Preface

The present volume contains three surveys in differential geometry, based on the thirtieth John H. Barrett Memorial Lectures delivered at the University of Tennessee, Knoxville in May 2000. Directed at researchers and advanced graduate students, the surveys introduce the background, context and main techniques of very recent developments in three distinct areas of geometry: (i) conformally invariant curvatures and operators in four dimensions, and the associated partial differential equations; (ii) the use of isometric group actions and metric geometry techniques to understand examples and the classification of classes of Riemannian manifolds, especially of positive curvature; (iii) variational problems for lagrangian immersions in symplectic manifolds, in particular the lagrangian volume minimization problem in a homology class.

The main theme of Chapter 1, written by S.-Y.A. Chang and Paul Yang, is conformal geometry in four dimensions. The integrand in the Chern-Gauss-Bonnet formula for the Euler characteristic of a closed four-manifold involves the conformally invariant Weyl tensor and a fourth-order curvature invariant Q. In section 1, the authors describe their work (joint with J.Qing) on conformal compactification of complete, non-compact locally conformally flat four-manifolds with integrable Q, including a Chern-Gauss-Bonnet formula with a 'defect' term related to the isoperimetric profile of the ends (Theorem 1.2). They then turn to a discussion of the geometry of the developing map image of a locally conformally flat manifold of positive scalar curvature (a domain on the sphere whose complement is the limit set of a Kleinian group), in particular their recent work relating geometric finiteness and Hausdorff dimension of the limit set.

The next three sections deal with the second elementary symmetric function $\sigma_2(A)$, where A is the 'conformal Ricci tensor'; the integral of $\sigma_2(A)$ is conformally invariant. The main theorem in section 3 (Theorem 2.1, joint with M.Gursky) states that a compact four-manifold where both this invariant and the Yamabe invariant are positive admits a conformally related metric for which $\sigma_2(A)$ is positive pointwise, and in particular has pinched positive Ricci curvature. This involves a PDE of Monge-Ampère type, the authors' work on functional determinants, and at a crucial step the Yamabe flow. The last section of this chapter explains the main steps in the difficult proof of a very recent result: under the same hypothesis, a further conformal deformation yields a metric of constant $\sigma_2(A)$. The proof involves degree theory for fully non-linear equations, as well as delicate continuity and blowing-up arguments.

The purpose of Chapter 2, written by Karsten Grove, is to present the 'fairly unexplored territory' of geometry and topology that symmetry groups of Riemannian manifolds both possess and reveal. The first section sketches the known structure of the quotient of a Riemannian manifold via a symmetry group. The discussion includes the Principal Orbit Theorem and the structure of such quotients when considered as Alexandrov spaces. Section 2 lays out a general classification program, namely that of classifying all positively or non-negatively curved manifolds with large isometry groups. Of course there are many possible meanings of 'large,' five of which are considered in more detail: high degree of symmetry, high symmetry rank (these two refer to high dimension and rank of the isometry group G), small dimension of the manifold M relative to that of G, small dimension of the quotient M/G (cohomogeneity), and fixed point set of dimension close to the dimension of M/G (fixed point cohomogeneity).

From this point on the chapter follows two themes: classification results and examples that arise when one tries to find classification results, especially in connection with non-negative curvature. In fact, all known methods for constructing manifolds of non-negative curvature are described in this chapter, together with several of the most important classifications of manifolds that arise from somehow restricting geometry and symmetry groups. The chapter concludes with a list of open problems and conjectures.

The geometry of lagrangian immersions into symplectic manifolds is the topic of Chapter 3, by Jon Wolfson. Section 1 introduces the definitions and examples of lagrangian submanifolds, and the basic topological results underlying the theory. The period and Maslov index of a lagrangian immersion are defined, and the interplay with Ricci curvature is discussed, in particular for Kähler-Einstein manifolds. The problem of minimizing volume among lagrangian cycles in a lagrangian homology class is introduced, with a brief discussion of potential applications to the Strominger-Yau-Zaslow mirror symmetry conjecture and to problems in non-linear elasticity.

The next two sections are devoted to Wolfson's recent work (joint with R.Schoen) on the lagrangian volume minimization problem. Section 2 deals with existence, including the geometric measure theory approach and the mapping problem. The former yields an integral current which is mass-minimizing among all integral cycles in a given lagrangian homology class (Theorem 2.5). This is followed by a detailed discussion of the 'bubbling behavior' for minimizing sequences of maps of surfaces, including the important distinctions with the classical (unconstrained) problem. Section 3 describes three aspects of the partial regularity theory: hamiltonian-stationary lagrangian cones, Hölder continuity and a new monotonicity formula for lagrangian stationary surfaces (which involves contact geometry). This leads to the main existence and partial regularity result, Theorem 3.7.

The John H. Barrett Memorial Lectures were established in 1970, in memory of a professor and mathematics department head at the University of Tennessee, through whose influence an active group in differential equations evolved. Each year from one to four leading researchers deliver survey lectures in an active area of mathematics. The 2000 Barrett Lectures received financial support from the National Science Foundation and Science Alliance. Information on the Barrett lecture series may be found in the University of Tennessee Mathematics Department web site: http://www.math.utk.edu.

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