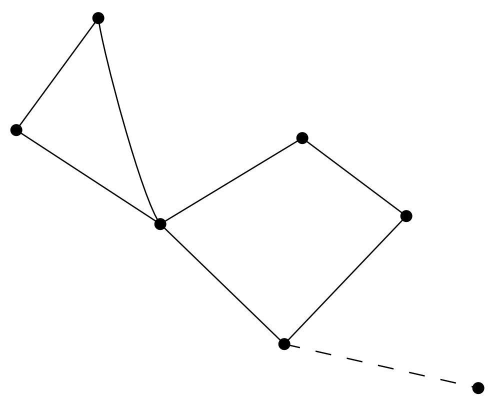
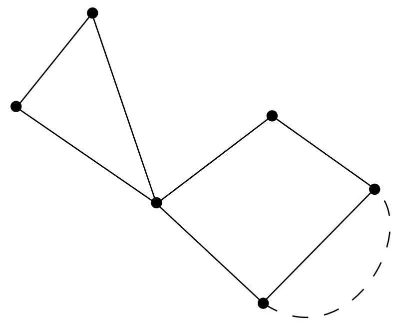


Figure 16

one end, and leaving the other free (the added edge is dotted in Figure 16), then the number of faces does not change, the number of edges is increased by one, and the number of vertices is increased by one. See Figure 16. Thus $V = V' + 1$, $E = D' + 1$, and $F = F'$. Since, by hypothesis, $V' - E' + F' = 2$, it follows that $V - E + F = 2$ as desired. If instead the edge is added by attaching *both ends* (again the added edge is dotted in Figure 17—there are two possibilities, as shown), then the number of faces is increased by one, the number of edges is increased by one, and the number of vertices does not change. Therefore $V = V'$, $E = E' + 1$, and $F = F' + 1$. Since, by hypothesis, $V' - E' + F' = 2$,



or

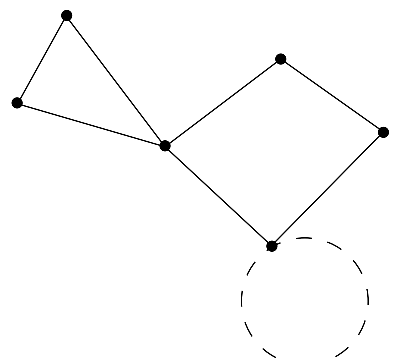


Figure 17

it follows that $V - E + F = 2$.

Since these are the only two ways that a new edge can be attached, we have established step (ii) of the inductive procedure. Our argument is complete. □

PROBLEM 1.4.4 *Assume that k is a positive integer. If $(k + 1)$ letters are delivered to k mailboxes, then show that one mailbox must contain at least two letters.*

Solution: Although many solutions are possible, we will use induction just to illustrate the method.

The statement $P(k)$ is "If $(k+1)$ letters are delivered to k mailboxes then some mailbox must receive at least two letters."

For the case $k = 1$, notice that if $k + 1 = 2$ letters are delivered to $k = 1$ mailboxes then some mailbox (namely the *only* mailbox) will receive two letters (indeed, all the letters).

Now assume that $P(j)$ has been proved. Assume that $(j + 1) + 1$ letters have been delivered to $(j + 1)$ mailboxes.

- If the last mailbox is empty, then all the letters have been delivered to the first j mailboxes. In particular, at least $(j + 1)$ (indeed, $(j + 2)$) letters have been delivered to these first j mailboxes. So the inductive hypothesis applies and one of these first j mailboxes contains at least two letters.

- If the last mailbox contains precisely one letter, then the remaining $(j + 1)$ letters have been delivered to the first j mailboxes. Once again, the inductive hypothesis applies to the first j mailboxes. So one of them will contain at least two letters.

- If the last mailbox contains two or more letters, then we are done because some mailbox (namely the last one) contains at least two letters.

That completes the verification of our statement. □

The principle embodied in the statement of the last problem is of pre-eminent importance in mathematics. It is commonly called the "pigeonhole principle." It was originally called the *Dirichletscher Schubfachschluss* (Dirichlet's drawer-shutting principle) because it was formulated by the German mathematician Peter Gustav Lejeune Dirichlet (1805-1859). In the next section, you will learn about arguing by contradiction. As an exercise, you can check then that the pigeonhole principle can easily be verified using the method of contradiction.

Challenge Problem 1.4.5 *A group of people gathers for a party. A lot of handshaking goes on. Show that the number of people who shake hands an odd number of times is even.*

PROBLEM 1.4.6 *Suppose that six people are in a room. Explain why either three of these people all know each other or else there are three of the people none of whom knows each other.*

Solution: Of course we are supposing that if A knows B then B knows A and vice versa. Call one of the people Joe. Of the other five people, either Joe knows three of them or he does not know three of them. Say that the first case obtains. In fact, say that he knows Harry, Mary, and Larry. Now if two of those people know each other (for instance, Harry knows Larry), then the triple {Joe, Harry, Larry} is a mutually acquainted triple. If, instead, it is not the case that two of these three people know each other, then {Harry, Mary, Larry} is a triple of people no two of whom know each other. $\square$

Challenge Problem 1.4.7 *We can reinterpret the last problem as follows.*

> *You have six points on a piece of paper. Each of the fifteen possible pairs of these points will be joined either by a red line segment or by a blue line segment. Show that there will either be a triangle that is all blue or a triangle that is all red.*

Explain why this formulation is equivalent to the original one.

Challenge Problem 1.4.8 *Refer to the last problem and the last challenge problem. Suppose now that you have three colors in which you can render the line segments: red, blue, and yellow. Then, with six points, it will not be the case that there must be a triangle that is all of one color. Explain why.*

In fact how many points do you need, with every pair connected by either a red, a blue, or a yellow line segment, so that you are guaranteed to have a triangle of just one color?

Challenge Problem 1.4.9 *Formulate a generalization of the last challenge problem. Suppose that you have k colors. How many points are required to guarantee that the process of joining all possible pairs of points with line segment segments of one of these colors will guarantee that there is a triangle of just one color?*

1.5 Problems of Logic

Logic plays a role in every problem that we solve. But some problems are primarily geometrical in nature, others are counting problems, and still others are analytical. In the present section we use logic itself as the chief tool both for formulating and for solving problems. We begin with a classic from the genre of "Liar and Truth Teller" problems.

PROBLEM 1.5.1 *You are on an island that is populated by two types of people: truth tellers and liars. When asked a* YES–NO *question, a truth teller always tells the truth and a liar always lies. There is no visual method for telling a truth teller from a liar. What single question could you ask anyone that you meet on the island to determine whether that person is a truth teller or a liar?*

Solution: If you ask a direct question such as "Are you a truth teller?" then a truth teller will answer "Yes" and a liar (who must lie) will also answer "Yes". You will get a similar result if you ask "Are you a liar?" Thus an elementary, direct question provides no basis for differentiation.

Therefore a compound question, such as a conditional, or an "or" question, or an "and" question is called for. One of the things that we learn in a basic logic course is that any question that is of one of these three types can be reformulated as a question of any one of the other three types (see [KRA1]). We concentrate on formulating an "if-then" question that will do the job.

The question that we formulate could be of the form "If it is raining then what would you say to ..." or "If you are a Doctor of Letters then what would be your answer to ...." However it is clear that these conditions have nothing to do with the matter at hand.

Probably more relevant would be a question of the form "If you were a truth teller then what would you say to ...." Likewise the concluding part of the question ought to have something to do with the problem that we are trying to solve. We now try the question

> **If you are a truth teller then how would you answer the question, 'Are you a liar?'**

Now we analyze how the two different types of island inhabitants would answer this question.

Obviously a truth teller would answer the question "Are you a liar?" by saying "No". If you pose the displayed question to a truth teller, then he will report truthfully on the answer just given, so he/she will say "No".

A liar can think just as clearly as a truth teller. He/she also knows that a truth teller, if asked whether he/she is a liar, will say "No". But the liar must lie. So he/she will say "Yes".

Thus we have found a question to which a truth teller will always answer "No" and a liar will always answer "Yes". This certainly gives a means for differentiating truth tellers from liars, and answers our problem. □

You might have some fun modifying the question that we formulated in the last problem and seeing what the results would be. Try the question "If you are a liar then how would you answer the question 'Are you a truth teller?' " There are other variations that you might try as well. What would happen if you used the question "Are you a duck?"

Here is a problem of the same sort that you can try as a challenge:

CHALLENGE PROBLEM 1.5.2 *You are on the island of truth tellers and liars. Two people walk up to you. Call them A and B. What single yes/no question can you ask A that will enable you to determine whether B is a liar or a truth teller?*

CHALLENGE PROBLEM 1.5.3 *An island is inhabited by Liars, Truth Tellers, and people who lie part of time and tell the truth part of the time. These people cannot be differentiated by studying their external appearance. If you meet an island inhabitant, what single question can you ask him/her to determine which of the three types of people you have encountered?*

The next problem has received a considerable amount of publicity in the last few years. It was inspired by the television game show LET'S MAKE A DEAL. The nature of the game show (a bit over-simplified) is as follows. The contestant is faced with three doors. He/she knows that behind one door is a very desirable prize—say a fancy car. Behind the other two doors are rather pesky and undesirable items—say that a goat is behind each. The contestant is to pick a door (blind), and is awarded the prize that is behind the door. But the game show host, Monty Hall, teases and cajoles and bribes the contestant, encouraging the contestant to change his/her mind and forcing the contestant to become confused over which is the most desirable door.

What has become known as the "Monty Hall" problem is this: The contestant picks a door. For the sake of argument, we say that he/she has picked Door Three. Before the door is opened, revealing what is behind it, Monty Hall says "I will now reveal to you what is behind one of the other doors." A door is opened and there stands a goat. Then Monty Hall says "Would you like to change *your* door selection?" Very interesting.

Clearly the contestant will not pick the door that Monty Hall has already opened, since that has a goat behind it. So the issue is whether the contestant will switch from the currently selected door to the remaining door (the one that the contestant has not chosen and Monty Hall did not open). A naive approach would be to say there is an equal probability for there to be a goat behind the remaining door and behind the door that the contestant has already selected—after all, one door has a goat and one has a car. What is the point of switching? However this naive approach does not take into account the fact that there are two distinct goats. A more careful analysis of cases occurs in our solution to the problem, and reveals a surprising answer.

PROBLEM 1.5.4 *Use a case-by-case analysis to solve the Monty Hall problem.*

Solution: We denote the goats by G_1 and G_2 (for goat one and goat two) and the car by C. For simplicity, we assume that the contestant will always select Door Three. We may not, however, assume that Monty Hall always reveals a goat behind Door One; for there may not be a goat behind Door One (it could be behind Door Two). Thus there are several cases to consider:

Door 1	Door 2	Door 3
G_1	G_2	C
G_2	G_1	C
G_1	C	G_2
G_2	C	G_1
C	G_1	G_2
C	G_2	G_1

As we know from Section 1.4, there are $6 = 3!$ possible permutations of three objects. That is why there are six rows in the array.

1. In the first case, Monty Hall will reveal a goat behind either Door 1 or Door 2. It is *not* to the contestant's advantage to switch, so we record **N**.

2. The second case is similar to the first, it is not to the contestant's advantage to switch, and we record **N**.

3. In the third case, Monty Hall will reveal a goat behind Door 1, and it *is* to the contestant's advantage to switch. We record **Y**.

4. The fourth case is like the third, and it is to the contestant's advantage to switch. We record **Y**.

5. In the fifth case, Monty Hall will reveal a goat behind Door 2. It *is* to the contestant's advantage to switch, so we record **Y**.

6. The sixth case is like the fifth, it is to the contestant's advantage to switch, and we record **Y**.

Observe that the tally of our case-by-case analysis is four **Y**'s and just two **N**'s. Thus the odds are two against one in favor of switching after Monty Hall reveals the goat. □

This last problem was in the nature of a probability problem. Many elementary probability problems are amenable to careful case-by-case, or counting, arguments. Chapters 3 and 8 will give you more experience with probability problems.

PROBLEM 1.5.5 *There are more adults than boys, more boys than girls, more girls than families. If no family has fewer than 3 children, then what is the least number of families that there could be?*

Solution: If there were just one family then there would be at least two girls, at least three boys, and at least four adults. But four adults make two families, and that is a contradiction.

If there were just two families, then there would be at least three girls, at least four boys, and at least five adults. But five adults means that there cannot be just two families; there are at least three. That is a contradiction.

If there were just three families, then there would be at least four girls, at least five boys, and at least six adults. That is not a contradiction. Thus three families *might* satisfy the conditions.

In fact suppose that there are three married couples. The first couple has two girls and a boy, the second has two girls and a boy, and the third has three boys. Then there are six adults, five boys, four girls and three families. All conditions of the problem are met.

The answer is that three families is the smallest number that there could be. □

The method of solution in the last problem is one of "exhaustion." It would not be a very satisfactory method to use if the answer to the problem were 357, for it would take a great deal of work to reach that answer. Nevertheless, exhaustion is an important and systematic tool to have in our arsenal.

PROBLEM 1.5.6 *Explain why there are infinitely many prime numbers.*

Solution: Recall that a prime number is a positive integer, not 1, that has no divisors other than 1 and itself. The first several primes are 2,3,5,7,11,13,17,19,23,.... Every positive integer is divisible by a prime number—indeed can be factored in a unique manner into prime factors. That is the content of the Fundamental Theorem of Arithmetic.

We solve the present problem by using the method of proof by contradiction. Suppose that there were in fact only finitely many primes in the world. Call these primes $p_1, p_2, \ldots, p_k$. Consider the number $N = (p_1 \cdot p_2 \cdots p_k) + 1$—the product of all these primes, with 1 added to it. Now N must be divisible by *some* prime number—as noted in the last paragraph. However it is not divisible by p_1, since division by p_1 results in a remainder of 1. Likewise, N is not divisible by p_2, because division by p_2 results in a remainder of 1. In fact we see that N is not divisible by any of $p_1, p_2, \ldots, p_k$. But these were all the primes in the universe. Yet N must be divisible by some prime! That is a contradiction.

We conclude that there cannot be just finitely many primes. There must be infinitely many. $\square$

"Proof by contradiction" is a simple but powerful device in mathematical and analytical reasoning. The scheme is as follows: We wish to prove a proposition P. Now P is either true or it is false. There is no "wait and see" or "in-between" status; it must be one or the other (see [KRA1] for further discussion of this concept). The strategy in "proof by contradiction" is to eliminate the possibility that P is false. So we *assume* that P is false and argue that that assumption is untenable (it leads to a contradiction). The only possible conclusion is that P is true. Problem 1.5.6 illustrates this method of reasoning.

The proof that there are infinitely many primes is usually attributed to Euclid; it is more than 2000 years old. It was one of the first proofs by contradiction. Interestingly, proof by contradiction did not become a standard mathematical tool until the twentieth century.

1.6 Issues of Parity

The most elementary example of parity is "oddness vs. evenness", but there are many others. We shall explore various aspects of parity in this section.

PROBLEM 1.6.1 *An $8' \times 8'$ bathroom is to have its floor tiled. Each tile is $2' \times 1'$. In one corner of the bathroom is a sink, and its plumbing occupies a $1' \times 1'$ square in the floor. In the opposite corner is a toilet, and its plumbing occupies a $1' \times 1'$ square in the floor. The situation*

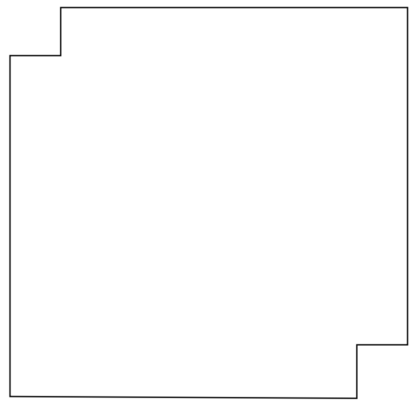


Figure 18

is shown in Figure 18. [Figure 19 exhibits a way to envision the floor broken up into one foot squares.]

How is it possible to achieve the required tiling of the floor?

Solution: The area to be tiled is $8' \times 8'$ less two square feet—in other words, we must tile an area of 62 square feet. Thus we will use 31 tiles.

Figure 20 shows one possible arrangement of tiles; notice that it fails to cover the entire floor. There are two squares remaining (on the lower right), and they cannot be covered by a single tile. Other attempts fail similarly. (Try this with your checkerboard; put coins on two of the opposite corners to signal that the tiles may not cover those squares).

We start to smell a rat. Perhaps the floor cannot be tiled. But how can we produce a cogent argument that explains why no tiling can work? There are many hundreds of ways to attempt to tile this floor, and it is distasteful to consider *all possible tilings*. The idea that we now introduce—one inspired by the concept of parity—is to color the bare floor of the bathroom like a checkerboard. Look at Figure 21. Notice that when we place a $2' \times 1'$ tile on the floor then it will cover two adjacent squares. One of these will be black and one will be white. Thus if we place two tiles on the floor, they will cover a total of two black squares and two white squares. In general, k tiles placed on the floor will cover k black squares and k white squares. Yet the bathroom floor that we are dealing with has 32 black squares and 30

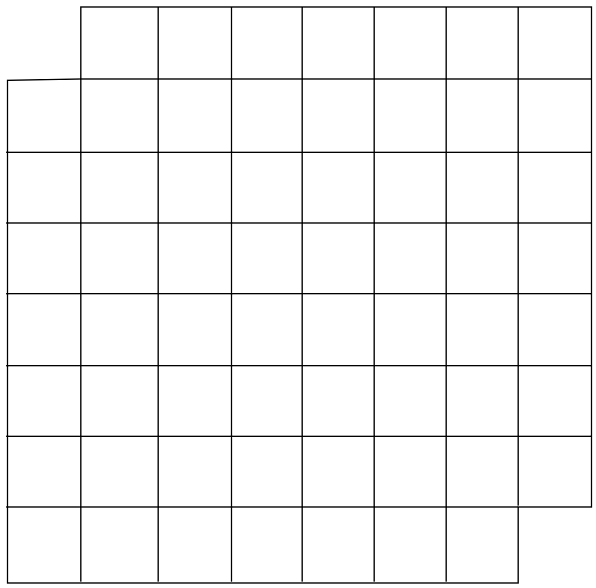


Figure 19

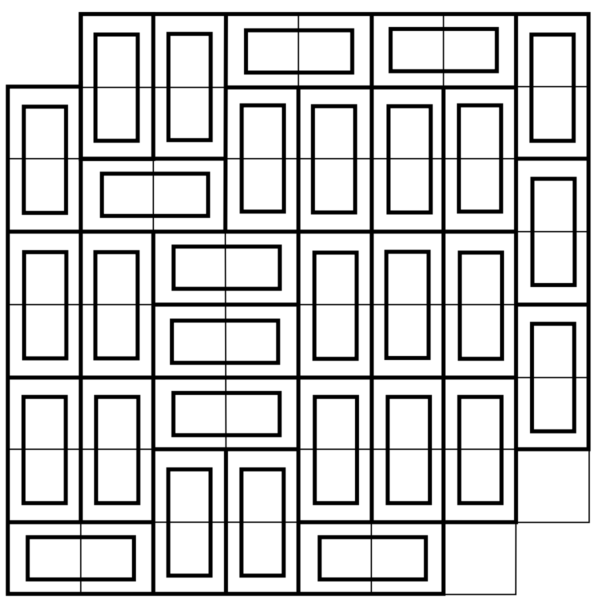


Figure 20

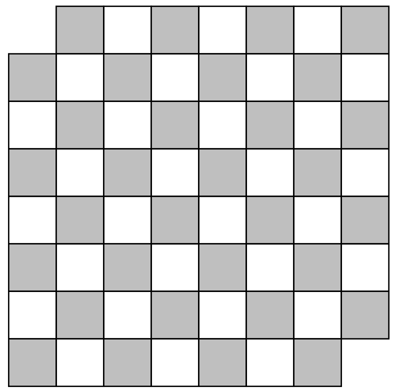


Figure 21

white squares. Since the number of black squares covered will always equal the number of white squares covered, we have an insurmountable problem. This bathroom floor can never be tiled! □

The use of parity that came into the last problem was the device of coloring the floor. It would have required intricate combinatorial reasoning to solve this problem without the intervention of the coloring.

PROBLEM 1.6.2 *We have a container that contains 6 quarts and another that contains 4 quarts. We fill these containers by immersing them in the river. How can we use them to fill one of the containers with 3 quarts of water?*

Solution: What is implicit in the problem, and what we make explicit now, is that the only moves we are allowed are (i) to fill a container, (ii) to empty a container, or (iii) to pour one container into the other. This being the case, our various pouring operations correspond to adding and subtracting multiples of 4 and 6. Now addition and subtraction of even numbers always results in an even number answer. Thus there is no way to obtain the number 3.

The problem cannot be solved. □

Challenge Problem 1.6.3 *Now suppose that you have a 9 quart container and a 4 quart container. How can you put exactly 6 quarts of water into the large container?*

We have given two examples in which the notion of parity taught us that a certain problem cannot be solved. Now we give one in which parity leads to a positive solution.

PROBLEM 1.6.4 (Maşek) *Imagine a polyhedron with 1981 vertices. [This is not as difficult as it sounds. Just place 1981 points on the unit sphere in three dimensional space. Now connect them with line segments in an obvious way to obtain a polyhedron.]*

Imagine that each edge *is assigned an electrical charge of* $+1$ *or* -1*. Explain why there must be a vertex such that the product of the charges of all the edges that meet at that vertex must be* $+1$*.*

Solution: Suppose that we multiply together all the products corresponding to all the vertices. Then every edge is counted twice (since each edge has *two* vertices on its ends) so every $+1$ is counted twice and every -1 is counted twice. Thus the product is $+1$.

But there are an odd number of vertices. Thus it cannot be the case that the product coming from each vertex is -1 (since the product of an odd number of -1's is equal to -1). Therefore at least one vertex has product equaling $+1$. □

PROBLEM 1.6.5 *A herd of cattle invades a barn dance. Suddenly the barn is running amok with both cattle and people. A quick count reveals 120 heads and 300 feet. How many cattle are there, and how many people?*

Solution: We write p for the number of people and c for the number of cattle. Then the total number of cattle plus people is $c + p$ while the total number of feet is $4c + 2p$ (since a head of cattle has four feet while a person has two). Thus

$$\begin{aligned} c + p &= 120 \\ 4c + 2p &= 200. \end{aligned}$$

We solve this system to find that $c = 30$ and $p = 90$. $\square$

CHALLENGE PROBLEM 1.6.6 *Suppose you are told in advance that 10 of the cattle present are lame, and only have three feet. But the count yields 120 heads and 300 feet. How many cattle and how many people are there?*

PROBLEM 1.6.7 (Halmos) *A party is held at the house of the Schlobodkins. There were four other couples present (besides Mr. and Mrs. Schlobodkin), and many, but not all, pairs of people shook hands. Nobody shook hands with anyone twice, and nobody shook hands with his/her spouse. Both the host and hostess shook some hands.*

At the end of the party, Mr. Schlobodkin polls each person present to see how many hands each person (other than himself) shook. Each person gives a different answer. Determine how many hands Mrs. Schlobodkin must have shaken.

Solution: We write S for the two Schlobodkins and we denote each of the other four couples by A, B, C, and D.

Nobody shook 9 hands, since nobody shook the hand of his/her spouse. Therefore the numbers 0 through 8 are used in describing the different numbers of handshakes performed by each of the nine people (other than Mr. Schlobodkin).

Someone shook 8 hands. Say that it is Mr. A. Then how many hands did Mrs. A shake? Everyone in couples B, C, D, S must have shaken Mr. A's hand, in order to account for 8 shakes. So each of the people in couples B, C, D, S shook hands at least once. But somebody shook hands zero times. It must be Mrs. A.

Now we eliminate Mr. and Mrs. A from our consideration. Someone shook exactly 7 hands. Say that it is Mrs. B. We know already that Mrs. B shook the hand of Mr. A. She did *not* shake the hand of Mrs. A, since nobody did. To obtain a total of 7 shakes, she must also have shaken the hands of all the people in couples C, D, S. But someone had to shake only one hand (the people in C, D, S have now each shaken at least two hands, since they each shook Mr. A's hand, as well as Mrs. B's hand). It must be Mr. B who shook only one hand.

Continuing in this fashion, we see that the person who shook 6 hands is betrothed to the person who shook 2. And the person who shook 5 hands is betrothed to the person who shook 3. That leaves only Mrs. Schlobodkin, who must have shaken 4 hands—four being the only remaining number (and note also that 4 is the only number that cannot be paired).

The answer to our problem is that Mrs. Schlobodkin shook four hands. □

CHALLENGE PROBLEM 1.6.8 *Go back to the beginning of the solution of the last problem. How can we be sure that it was not Mrs. Schlobodkin who shook 8 hands?*

PROBLEM 1.6.9 *A sheep can clear a certain field, eating the grass, in one day. A cow can clear the same field in half a day. How long does it take the sheep and the cow, working together, to clear the field?*

Solution: According to the information in the problem, a cow is like two sheep (when it comes to clearing the field, that is). So the cow and the sheep together is like three sheep. Thus they will clear the field in a third of a day. □

CHALLENGE PROBLEM 1.6.10 *A wildebeest can clear a certain field in two days. A llama can clear it in three days. And a goat can clear it in four days. How long does it take the three animals together to clear the field?*

PROBLEM 1.6.11 *What is the last digit of* 3^{4798}*?*

Solution: Clearly the last thing we want to do is to calculate this number. Even using a system like MATHEMATICA we are liable to encounter memory or storage problems. Instead let us think.

Notice that $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243$, etc. The only possible last digits are 3,9,7, and 1; then the pattern repeats. In this list, 1 is special because $1 \cdot 1 = 1$. And the digit 1 occurs when we raise 3 to the *fourth* power. This suggest that we write

$$3^{4796} = \left[3^4\right]^{1199} \cdot 3^2.$$

Here we simply divided the exponent 4798 by 4, obtaining a quotient of 1199 and a remainder of 2. Then we used elementary laws of exponents.

Now, by what we have already observed, the expression in brackets will terminate with a 1. If we raise it to the 1199 power, it will still terminate with a 1. On the other hand, 3^2 is 9. We conclude that 3^{4796} terminates with a 9. $\square$

Challenge Problem 1.6.12 *What is the last digit of 7^{65432}?*

Challenge Problem 1.6.13 (This is tricky.) *What are the last three digits of 3^{4798}?*

EXERCISES for Chapter 1

1. Show that any positive integral power of $(\sqrt{2}-1)$ can be written in the form $\sqrt{N}-\sqrt{N-1}$ for N a positive integer. [*Hint:* Use induction. Consider separately the cases of even powers of $(\sqrt{2}-1)$ and odd powers of $(\sqrt{2}-1)$.]
2. Calculate the sum of the first k odd integers.
3. Calculate the sum of the first k cubes of integers.
4. Demonstrate that, in any collection of 52 distinct positive integers, there are two distinct numbers whose sum or whose difference is divisible by 100.
5. Find all pairs of integers m, n such that $m \cdot n = m + n$.
6. How many zeros end the number (200!) ?
7. How many zeros end the number $2^{300} \cdot 5^{600} \cdot 4^{400}$?
8. A bunch of people are in a room. Some of them shake hands. Some do not. What can you say about the number of people who shake hands an even number of times?
9. How many digits are used to number the pages of a book having 100 pages—numbered from 1 to 100?
10. How many positive integers k are there with the property that $k!$ does *not* end in a zero?
11. A certain number k is a multiple of 9. Add the digits together. If the result has more than one digit, add those together. Continue adding digits together until you have a one digit answer. It will be a 9. Can you explain why this is so?
12. Refer to Exercise 11. Give the following instructions to a friend: "Pick an integer from 1 to 10. Multiply it by 9. Add the digits together. Subtract 5. Now you have a single digit. Think of the letter of the alphabet that corresponds to that digit—1 is A, 2 is B, and so forth. Think of a country whose name begins with that letter. Now take the second letter of that country name. Think of an animal whose name begins with that letter." Let the friend think about this for a moment. Then say "But there are no elephants in Denmark!"

    What is the joke? Why does it work?
13. Fix a positive prime integer p. Suppose that n is any positive integer. Find a formula for the number of factors of p that occur in $n!$
.

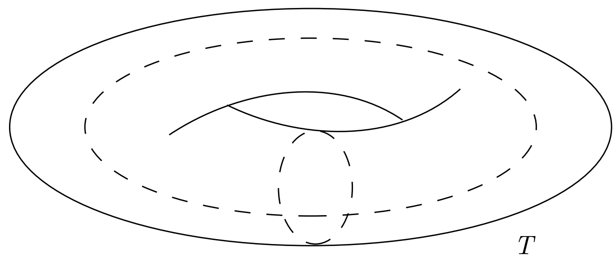


Figure 22

14. [Halmos] A watermelon weighs 500 pounds. It turns out that 99% of the weight of the watermelon is due to water in the watermelon. After the watermelon has sat in a drying room for a while, it turns out that it is only 98% water by weight. How much does it weigh now?

15. Fifteen teams play in a tournament. Each team plays every other team exactly once. A team receives 3 points for a win, 2 points for a draw, and 1 point for a loss. Every team ends up with a different total score. The team with the lowest total scored 21 points. Explain why the best team had at least one loss.

16. Suppose that T is a torus, as illustrated in Figure 22.

Determine the correct number γ so that the formula $V - E + F = \gamma$ will hold for any admissible graph on the surface of the torus T (recall that, for a sphere, the number γ was 2, but for the torus it will be a different number). The number γ is called the *Euler characteristic* of the torus. We learned in the text that the Euler characteristic of the sphere is 2.

Devise a proof that this number γ will work for any admissible graph in T.

17. Suppose that S is a sphere with k handles (see Figure 23).
[In a certain sense, a torus is a sphere with one handle. Can you explain this assertion?] What is the correct number γ so that the formula $V - E + F = \gamma$ will be true for any admissible graph on the surface S? Once you find γ, can you explain why this formula is valid?

18. Each point of the cartesian plane is colored either red or blue or yellow. Explain why we can conclude that some unit segment in the plane has both ends the same color.

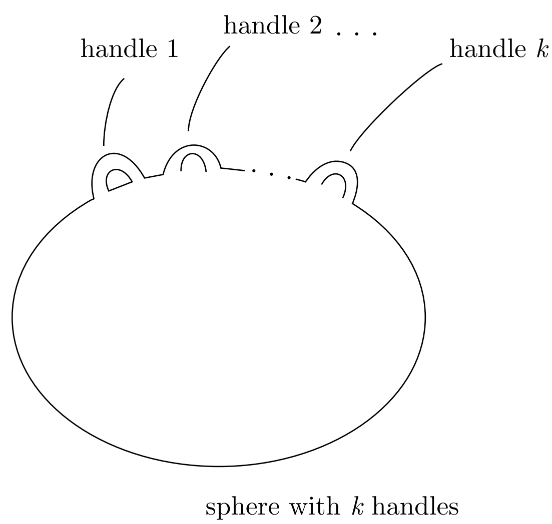


Figure 23

19. Show that if each point in the plane is colored one of seven colors—red, blue, yellow, green, purple, orange, magenta—then it is possible that no segment of unit length has both ends the same color.

20. A certain primitive village has the following social rules: Whenever a husband is unfaithful to his wife, every other wife knows immediately, but not the wife who has been cheated on. The wives never talk to anyone about these matters, and neither do the husbands. As soon as a wife can determine irrefutably that her husband has cheated, she tatoos the letter "A" on his forehead before sundown of that same day.

On a given day the mayor announces that there is at least one unfaithful husband in the village (he does *not*, however, say how many unfaithful husbands there are). If in fact there are 37 unfaithful husbands in the village, what will transpire? [*Hint:* First think about the situation where the mayor makes his announcement but there is in fact just one unfaithful husband in the village. Then try the case where there are just two unfaithful husbands in the village. Now use induction.] This problem comes from [HAL].

21. How does the last problem change if the mayor actually announces the number of unfaithful husbands?

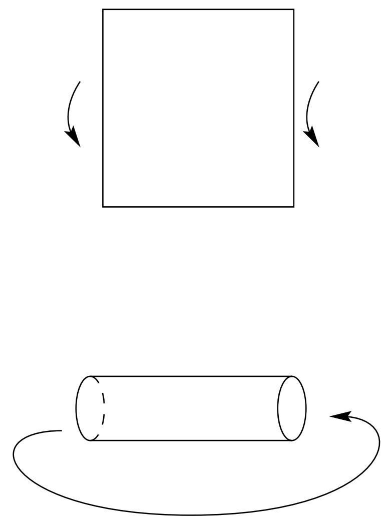


Figure 24

22. Say that you have a square piece of paper. Imagine pasting the upper edge to the lower edge (Figure 24) and the left edge to the right edge. Do this in such a way that the orientations of these edges are preserved. What geometric figure results?

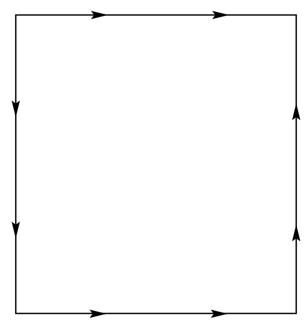
Now imagine that, when the left and right edges are glued we first give the left edge a twist (Figure 25 suggests what is meant by this). The result is an object called the Klein bottle; the Klein bottle cannot actually be realized as a surface in space, but we may think about it mathematically from the description just given.

What is the correct number γ so that the formula $V - E + F = \gamma$ will hold for any admissible graph on the surface of the Klein bottle? Can you prove that your choice of γ is correct?

23. Calculate the sum in closed form:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}.$$

24. Assume that S is a set with k elements. We know that S has 2^k subsets. What does this statement have to do with the binomial coefficients? [*Hint:* Look at $(1+x)^k$.]



match the orientation of the arrows

Figure 25

25. An urn contains a white balls and b black balls; we know that $a + b \geq 3$. Players A and B play a game with the urn and the balls. Consider two strategies:

1. Player A draws a ball at random. If it is white he wins, otherwise he loses.

2. Player A draws a ball and throws it away without looking at it. Player B then draws a black ball. Next, A draws another ball. If this second ball for A is white then he wins. Otherwise he loses.

Show that, with the first strategy, player A wins with probability $a/(a+b)$. But with the second strategy he wins with probability $a/(a+b) + a/[(a+b)(a+b-2)]$. Clearly the second strategy is preferable.

What does this problem have to do with the Monte Hall problem?

26. You have a wooden cube that is $3'' \times 3'' \times 3''$. By drawing a 3×3 grid on each face of the cube, you can indicate how the cube subdivides into 27 sub-cubes of equal size. Refer to Figure 26.

Is it possible for a termite to eat his way through each of the outside cubes just once, and then finish his journey in the middle cube?

27. Examine the example from the text about tiling the bathroom floor. What happens if the two omitted squares are in adjacent corners of the bathroom instead of opposite corners? What happens if the two omitted squares are adjacent to each other (in this last case, does it

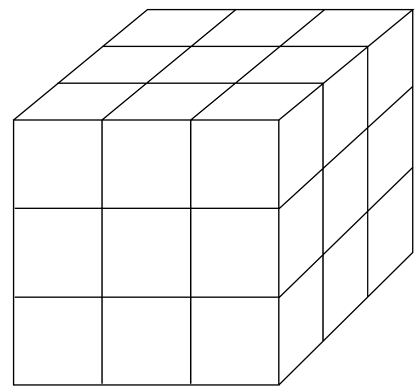


Figure 26

matter *where* the two omitted squares are—in a corner, in the middle of a side, or in the middle of the floor?). Do some experimenting!

28. Find at least two methods for calculating the sum $101+102+103+\cdots+200$. [*Note:* Just adding them up does not count as a "method."]

29. An auditorium has 500 seats. The decorator has three different colors of fabric—red, blue, and yellow, and will randomly upholster each seat with one of the three colors. So some will end up red, some blue, and some yellow, with no particular pattern. In how many different ways can this be done?

30. Five people play "seven card stud" poker. In seven card stud, you are dealt five cards face down and two face up. Joe is showing two aces for his face up cards. No other aces show on the table. What are the odds that he has another ace among his five face down cards?