University
LECTURE
Series

Volume 20

An Introduction to Superprocesses

Alison M. Etheridge

American Mathematical Society
Providence, Rhode Island

Contents

Preface

Much current research in probability theory is concerned with stochastic processes taking values in infinite dimensional state spaces, making the transition from the highly successful theory of stochastic (ordinary) differential equations to that of stochastic partial differential equations. These generalisations are essential for the modelling of systems (evolving in time) for which the spatial structure plays a significant rôle. One of the difficulties is that many spatial stochastic processes are not function valued and so much of the theory that has been developed simply does not apply. Superprocesses represent a rich class of models that do not, in general, fall into the framework of stochastic partial differential equations, but for which, nonetheless, a considerable body of techniques have been developed.

We use the term superprocess for two classes of measure-valued stochastic process, both of which are most easily understood as models of evolving populations. A familiar model for the size of an evolving population is the Galton-Watson branching process. In 1951, Feller observed that for large populations one can employ a model obtained from the Galton Watson process by rescaling and passing to a limit. The *Feller diffusion approximation* is now a key tool in mathematical population genetics. Superprocesses arise as the extension of this idea to models that record not only the size of the population, but also the location of individuals within it. There are two fundamental examples: the Dawson-Watanabe superprocess and the Fleming-Viot superprocess. In the first, the size of the population evolves as the Feller diffusion, while in the second, although individuals die and reproduce, the total population size is constrained to be constant.

One of the strengths of the superprocess models is their generality. For example, the location of an individual could be a spatial position in $\mathbb{R}^d$, say, or her genetic type. In the genetic setting, the 'spatial motion' of individuals is a model of mutation between types. However, with the exception of Chapter 5 and part of Chapter 7, we shall always assume that individuals in our population move around according to Brownian motion in $\mathbb{R}^d$. In addition, we deal almost exclusively with the simplest of the superprocess models. This allows us to introduce some of the important ideas without obscuring them with notation or technical assumptions.

A number of different ways of thinking about superprocesses have emerged over the last ten years, and the principal aim of this book is to describe them in an accessible way. Often we have sacrificed rigour for intuition. The philosophy is that once one understands why a result should be true, the literature will seem less daunting. We will also settle for a proof of a weaker result if that proof has wider applicability.

There are many omissions. In particular, we have shied away from techniques from nonstandard analysis, even though they are especially well-suited to this setting. Nowhere do we discuss large deviations for the superprocess. Nor have we discussed the use of Dirichlet forms. The list is endless, but the book is not.

We have also avoided a discussion of the history of the subject. The choice of the (now widely accepted) name Dawson-Watanabe superprocess reflects the significance of Watanabe's paper of 1968, and of a series of papers by Dawson, beginning in the late 1970's. The name superprocess was coined by Dynkin in the 1980's. Before that, the Dawson-Watanabe superprocess was known as 'critical measure-valued branching Brownian motion', 'the critical measure diffusion' and other variants on that theme. Unfortunately, the term 'superprocess' leads to some confusion when one begins to do potential theory. (Harmonic functions for the superprocess are not the same as superharmonic functions.) However, mathematics is riddled with such inconsistencies, and the new name is certainly an improvement on the old. There is still not complete agreement in the literature. In particular, the special process that we have chosen to call the Dawson-Watanabe superprocess is often called *superBrownian motion*.

There are excellent sets of notes on superprocesses by Dawson, Le Gall and Perkins ([**Daw93**], [**LeG99**], [**Per00**]), each with its own emphasis. I am particularly grateful to Ed Perkins for letting me have a preview of his St Flour notes. A (slightly older) review of Fleming-Viot superprocesses is provided by [**EK93**]. The forthcoming book of Donnelly and Tavaré ([**DT00**]) explains the applications in genetics.

Although I had already decided to write a book on superprocesses, I only began to think seriously about this project when Steve Evans and Ruth Williams invited me to join a double act with Rob Adler, to give a series of introductory lectures at MSRI, in the Fall of 1997. I should like to thank Steve, Ruth and Rob for this opportunity and the audience for their feedback. An early version of the first few chapters was read by Patrick Fitzsimmons, Vlada Limic, Amber Puha and Ruth Williams in UC San Diego and Anja Sturm in Oxford and I am very grateful to all these people for their comments. Finally, in the Summer of 1999, I used the same early draft to give a series of lectures to the Probability Intern Program at the University of Madison, Wisconsin. Thanks to Tom Kurtz for that opportunity and to the audience for still more feedback. Needless to say, the many errors and obscurities that remain are my own.

An Advanced Fellowship from EPSRC has freed up enough time for the project to be completed and Magdalen College has provided unbeatable surroundings in which to work. I am grateful to Ed Dunne for his help, and above all for his patient reminders every time I missed another promised completion date. Lionel Mason has provided invaluable support and encouragement (and sustenance).