Foreword

Noncommutative geometry nowadays looks as a vast building site.

On the one hand, practitioners of noncommutative geometry (or geometries) already built up a large and swiftly growing body of exciting mathematics, challenging traditional boundaries and subdivisions.

On the other hand, noncommutative geometry lacks common foundations: for many interesting constructions of "noncommutative spaces" we cannot even say for sure which of them lead to isomorphic spaces, because they are not objects of an all-embracing category (like that of locally ringed topological spaces in commutative geometry).

Matilde Marcolli's lectures reflect this spirit of creative growth and interdisciplinary research.

She starts Chapter 1 with a sketch of philosophy of noncommutative geometry à la Alain Connes. Briefly, Connes suggests imagining C^*-algebras as coordinate rings. He then supplies several bridges to commutative geometry by his construction of "bad quotients" of commutative spaces via crossed products and his treatment of noncommutative Riemannian geometry. Finally, algebraic tools like K-theory and cyclic cohomology serve to further enhance geometric intuition.

Marcolli then proceeds to explaining some recent developments drawing upon her recent work with several collaborators. A common thread in all of them is the study of various aspects of uniformization: classical modular group, Schottky groups. The modular group acts upon the complex half plane, partially compactified by cusps: rational points of the boundary projective line. The action becomes "bad" at irrational points, and here is where noncommutative geometry enters the game. A wealth of classical number theory is encoded in the coefficients of modular forms, their Mellin transforms, Hecke operators and modular symbols. Their counterparts living at the noncommutative boundary have only recently started to unravel themselves, and Marcolli gives a beautiful overview of what is already understood in Chapters 2 and 3.

Schottky uniformization provides a visualization of Arakelov's geometry at arithmetic infinity, which serves as the main motivation of Chapter 4.

Among the most tantalizing developments is the recurrent emergence of patches of common ground for number theory and theoretical physics.

In fact, one can present in this light the famous theorem of young Gauss characterising regular polygons that can be constructed using only ruler and compass. In his *Tagebuch* entry of March 30 he announced that a regular 17-gon has this property.

Somewhat modernizing his discovery, one can present it in the following way.

In the complex plane, roots of unity of degree n form vertices of a regular n-gone. Hence it makes sense to imagine that we study the ruler and compass constructions as well not in the Euclidean, but in the complex plane. This has an unexpected consequence: we can characterize the set of *all points* constructible in this way as the maximal Galois 2-extension of $\mathbb{Q}$. It remains to calculate the Galois group of $\mathbb{Q}(e^{2\pi i/17})$: since it is cyclic of order 16, this root of unity is constructible. Moreover, the same is true for all p-gons where p is a prime of the form $2^n + 1$ but not for other primes.

A remarkable feature of this result is the appearance of a hidden symmetry group. In fact, the definitions of a regular n-gon and ruler and compass constructions are initially formulated in terms of Euclidean plane geometry and suggest that the relevant symmetry group must be that of rigid rotations $SO\,(2)$, eventually extended by reflections and shifts. This conclusion turns out to be totally misleading: instead, one should rely upon $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$. The action of the latter group upon roots of unity of degree n factors through the maximal abelian quotient and is given by $\zeta \mapsto \zeta^k$, with k running over all $k \mod n$ with $(k, n) = 1$, whereas the action of the rotation group is given by $\zeta \mapsto \zeta_0\zeta$ with ζ_0 running over all n-th roots. Thus, the $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$-symmetry does not conserve angles between vertices which seem to be basic for the initial problem. Instead, it is compatible with addition and multiplication of complex numbers, and this property proves to be crucial.

With some stretch of imagination, one can recognize in the Euclidean avatar of this picture a physics flavor (putting it somewhat pompously, it appeals to the kinematics of 2-dimensional rigid bodies in gravitational vacuum), whereas the Galois avatar definitely belongs to number theory.

In the Marcolli lectures, stressing number theory, physics themes pop up at the end of Chapter 2 (Chaotic Cosmology in general relativity), the beginning of Chapter 3 (formalism of quantum statistical mechanics), and finally, sec. 5 of Chapter 4 where some models of black holes in general relativity turn out to have the same mathematical description as ∞-adic fibers of curves in Arakelov geometry. The reemergence of Gauss' Galois group $\mathrm{Gal}^{ab}(\overline{\mathbb{Q}}/\mathbb{Q})$ in Bost–Connes symmetry breaking, and of Gauss' statistics of continued fractions in the Chaotic Cosmology models, shows that connections with classical mathematics are as strong as ever.

Hopefully, this lively exposition will attract young researchers and incite them to engage themselves in exploration of the rich new territory.

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