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University  
**LECTURE**  
Series

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Volume 21

Lectures on Tensor  
Categories and  
Modular Functors

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**American Mathematical Society**  
Providence, Rhode Island

# Contents

Introduction	1
Chapter 1. Braided Tensor Categories	9
1.1. Monoidal tensor categories	9
1.2. Braided tensor categories	15
1.3. Quantum groups	19
1.4. Drinfeld's category	21
Chapter 2. Ribbon Categories	29
2.1. Rigid monoidal categories	29
2.2. Ribbon categories	33
2.3. Graphical calculus for morphisms	35
2.4. Semisimple categories	44
Chapter 3. Modular Tensor Categories	47
3.1. Modular tensor categories	47
3.2. Example: Quantum double of a finite group	59
3.3. Quantum groups at roots of unity	62
Chapter 4. 3-Dimensional Topological Quantum Field Theory	71
4.1. Invariants of 3-manifolds	71
4.2. Topological quantum field theory	76
4.3. 1+1 dimensional TQFT	78
4.4. 3D TQFT from MTC	82
4.5. Examples	87
Chapter 5. Modular Functors	93
5.1. Modular functors	93
5.2. The Lego game	99
5.3. Ribbon categories via the Hom spaces	108
5.4. Modular functors in genus zero and tensor categories	116
5.5. Modular categories and modular functors for zero central charge	119
5.6. Towers of groupoids	124
5.7. Central extensions of modular functors	130
5.8. From 2D MF to 3D TQFT	135
Chapter 6. Moduli Spaces and Complex Modular Functors	141
6.1. Moduli spaces and complex Teichmüller tower	141
6.2. Compactification of the moduli space and gluing	147
6.3. Connections with regular singularities	152
6.4. Complex-analytic modular functors	158

6.5. Example: Drinfeld's category	161
6.6. Twisted $\mathcal{D}$ -modules	164
6.7. Complex modular functors with central charge	167
Chapter 7. Wess–Zumino–Witten Model	173
7.1. Preliminaries on affine Lie algebras	174
7.2. Reminders from algebraic geometry	175
7.3. Conformal blocks: definition	177
7.4. Flat connection in the bundle of conformal blocks	181
7.5. From local parameters to tangent vectors	188
7.6. Families of curves over a formal base	191
7.7. Coinvariants for singular curves	194
7.8. The gluing axiom for the bundle of coinvariants	195
7.9. WZW modular functor	202
7.10. Calculation of the central charge	203
Bibliography	213
Index	217
Index of Notation	219

## Introduction

The goal of this book is to give a comprehensive exposition of the relations among the following three topics: tensor categories (such as a category of representations of a quantum group), 3-dimensional topological quantum field theory (which, as will be explained below, includes invariants of links), and 2-dimensional modular functors (which arise in 2-dimensional conformal field theory).

The idea that these subjects are somehow related first appeared in the physics literature, in the study of quantum field theory. The pioneering works of Witten [W1, W2] and Moore and Seiberg [MS1] triggered a real avalanche of papers, both physical and mathematical, exploring various aspects of these relations. Among the more important milestones we should name the papers of Segal (who invented the notion of modular functor) [S], Reshetikhin–Turaev [RT1, RT2], Tsuchiya–Ueno–Yamada [TUY], Drinfeld [Dr3, Dr4], Beilinson–Feigin–Mazur [BFM], and many others.

By the late 1990s it had become a commonplace that these topics are closely related. However, when the second author decided to teach (and the first author to take) a class on this topic at MIT in the Spring of 1997, they realized that finding precise statements in the existing literature was not easy, and there were some gaps to be filled. Moreover, the only work giving a good exposition of all these notions was Turaev’s book [Tu], which however did not cover some important (from our point of view) topics, such as the complex-analytic approach to modular functors, based on connections on the moduli spaces. There were also several unpublished expository texts, such as a manuscript by Kevin Walker, a preprint by Graeme Segal, and another manuscript by Beilinson, Feigin, and Mazur [BFM]. Unfortunately, all of them were (and still are) unfinished, and it is not clear if they will ever be finished. Thus, it was natural that after the course was completed, we decided to turn it into a book which would provide a comprehensive exposition. Needless to say, this book is almost completely expository, and contains no new results — our only contribution was putting known results together, filling the gaps, and sometimes simplifying the proofs.

To give the reader an idea of what kind of relations we are talking about, we give a quick introduction. Let us first introduce the main objects of our study:

**Tensor categories:** These are abelian categories with associative tensor product, unit object, and some additional properties, such as rigidity (existence of duals). We will be interested only in semisimple categories, such as a category of complex representations of a compact group. However, we weaken the commutativity condition: namely, we require the existence of a functorial isomorphism  $\sigma_{VW}: V \otimes W \rightarrow W \otimes V$ , but — unlike the classical representation theory — we do not require  $\sigma^2 = \text{id}$ . The best

known example of such a category is a category of representations of a quantum group; however, there are also other examples.

We will also need a special class of tensor categories which are called *modular tensor categories* (MTC); these are semisimple tensor categories with a finite number of simple objects and certain non-degeneracy properties. The main example of such a category is provided by a suitable semisimple quotient of the category of representations of a quantum group at a root of unity.

**3-dimensional topological quantum field theory (3D TQFT):**

Despite its physical name, this is a completely mathematical object (to such an extent that some physicists question whether it has any physical meaning at all). A simple definition is that a *3D TQFT* is a rule that assigns to every 2-dimensional manifold  $N$  a finite-dimensional vector space  $\tau(N)$ , and to every cobordism — i.e., a 3-manifold  $M$  such that its boundary  $\partial M$  is written as  $\partial M = \overline{N}_1 \sqcup N_2$  — a linear operator  $\tau(N_1) \rightarrow \tau(N_2)$  (here  $\overline{N}$  is  $N$  with reversed orientation). In particular, this should give a linear operator  $\tau(M): \mathbb{C} \rightarrow \mathbb{C}$ , i.e., a complex number, for every closed 3-manifold  $M$ .

We will, however, need a somewhat more general definition. Namely, we will allow 2-manifolds to have marked points with some additional data assigned to them, and 3-manifolds to have framed tangles inside, which should end at the marked points. In particular, taking a 3-sphere with a link in it, we see that every such extended 3D TQFT defines invariants of links.

**2-dimensional modular functor (2D MF): topological definition:**

By definition, a *topological 2D modular functor* is the assignment of a finite-dimensional vector space to every 2-manifold with boundary and some additional data assigned to the boundary components, and assignment of an isomorphism between the corresponding vector spaces to every isotopy class of homeomorphisms between such manifolds. In addition, it is also required that these vector spaces behave nicely under *gluing*, i.e., the operation of identifying two boundary circles of a surface to produce a new surface.

**2-dimensional modular functor (2D MF): complex-analytic definition:**

A *complex-analytic 2D modular functor* is a collection of vector bundles with flat connections on the moduli spaces of complex curves with marked points, plus the gluing axiom which describes the behavior of these flat connections near the boundary of the moduli space (in the Deligne–Mumford compactification). Such structures naturally appear in conformal field theory: every rational conformal field theory gives rise to a complex-analytic modular functor. The most famous example of a rational conformal field theory — and thus, of a modular functor — is the Wess–Zumino–Witten model, based on representations of an affine Lie algebra.

The main result of this book can be formulated as follows: *the notions of a modular tensor category, 3D TQFT and 2D MF (in both versions) are essentially equivalent.*

Below we will provide a simple example that illustrates how one fact — the quantum Yang–Baxter equation — looks in each of these setups. Let us fix a semisimple abelian category  $\mathcal{C}$  and a collection of objects  $V_1, \dots, V_n \in \mathcal{C}$ .

**Tensor category setup.** Assume that we have a structure of a tensor category on  $\mathcal{C}$ . Denote by  $\sigma_i$  the commutativity isomorphisms

$$\sigma_{V_i V_{i+1}}: V_1 \otimes \dots \otimes V_i \otimes V_{i+1} \otimes \dots \otimes V_n \rightarrow V_1 \otimes \dots \otimes V_{i+1} \otimes V_i \otimes \dots \otimes V_n.$$

Then the axioms of a tensor category imply that

$$(QYBE) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

where both sides are isomorphisms  $\dots \otimes V_i \otimes V_{i+1} \otimes V_{i+2} \otimes \dots \rightarrow \dots \otimes V_{i+2} \otimes V_{i+1} \otimes V_i \otimes \dots$ . This identity is known as the *quantum Yang–Baxter equation*.

**3D TQFT setup.** Consider the 2-sphere  $S^2 = \mathbb{R}^2 \cup \{\infty\}$  with marked points  $p_1 = (1, 0), p_2 = (2, 0), \dots, p_n = (n, 0)$  and with objects  $V_1, \dots, V_n$  assigned to these points; this defines a vector space  $\tau(S^2; V_1, \dots, V_n)$ . Consider the 3-manifold  $M = S^2 \times [0, 1]$  with a tangle inside as shown in Figure 0.1 (which only shows two planes; to get the spheres, the reader needs to add an infinite point to each of them).

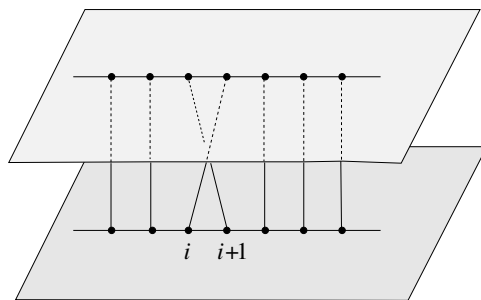


FIGURE 0.1. A 3-manifold with a braid inside.

This gives operators

$$\sigma_i^{\text{TQFT}}: \tau(S^2; V_1, \dots, V_n) \rightarrow \tau(S^2; V_1, \dots, V_{i+1}, V_i, \dots, V_n)$$

which also satisfy the quantum Yang–Baxter equation (QYBE). This follows from the fact that the two 3-manifolds  $S^2 \times [0, 1]$ , with the tangles shown in Figure 0.2 inside, are homeomorphic.

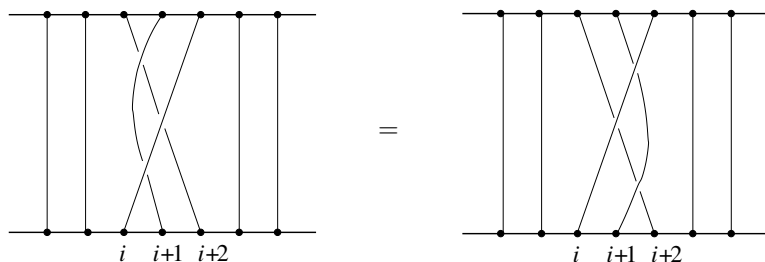


FIGURE 0.2. Two tangles inside 3-manifolds.

**2D MF (topological) setup.** Here, again, we take  $N = S^2 = \mathbb{R}^2 \cup \{\infty\}$  with small disks around the points  $p_1, \dots, p_n$  removed, and with objects  $V_1, \dots, V_n$  assigned to the boundary circles. The modular functor assigns to such a surface a vector space  $\tau(S^2; V_1, \dots, V_n)$ . Consider the homeomorphism  $b_i$  shown in Figure 0.3. This defines operators

$$(b_i)_* : \tau(S^2; V_1, \dots, V_n) \rightarrow \tau(S^2; V_1, \dots, V_{i+1}, V_i, \dots, V_n)$$

which also satisfy the quantum Yang–Baxter equation. Now this follows from the fact that the homeomorphisms  $b_i b_{i+1} b_i$  and  $b_{i+1} b_i b_{i+1}$  are isotopic.

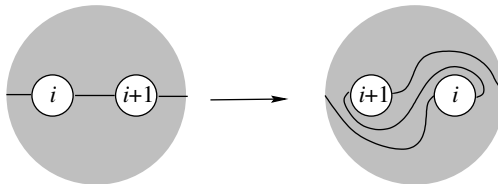


FIGURE 0.3. Braiding for topological modular functor.

**2D MF (complex-analytic) setup.** We consider the moduli space of spheres with  $n$  marked points. A 2D MF defines a local system on this moduli space; denote the fiber of the corresponding vector bundle over the surface  $\Sigma = \mathbb{P}^1$  with marked points  $p_1 = 1, \dots, p_n = n$  by  $\tau(\mathbb{P}^1; V_1, \dots, V_n)$ . Then the operator of holonomy along the path  $b_i$ , shown in Figure 0.4, gives a map

$$(b_i)_* : \tau(\mathbb{P}^1; V_1, \dots, V_n) \rightarrow \tau(\mathbb{P}^1; V_1, \dots, V_{i+1}, V_i, \dots, V_n),$$

and the quantum Yang–Baxter equation follows from the identity

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$$

in the fundamental group of the moduli space of punctured spheres.

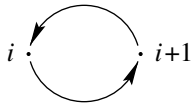
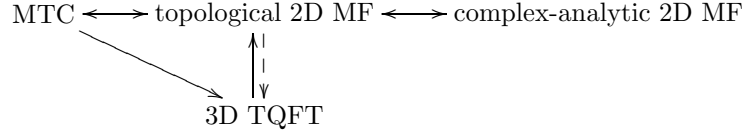


FIGURE 0.4. Braiding for complex-analytic modular functor.

This simple example should convince the reader that indeed there is some common algebraic structure playing a pivotal role in all of these subjects. In this particular example, it is not too difficult to show that this underlying algebraic structure is nothing but the braid group. However, when we try to include the notion of a dual representation on the tensor category side and of higher genus surfaces on the topological side, the situation gets more complicated. Still, the main result holds: under some (not too restrictive) assumptions, the notions of modular tensor category (MTC), 3D TQFT and 2D MF (topological and complex-analytic) are essentially equivalent. Schematically, this can be expressed by the

following diagram:



Here is a brief description of these equivalences, along with precise references:

**Tensor categories  $\rightarrow$  3D TQFT:** This equivalence is given by Reshetikhin–Turaev invariants of links and 3-manifolds [RT1, RT2] and their generalization to surfaces with boundaries [Tu]. In particular, in the example of a sphere with  $n$  marked points described above, this correspondence is given by  $\tau(S^2; V_1, \dots, V_n) = \text{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \dots \otimes V_n)$ ,  $\sigma_i = \sigma_i^{\text{TQFT}}$ . Precise statements can be found in Chapter 4, in particular, in Theorem 4.4.3. These invariants have a long history, which we cannot describe here; it suffices to say that the idea that path integrals in conformal field theory give rise to invariants of links was suggested by Witten [W1, W2]. Unfortunately, path integral technique is still far from being rigorous from a mathematical point of view, and so Reshetikhin and Turaev do not use it; instead, they use a presentation of a 3-manifold as a result of surgery along a framed link.

**3D TQFT  $\rightarrow$  2D MF (topological):** This arrow is almost tautological: all the axioms of 2D MF are contained among the axioms of 3D TQFT, except for the gluing axiom, which is also rather easy to prove. Details are given in Section 5.8.

**2D MF (topological)  $\rightarrow$  3D TQFT:** A complete construction of such a map is not yet known (at least to the authors); some partial results in this direction, due to Crane [C] and Kohno [Ko], are given in Section 5.8. They are based on the Heegaard splitting.

**2D MF (topological)  $\leftrightarrow$  tensor categories:** This is based on the results of Moore and Seiberg [MS1], who showed (with gaps, which were filled in [BK], [FG]) that the axioms of a modular tensor category, when rewritten in terms of the vector spaces of homomorphisms, almost coincide with the axioms of a 2D topological modular functor. (The word “almost” refers to a minor difficulty in dealing with duality, or rigidity, in a tensor category.) This is discussed in detail in Chapter 5; in particular, the main result is given in Theorem 5.5.1, or, in a more abstract language, in Theorem 5.6.19.

**2D MF (topological)  $\leftrightarrow$  2D MF (complex-analytic):** This is based on the Riemann–Hilbert correspondence, which, in particular, claims that the categories of local systems (= locally constant sheaves) and vector bundles with flat connections with regular singularities are equivalent. Applying this to the moduli space of Riemann surfaces with marked points, and using the fact that the fundamental group of this moduli space is exactly the mapping class group, we get the desired equivalence. We also have to check that this equivalence preserves the gluing isomorphisms. All this is done in Chapter 6; in particular, the main result is contained in Theorem 6.4.2.

The book is organized as follows.



In **Chapter 1**, we give basic definitions related to braided tensor categories, such as commutativity and associativity isomorphisms, and state various coherence theorems. We also give two basic examples of tensor categories: the category  $\mathcal{C}(\mathfrak{g})$  of representations of a quantum group  $U_q(\mathfrak{g})$  (for formal  $q$ , i.e., over the field of rational functions in  $q$ ) and Drinfeld’s category  $\mathcal{D}(\mathfrak{g}, \varkappa)$ ,  $\varkappa \notin \mathbb{Q}$ , which as an abelian category coincides with the category of finite-dimensional complex representations of a simple Lie algebra  $\mathfrak{g}$ , but has commutativity and associativity isomorphisms defined in terms of asymptotics of the Knizhnik–Zamolodchikov equations.

In **Chapter 2**, we continue the study of the theory of tensor categories. We define the notion of ribbon category (in other terminology, rigid balanced braided tensor category) as a category in which every object has a dual satisfying some natural properties, and in which there are functorial isomorphisms  $V^{**} \simeq V$  compatible with the tensor product. We develop “graphical calculus” allowing one to present morphisms in a ribbon category by ribbon (framed) tangles. In particular, this shows that every ribbon category gives rise to invariants of links (Reshetikhin–Turaev invariants). We also show that both examples of Chapter 1 — that is, the categories  $\mathcal{C}(\mathfrak{g})$  and  $\mathcal{D}(\mathfrak{g}, \varkappa)$  — are ribbon.

In **Chapter 3**, we introduce one more refinement of the notion of tensor category: that of a modular tensor category. By definition, this is a semisimple ribbon category with a finite number of simple objects satisfying a certain non-degeneracy condition. It turns out that these categories have a number of remarkable properties; in particular, we prove that in such a category one can define a projective action of the group  $\mathrm{SL}_2(\mathbb{Z})$  on an appropriate object, and that one can express the tensor product multiplicities (fusion coefficients) via the entries of the  $S$ -matrix (this is known as the Verlinde formula). We also give two examples of modular tensor categories. The first one, the category  $\mathcal{C}(\mathfrak{g}, \varkappa)$ ,  $\varkappa \in \mathbb{Z}_+$ , is a suitable semisimple subquotient of the category of representations of the quantum group  $U_q(\mathfrak{g})$  for  $q$  being a root of unity,  $q = e^{\frac{2\pi i}{m\varkappa}}$ . The second one is the category of representations of a quantum double of a finite group  $G$  or, equivalently, the category of  $G$ -equivariant vector bundles on  $G$ . (We do not explain here what is the proper definition of Drinfeld’s category  $\mathcal{D}(\mathfrak{g}, \varkappa)$  for  $\varkappa \in \mathbb{Z}_+$ , which would be a modular category — this will be done in Chapter 7.)

In **Chapter 4**, we move from algebra (tensor categories) to topology, namely, to invariants of 3-manifolds and topological quantum field theory (TQFT). We start by showing how one can use Reshetikhin–Turaev invariants of links to define, for every modular tensor category, invariants of closed 3-manifolds with a link inside. This construction is based on presenting a manifold as a result of surgery of  $S^3$  along a framed link, and then using Kirby’s theorem to check that the resulting invariant does not depend on the choice of such a presentation. Next, we give a general definition of a topological quantum field theory in any dimension and consider a “baby” example of a 2D TQFT. After this, we return again to the dimension 3 case and define “extended” 3D TQFT, in which 3-manifolds may contain framed tangles whose ends must be on the boundary; thus, the boundary becomes a surface with marked points and non-zero tangent vectors assigned to them. The main result of this chapter is that every MTC defines an extended 3D TQFT (up to a suitable “central extension”). This, in particular, explains the action of  $\mathrm{SL}_2(\mathbb{Z})$  which was introduced in Chapter 3: this action corresponds to the natural action of  $\mathrm{SL}_2(\mathbb{Z})$  on the torus with one marked point.

In **Chapter 5**, we introduce topological 2D modular functors and discuss their relation with the mapping class groups. We also introduce the proper formalism: that of towers of groupoids. The main part of this chapter is devoted to describing the tower of mapping class groups — and thus, the modular functor — by generators and relations, as suggested by Moore and Seiberg. Our exposition follows the results of [BK]. Once such a description is obtained, as an easy corollary we get that every modular tensor category defines a 2D topological modular functor (with central charge — see below), and conversely, every 2D MF defines a tensor category that is “weakly rigid”. Unfortunately, we were unable to prove — and we do not know of such a proof — that the tensor category defined by a 2D MF is always rigid. However, if it is actually rigid then we prove that it is an MTC, and we provide a criterion for rigidity which is satisfied in all examples known to us. We also describe accurately the central charge phenomenon. As was said before, an MTC defines only a projective representation of  $SL_2(\mathbb{Z})$ , or, equivalently, a representation of a central extension of  $SL_2(\mathbb{Z})$ , while a 2D MF should give a true representation of  $SL_2(\mathbb{Z})$  and all other mapping class groups. To account for projective representations, we introduce the notion of a modular functor with central charge, which can be thought of as a “central extension” of the modular functor, and show how the central charge can be calculated for a given MTC.

In **Chapter 6**, we introduce the complex-analytic version of modular functors. We start by giving all the necessary preliminaries, both about flat connections with regular singularities (mostly due to Deligne) and about the moduli space of punctured curves and its compactification (Deligne–Mumford). Unfortunately, this presents a technical problem: the moduli space is not a manifold but an algebraic stack; we try to avoid actually defining algebraic stacks, thus making our exposition accessible to people with limited algebraic geometry background. After this, we define the complex-analytic MF as a collection of local systems with regular singularities on the moduli spaces of punctured curves, formulate the gluing axiom, which now becomes the statement that these local systems “factorize” near the boundary of the moduli space (the accurate definition uses the specialization functor), and prove that the notions of topological and complex analytic MF are equivalent. We also return to Drinfeld’s category  $\mathcal{D}(\mathfrak{g}, \varkappa)$  and show that its definition in terms of Knizhnik–Zamolodchikov equations is nothing but an example of a complex-analytic modular functor in genus zero.

Finally, in **Chapter 7**, we consider the most famous example of a modular functor, namely the one coming from the Wess–Zumino–Witten model of conformal field theory. This modular functor is based on integrable representations of an affine Lie algebra  $\hat{\mathfrak{g}}$ ; the vector bundle with flat connection is defined as the dual to the bundle of coinvariants with respect to the action of the Lie algebra of meromorphic  $\mathfrak{g}$ -valued functions (in the physics literature, this bundle is known as the bundle of conformal blocks). The main result of this chapter is the proof that this bundle indeed satisfies the axioms of a complex-analytic modular functor. The most difficult part is proving the regularity of the connection at the boundary of the moduli space, which was first done by Tsuchiya, Ueno, and Yamada [TUY]. The proof presented in this chapter is based on the results of the unpublished manuscript [BFM], with necessary changes.

**History.** Even though the theory described in this book is relatively new, the number of related publications is now measured in thousands, if not tens of

thousands. We tried to list some of the most important references in the beginning of each chapter; however, this selection is highly subjective and does not pretend to be complete in any way. If you find that we missed some important result or gave an incorrect attribution, please let us know and we will gladly correct it in the next edition.

**Acknowledgments.** This book grew out of the course of lectures on tensor categories given by the second author at MIT in the Spring of 1997. Therefore, we would like to thank all participants of this class — without them, this book would have never been written.

Second, we want to express our deep gratitude to all those who helped us in the work on this manuscript — by explaining to us many things which we did not fully understand, by reading preliminary versions and pointing out our mistakes, and much more. Here is a partial list of them: Alexander Beilinson, Pierre Cartier, Pierre Deligne, Pavel Etingof, Boris Feigin, Michael Finkelberg, Domenico Fiorenza, Victor Kac, David Kazhdan, Tony Pantev.

Most of this work was done at MIT, and we also enjoyed the hospitality of several other institutions: ENS (Paris), ESI (Vienna), IAS (Princeton), and IHES (Paris). We acknowledge the partial support of the National Science Foundation and the Alfred P. Sloan Foundation. This research was partially conducted by the first author for the Clay Mathematics Institute. We thank the American Mathematical Society, and especially the editor Elaine W. Becker, for the final materialization of this project.