Preface

The notion of a vertex algebra was introduced ten years ago by Richard Borcherds [**B1**]. This is a rigorous mathematical definition of the chiral part of a 2-dimensional quantum field theory studied intensively by physicists since the landmark paper of Belavin, Polyakov and Zamolodchikov [**BPZ**]. However, implicitly this notion was known to physicists much earlier. Some of the most important precursors are Wightman axioms [**W**] and Wilson's notion of the operator product expansion [**Wi**]. In fact, as I show in Sections 1.1 and 1.2, the axioms of a vertex algebra can be deduced from Wightman axioms. The exposition of these two sections is somewhat terse. The rest of the book, written at a more relaxed pace, is motivated by these sections but can be read independently of them.

Axioms of a vertex algebra used in this book are essentially those of [**FKRW**] and were inspired by Goddard's lectures [**G**]. These axioms are much simpler than the original Borcherds' axioms and are very easy to check. One of the objectives of this book is to show that these systems of axioms are equivalent (see Section 4.8).

Another objective of the book is to lay rigorous grounds for the notion of the operator product expansion (OPE) and demonstrate how to use it to perform calculations that are otherwise very painful. The classical Wick theorem allows one to compute OPE in free field theories. A "non-commutative" generalization of Wick's formula allows one to compute OPE of arbitrary fields (see Section 3.3).

The main objective of the book is to show how to construct a variety of examples of vertex algebras, and how to perform calculations using the formalism of vertex algebras to get applications in many different directions (Chapter 5).

In Sections 2.7 and 5.10, I present some new material on a topic closely related to vertex algebras — the theory of conformal superalgebras.

These notes represent a part of the course given at MIT in 1994 and 1995. Unfortunately, I didn't have time to write down the chapters on representation theory of vertex algebras and some other applications. (Most quoted literature is related to these unwritten chapters, and I hope that the present book will facilitate the reading of these papers.) In fact, another important application of vertex algebra theory is that it picks out the most interesting representations of infinite-dimensional Lie (super)algebras and provides means for their detailed study.

There is nothing in this book on the application to the Monster simple group (there is a book [**FLM**] on this, after all), nothing on Borcherds' solution of the Conway-Norton problem [**B2**], and nothing on Borcherds' marvelous applications to generalized Kac-Moody algebras and automorphic forms [**B3**].

A technical remark is in order. What I call a "vertex algebra" should probably be called a "$N = 0$ vertex superalgebra" (see Section 5.9 for the definition of a $N = n$ vertex superalgebra), but I decided on this simpler name. (Also, I call a "conformal vertex algebra" what is called in [**FLM**], with some additional restrictions, a "vertex operator algebra.") The reader who detests "supermathematics" may assume that the $\mathbb{Z}_2$-gradation is trivial, that "Lie superalgebra" means "Lie algebra", etc. But then he skips fermions and beautiful applications to identities and to soliton equations, the rich variety of superconformal theories, etc.

The bibliography is by no means complete. It is already quite a task to compile a complete list that would include all the relevant work done by physicists. However, it includes all items that influenced my thinking on the subject. One may also find there further references.

In addition to the sources mentioned above, the most important for the present book were the work of Todorov on the Wightman axioms point of view on CFT, the paper by Li from which I learned the unified formula for n-th products and Dong's lemma, the paper by Getzler from which I learned the "non-commutative" Wick formula, and the work of Lian and Zuckerman on "quantum operator algebras."

A preliminary version of these notes has been published in the proceedings of the summer school in Bulgaria in 1995 where I lectured on this subject. I am grateful to Ivan Todorov and Kiyokazu Nagatomo for reading the manuscript and correcting errors, and to Maria Golenishcheva-Kutuzova, Mike Hopkins, Andrey Radul, and Ivan Todorov for numerous illuminating discussions.

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Preface to the second edition

This improved and enlarged edition is based on a course given at M.I.T. in the spring of 1997 and in Rome University in May and June of 1997. Below is a list of the most important improvements and additions.

Chapter 2. The notion of formal Fourier transform is introduced in Section 2.2. This reduces significantly the calculations and leads to the important notion of λ-bracket in the theory of conformal algebras. Four new Sections 2.8–2.11 on the theory of conformal algebras are added and Section 2.7 is reworked. Thus, Sections 2.7–2.11 present the foundations of this rapidly developing area of algebraic conformal field theory.

Conformal algebra is an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory. It is, to some extent, related to a vertex algebra in the same way Lie algebra is related to its universal enveloping algebra. A structure theory of vertex algebras, similar, for example, to the structure theory of finite-dimensional Lie algebras, seems to be far away. Conformal algebras turned out to be a much more tractable object; as shown in Sections 2.7–2.11, for finite conformal algebras such a theory can be developed.

In Section 2.7 an explicit correspondence between an important class of infinite-dimensional Lie algebras, called formal distribution Lie algebras, and certain new structures, called conformal algebras, is established and a classification of finite conformal algebras is outlined. In Sections 2.8 and 2.9 representation theory of conformal algebras is developed, and in Section 2.11 the corresponding cohomology theory is explained. In Section 2.10 elements of conformal linear algebra are presented.

Chapter 3. The "non-commutative" Wick formula is expressed via λ-bracket (formula (3.3.12)), which greatly facilitates the use of this formula.

Chapter 4. The exposition of Sections 4.4-4.6 is simplified by making a more systematic use of the Uniqueness Theorem (a similar simplification was independently found in [**MN**]). Section 4.11 on field algebras is corrected.

Chapter 5. A new Section 5.8 on super boson-fermion correspondence is added. Comparing characters leads to a beautiful identity, whose specializations give classical results on sums of squares which go back to Gauss and Jacobi. In Section 5.10 a complete list of finite simple conformal superalgebras is given.

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