

## Preface

This book is devoted to the applications of the probability theory to the theory of nonlinear partial differential equations. More precisely, we investigate the class  $\mathcal{U}$  of all positive solutions of the equation  $Lu = \psi(u)$  in  $E$  where  $L$  is an elliptic differential operator of the second order,  $E$  is a bounded smooth domain in  $\mathbb{R}^d$  and  $\psi$  is a continuously differentiable positive function.

The progress in solving this problem till the beginning of 2002 was described in the monograph [D]. [We use an abbreviation [D] for [Dy02].] Under mild conditions on  $\psi$ , a trace on the boundary  $\partial E$  was associated with every  $u \in \mathcal{U}$ . This is a pair  $(\Gamma, \nu)$  where  $\Gamma$  is a subset of  $\partial E$  and  $\nu$  is a  $\sigma$ -finite measure on  $\partial E \setminus \Gamma$ . [A point  $y$  belongs to  $\Gamma$  if  $\psi'(u)$  tends sufficiently fast to infinity as  $x \rightarrow y$ .] All possible values of the trace were described and a 1-1 correspondence was established between these values and a class of solutions called  $\sigma$ -moderate. We say that  $u$  is  $\sigma$ -moderate if it is the limit of an increasing sequence of moderate solutions. [A moderate solution is a solution  $u$  such that  $u \leq h$  where  $Lh = 0$  in  $E$ .] In the Epilogue to [D], a crucial outstanding question was formulated: *Are all the solutions  $\sigma$ -moderate?* In the case of the equation  $\Delta u = u^2$  in a domain of class  $C^4$ , a positive answer to this question was given in the thesis of Mselati [Ms02a]—a student of J.-F. Le Gall.<sup>1</sup> However his principal tool—the Brownian snake—is not applicable to more general equations. In a series of publications by Dynkin and Kuznetsov [Dy04b], [Dy04c], [Dy04d], [Dy04e],[DK03], [DK04], [Ku04], Mselati's result was extended, by using a superdiffusion instead of the snake, to the equation  $\Delta u = u^\alpha$  with  $1 < \alpha \leq 2$ . This required an enhancement of the superdiffusion theory which can be of interest for anybody who works on applications of probabilistic methods to mathematical analysis.

The goal of this book is to give a self-contained presentation of these new developments. The book may be considered as a continuation of the monograph [D]. In the first three chapters we give an overview of the theory presented in [D] without duplicating the proofs which can be found in [D]. The book can be read independently of [D]. [It might be even useful to read the first three chapters before reading [D].]

In a series of papers (including [MV98a], [MV98b] and [MV04]) M. Marcus and L. Véron investigated positive solutions of the equation  $\Delta u = u^\alpha$  by

---

<sup>1</sup>The dissertation of Mselati was published in 2004 (see [Ms04]).

purely analytic methods. Both analytic and probabilistic approaches have their advantages and an interaction between analysts and probabilists was important for the progress of the field. I take this opportunity to thank M. Marcus and L. Véron for keeping me informed about their work.

The Choquet capacities are one of the principal tools in the study of the equation  $\Delta u = u^\alpha$ . This class contains the Poisson capacities used in the work of Dynkin and Kuznetsov and in this book and the Bessel capacities used by Marcus and Véron and by other analysts. I am very grateful to I. E. Verbitsky who agreed to write Appendix B, where the relations between the Poisson and Bessel capacities are established, thus allowing to connect the work of both groups.

I am indebted to S. E. Kuznetsov who provided me with several preliminary drafts of his paper [Ku04] used in Chapters **8** and **9**. I am grateful to him and to J.-F. Le Gall and B. Mselati for many helpful discussions. It is my pleasant duty to thank J.-F. Le Gall for permission to include in the book as Appendix A his note which clarifies a statement used but not proved in Mselati's thesis (we use it in Chapter **8**).

I am especially indebted to Yuan-chung Sheu for reading carefully the entire manuscript and suggesting many corrections and improvements.

The research of the author reported in this book was supported in part by the National Science Foundation Grant DMS-0204237.