Contents

Preface

The first two chapters of these notes reproduce, with additional detail here and there, the content of the oral lectures delivered in March 1993. For example, I have included the original proof of the existence theorem (Chapter II, §§6 and 7) which previously was available only in preprint form.

The postscript (Chapter III) perhaps requires a word of explanation. During the period that has elapsed since these lectures were given, the subject has enjoyed considerable advance and clarification, mainly thanks to the work of Ivan Cherednik, who perceived that the affine Hecke algebras provided a key to unlock its mysteries. Chapter III provides a brief survey of these recent developments.

Introduction

The aim of this Introduction is to provide some background and perspective to the subject of these lectures for the benefit of readers who are not already well-versed in these matters and to explain its connections with other branches of mathematics, such as combinatorics and representation theory. References in square brackets are to the bibliography at the end of this Introduction, *not* to that at the end of the book.

Symmetric functions

The theory of symmetric functions is one of the most classical parts of algebra, going back to the 16th and 17th centuries and the attempts of mathematicians of that epoch to solve polynomial equations of degree higher than the second. The coefficients of a polynomial equation in one unknown are, up to sign, just the elementary symmetric functions of the roots of that equation, and hence any symmetric polynomial in the roots can be expressed uniquely as a polynomial in the coefficients. For example, the equations

$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r},$$

which serve to express the power sums p_r in terms of the elementary symmetric functions e_r, were first found by Isaac Newton in the 17th century. The first part of Chapter I (§§1–6) provides a rapid survey, from a modern viewpoint, of this classical material.

Schur functions and their generalizations

Of the various families of symmetric functions, the most significant are undoubtedly the Schur functions, because of their intimate relationship with the irreducible characters of both the symmetric groups and the general linear groups, and for their combinatorial applications.

Let us consider the representation-theoretic aspect first.

(i) The irreducible (complex) characters χ^λ of the symmetric group S_n are indexed by the partitions λ of n. So also are the conjugacy classes in S_n, because the conjugacy class of a permutation is determined by its cycle-type, which is a partition of n. The value χ^λ_μ of the character χ^λ at an element of cycle-type μ is then given by the scalar product

$$\chi^\lambda_\mu = \langle s_\lambda, p_\mu \rangle.$$

Equivalently, χ^λ_μ is the coefficient of s_λ in the expression of p_μ as a sum of Schur functions.

(ii) A matrix representation ρ of the general linear group $\mathrm{GL}_n(\mathbb{C})$ is said to be *polynomial* if the matrix coefficients $\rho_{ij}(X)$, $X \in \mathrm{GL}_n(\mathbb{C})$, are polynomial functions of the entries of the matrix X. The irreducible polynomial representations ρ_λ of $\mathrm{GL}_n(\mathbb{C})$ are indexed by the partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ of length $\leq n$, and the character of ρ_λ is

$$\operatorname{trace} \rho_\lambda(X) = s_\lambda(\xi_1, \dots, \xi_n)$$

where $\xi_1, \dots, \xi_n$ are the eigenvalues of the matrix $X \in \mathrm{GL}_n(\mathbb{C})$.

The Schur functions also arise naturally in various combinatorial contexts: not only those associated with the combinatorics of the symmetric groups, but also, for example, in the enumeration of plane partitions. A *plane partition* may be thought of as an infinite matrix $\pi = (\pi_{ij})_{i,j \geq 1}$ in which each entry π_{ij} is a nonnegative integer, such that $\pi_{ij} \geq \pi_{i+1,j}$ and $\pi_{ij} \geq \pi_{i,j+1}$ for all (i,j), and such that $|\pi| = \sum_{i,j} \pi_{ij}$ is finite (so that all but a finite number of the π_{ij} are zero). Thus a plane partition is not so very different from a tableau, as defined in Chapter I, §7; and the expression of the Schur function s_λ as a sum of monomials indexed by tableaux (loc. cit.) can therefore be exploited, as was first realized by R. Stanley [**S**], to derive generating functions for various classes of plane partitions, such as (for example) MacMahon's famous generating function for all plane partitions:

$$\sum_\pi t^{|\pi|} = \prod_{n \geq 1} (1 - t^n)^{-n}.$$

The Schur functions lend themselves readily to generalization in various directions, and some (but by no means all) of these generalizations are briefly surveyed in §§8–12 of Chapter I. Of these, both the zonal polynomials (§8) and the Hall-Littlewood functions (§10) have close connections with representation theory: they may be interpreted as the values of zonal spherical functions on the general linear group $\mathrm{GL}_n(F)$, where F is a local field, relative to a maximal compact subgroup K. In the case of the zonal polynomials, F is the real field and K is the orthogonal group; and in the case of the Hall-Littlewood functions, F is a p-adic field and $K = \mathrm{GL}_n(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers in F (and the parameter t is equal to $1/q$, where q is the cardinality of the residue field of F). Moreover, just as the Schur functions determine the character tables of the symmetric groups, as we have seen above, so the Hall-Littlewood functions play a crucial role in the determination of the irreducible complex characters of the finite general linear groups, as was first realized by J. A. Green [**G1**].

Finally, both the zonal polynomials and the Hall-Littlewood functions are subsumed as limiting cases of the two-parameter symmetric functions that are the subject of §§11 and 12. As remarked at the end of §12, the structure of some of the formulas suggests strongly that these symmetric functions should be regarded as attached to root systems of type A and that they should have counterparts attached to other root systems.

This more general theory is the subject of Chapter II. Apart from symmetric functions, there are two other forerunners of this theory, namely the theory of hypergeometric functions and Jacobi polynomials attached to root systems developed by G. Heckman and E. Opdam [**HO**], and the succession of constant term conjectures (now, fortunately, all theorems) starting with those of F. J. Dyson [**D**] and G. E. Andrews [**A**]. We consider these inputs in turn.

Jacobi polynomials attached to root systems

The Jacobi polynomials $P_\lambda(k)$ of Heckman and Opdam (loc. cit.) attached to a root system R are a limiting case of our $P_\lambda(q,t)$, where the parameters q and t (thought of as real or complex numbers) both tend to 1 in such a way that $(1-t)/(1-q)$ tends to a definite limit k. They were defined in loc. cit. as the simultaneous eigenfunctions of a second-order differential operator that extrapolates the radial part of the Laplace operator on a symmetric space G/K, where G is a connected noncompact semisimple Lie group and K is a maximal compact subgroup of G; the root system R is the restricted root system of G/K, and the parameter k is half the root multiplicity (assumed constant, for simplicity of description). When the root system is of type A, the Jacobi polynomials coincide with the Jack symmetric functions of Chapter I, §9.

Constant term identities

The constant term identities now to be described constitute another antecedent of the orthogonal polynomials of Chapter II, of a more combinatorial flavour. In 1962 F. J. Dyson [**D**] was led by considerations of statistical mechanics to conjecture that the constant term in the expansion of the product

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-x_ix_j^{-1})^k,$$

where $x_1,\dots,x_n$ are independent variables and k is a positive integer, should be

$$(nk)!/(k!)^n. \tag{1}$$

This conjecture was soon proved true, by J. Gunson [**G2**] and K. Wilson [**W**], who showed more generally that the constant term in the expansion of

$$\prod_{i\neq j}(1-x_ix_j^{-1})^{k_i}, \tag{2}$$

where $k_1,\dots,k_n$ are nonnegative integers, is the multinomial coefficient

$$\frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}. \tag{3}$$

Next, in 1975 G. E. Andrews [**A**] conjectured a q-analogue of this result. Write

$$(x;q)_n=\prod_{i=1}^{n}(1-q^{i-1}x),$$

where q,x are independent variables and n is an integer ≥ 0. Then Andrews' conjecture was that the polynomial ($\in\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1},q]$)

$$\prod_{i\neq j}(\varepsilon_{ij}x_ix_j^{-1};q)_{k_i}, \tag{4}$$

where $\varepsilon_{ij}=1$ or q according as $i<j$ or $i>j$, should have constant term (i.e., independent of $x_1,\dots,x_n$) equal to

$$\frac{(q;q)_{k_1+\cdots+k_n}}{(q;q)_{k_1}\cdots(q;q)_{k_n}} \tag{5}$$

when $q=1$, this reduces to the previous result. However, (5) proved a harder nut to crack, and was finally proved correct by Zeilberger and Bressoud [**ZB**] in 1985.

These results and conjectures in turn prompted analogous conjectures [**M**] for an arbitrary root system R. To state these in their simplest form, assume that R is reduced and irreducible, and for each $\alpha \in R$ let e^α denote the corresponding formal exponential, as in Chapter II, §3. Then the constant term in the expansion of the product

$$(6) \qquad \prod_{\alpha \in R} (1 - e^\alpha)^k$$

(where as before k is a positive integer) should be

$$(7) \qquad \prod_{i=1}^{r} \binom{kd_i}{k},$$

where $d_1, \dots, d_r$ are the degrees of the fundamental polynomial invariants of the Weyl group of R; and, more generally, the constant term (i.e., involving q but no exponentials) in

$$(8) \qquad \prod_{\alpha \in R^+} (e^\alpha; q)_k (qe^{-\alpha}; q)_k$$

should be

$$(9) \qquad \prod_{i=1}^{r} \begin{bmatrix} kd_i \\ k \end{bmatrix},$$

where $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ is the q-binomial coefficient or Gaussian polynomial, defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = (q;q)_n / (q;q)_r (q;q)_{n-r}.$$

Clearly (9) implies (7) by setting $q = 1$. Moreover, when the root system R is of type A_{n-1}, the degrees d_i are $2, 3, \dots, n$, and so (9) reduces to the special case of (5) in which $k_1 = \cdots = k_n = k$. Whether the general form of (5), with n distinct parameters k_i, has counterparts for other root systems is still an open question.

The combinatorialists made some progress towards establishing (9) on a case-by-case basis—see [**K**] for the classical root systems, [**GG**] for F_4 and [**H**] for G_2—but were unable to handle the cases of the exceptional root systems E_6, E_7, and E_8. The first uniform proof, for all root systems R, of the conjecture (7) (the case $q = 1$ of (9)) was found by E. Opdam [**O**], by exploiting the shift operators he and Heckman had constructed as part of their theory of hypergeometric functions and Jacobi polynomials alluded to earlier in this introduction; and finally (9) was proved in full generality by Cherednik [**C**] in 1995.

In the context of the theory of orthogonal polynomials attached to root systems developed in Chapter II, the conjecture (9) essentially appears as the simplest case ($\lambda = 0$) of the norm formula (8.3′). (The function Δ (with $t = q^k$) defined in Chapter II, §4, is not quite the same as (8) above, but it is not difficult to switch from one to the other.)

Finally, it should be said that the account in Chapter II of these notes reflects the state of knowledge at the time (March 1993) the lectures were delivered. At that time, for example, the norm formula referred to above was still conjectural. In Chapter III I have attempted to remedy this defect to some extent, by surveying some of the more recent developments involving the affine Hecke algebra, such as the nonsymmetric orthogonal polynomials E_λ (Chapter III, §6), a proof of the norm formula (and hence in particular of (9) above) in §7, and Cherednik's Fourier transform (§9). However, as the subject is still evolving rapidly, this survey is necessarily incomplete.

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