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Riemannian Geometry During the Second Half of the Twentieth Century

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-1. Introduction

From the 50's to the 70's the task of describing the evolution, the techniques and the results of Riemannian geometry would have been easy enough. But since the 70's Riemannian geometry has experienced such a dramatic increase that this task has now become almost impossible. This is not only for reasons of volume, but also because of the difficulties of forming an ordering. In fact today's results, techniques and examples are so strongly interrelated that we have been forced to take several steps.

First: not to be exhaustive. Secondly we have adopted quite an artificial classification. As a consequence this text is not really a complete survey. On the contrary, we will refer to existing partial surveys whenever we know of their existence. Finally, we will mention briefly at the end some topics which are important but are not treated in this text explicitly or even in general, due to the lack of space. We still hope that we will be clear and fair enough. And also that our text is, at the present time, reasonably up-to-date.

There is also the problem of defining "Riemannian geometry". For obvious reasons of dimension and competence we restrict it quite strongly to *what's really happening on the Riemannian manifold itself, in particular as a metric space*. In particular we are going to ignore completely most of the topics of *differential geometry*, even *Riemannian bundles.* This is a field which has to this day undergone tremendous developments, including among others: differential topology, fiber bundles and connections, singularities and transversality, contact manifolds, symplectic geometry, minimal surfaces and generalizations of them, Yang-Mills (gauge) theory, twistors, foliations, submanifolds of $\mathbf{R}^d$, conformal geometry. For example, despite their great importance among various Riemannian bundles, Yang-Mills Yang-Mills fields are not considered, at least here, as Riemannian geometry but of course they are an important part of differential geometry. We will of course mention them, though only very briefly in TOP. 6. The tangent bundle, in particular the unit tangent bundle, will be used, but typically in connection with the geodesic flow. Some people may find our "definition" of Riemannian geometry too narrow. They might be right, as it reflects the (biased) "elementary geometry" temper of the present author as well as his liking for results which are simple to state. Because of their basic use in Riemannian geometry and the way they are constructed, we will explain exterior differential forms and spinors in a little more detail in TOP. 6.A and B. One conclusion is the desire that some form of "Handbook of differential geometry" will appear quite soon; for the moment one can consult the three volumes of (Greene & Yau, 1993).

We also took one more gamble: we aim to give an historical overview, but, at the same time, describe the state of affairs as it is today. This will not take too much space since we will use an "author-date" reference system, which will hence give credits automatically. Concerning references we will be far from exhaustive, since we are not presenting a complete survey and also because our bibliography would have become unreasonably extensive. In most cases what we will do is to give a reference which is good with respect to date and will, at the same time, enable the reader to go backwards using one or more bibliographies by induction. This will be especially helpful for the topics which are only mentioned briefly.

$$\begin{aligned} &\textit{manifold (unless otherwise stated)} = \\ &\textit{Riemannian manifold} = \textit{a (the) ``Riemannian metric''} \\ &\qquad\qquad\qquad\quad = \textit{metric} \end{aligned}$$

most manifolds will be COMPACT
and never with a boundary (closed)
and if not compact then always complete

Acknowledgments are very important for the present text. First, to the editors of a book to appear on mathematics during the second half of the century, who asked me to write such a text for the differential geometry part. I soon started writing and very quickly found myself embarked upon a project whose length would exceed by far (as the reader can now see) the twenty page allowance I was offered. Therefore I decided to submit it to the Jahresbericht of the DMV.

I was able to write such a text with enthusiasm thanks to the Rome University La Sapienza, the Indian Institute of Technology at Powai-Bombay, the University of Pennsylvania, and the Zürich Polytechnicum, each of which invited me to give lectures, Roma in 1992, Bombay in 1993, Penn in the fall of 94, and Zürich in the winter semester 95–96. Most important is the fact that these four departments permitted me to give lectures entitled "Topics in Riemannian Geometry" in which I covered a lot of material but with almost no proofs, simply sketching ideas and ingredients. I would like to thank those four mathematics departments very much for having accepted my lectures which did not fit into any classical framework.

Last but not least, a preliminary version was sent out to fifty colleagues. I got many answers, all of them important. I will now give a list of names and for obvious reasons I will do this without trying to state the importance of the individuals' help. Those who helped me a lot, and whom I even bothered many, many times, will recognize themselves. All of them gave help which was very valuable: Stephanie Alexander, Michael Anderson, Ivan Babenko, Victor Bangert, Pierre Bérard, Lionel Bérard Bergery, Gérard Besson, Armand Borel, Jean-Benoît Bost, Jean-Pierre Bourguignon, Robert Bryant, Dan Burghelea, Peter Buser, Jeff Cheeger, Shiing-shen Chern, Tobias Colding, Alain Connes, Yves Colin de Verdière, Thibault Damour, Jost Eschenburg, Kenji Fukaya, Jacques Gasqui de Saint-Joachim, Paul Gauduchon, Robert Greene, Mikhael Gromov, Nancy Hingston, Dominique Hulin, Mikhail Katz, Ruth Kellerhals, Bruce Kleiner, Horst Knörrer, Jean-Louis Koszul, Jacques Lafontaine, Rémi Langevin, Blaine Lawson, André Lichnerowicz, Paul Malliavin, Wolfgang Meyer, René Michel, Robert Osserman, Pierre Pansu, Peter Petersen, Hans-Bert Rademacher, Takashi Sakai, Katsuhiro Shiohama, Shanta Shrinivasan, Alain-Sol Sznitman, Iskander Taimanov, Domingo Toledo, Lieven Vanhecke, Takao Yamaguchi, Wolfgang Ziller.

I do hope that, whatever its imperfections are, the present text will be of help to Riemannian geometers of every age and might even be pleasant to read in part or in totality for nonexperts.

Finally I heartfully thank the DMV, the editors of the Jahresbericht, in particular Ernst Heintze, for having made very special efforts to publish the present text which does not fit into any classical category. Special thanks are due to Karin Seeger for the difficult and lengthy job of transforming my inaccurate English into the present form, as well as for converting my old fashioned Word 5 into LaTeX.

The present text is essentially the same as (Berger, 1998). We just added new references which appeared since 1998. I am heartfully thanking the AMS editorial board for offering to publish the initial text in book form.

The interested reader who demands more details, ideas of proofs, and a lot of pictures can look for the forthcoming book (Berger, 2000).