CHAPTER I

Differentiable Quasiconformal Mappings

Introduction

There are several reasons why quasiconformal mappings have recently come to play a very active part in the theory of analytic functions of a single complex variable.

1. The most superficial reason is that q.c. mappings are a natural generalization of conformal mappings. If this were their only claim they would soon have been forgotten.

2. It was noticed at an early stage that many theorems on conformal mappings use only the quasiconformality. It is therefore of some interest to determine when conformality is essential and when it is not.

3. Q.c. mappings are less rigid than conformal mappings and are therefore much easier to use as a tool. This was typical of the utilitarian phase of the theory. For instance, it was used to prove theorems about the conformal type of simply connected Riemann surfaces (now mostly forgotten).

4. Q.c. mappings play an important role in the study of certain elliptic partial differential equations.

5. Extremal problems in q.c. mappings lead to analytic functions connected with regions or Riemann surfaces. This was a deep and unexpected discovery due to Teichmüller.

6. The problem of moduli was solved with the help of q.c. mappings. They also throw light on Fuchsian and Kleinian groups.

7. Conformal mappings degenerate when generalized to several variables, but q.c. mappings do not. This theory is still in its infancy.

A. The Problem and Definition of Grötzsch

The notion of a quasiconformal mapping, but not the name, was introduced by H. Grötzsch in 1928. If Q is a square and R is a rectangle, not a square, there is no conformal mapping of Q on R which maps vertices on vertices. Instead, Grötzsch asks for the most nearly conformal mapping of this kind. This calls for a measure of approximate conformality, and in supplying such a measure Grötzsch took the first step toward the creation of a theory of q.c. mappings.

All the work of Grötzsch was late to gain recognition, and this particular idea was regarded as a curiosity and allowed to remain dormant for several years. It reappears in 1935 in the work of Lavrentiev, but from the point of view of partial differential equations. In 1936 I included a reference to the q.c. case in my theory of covering surfaces. From then on the notion became generally known, and in 1937 Teichmüller began to prove important theorems by use of q.c. mappings, and later theorems about q.c. mappings.

We return to the definition of Grötzsch. Let $w = f(z)$ $(z = x + iy,\ w = u + iv)$ be a C^1 homeomorphism from one region to another. At a point z_0 it induces a linear mapping of the differentials

$$du = u_x\,dx + u_y\,dy \qquad (1)$$
$$dv = v_x\,dx + v_y\,dy$$

which we can also write in the complex form

$$dw = f_z dz + f_{\overline{z}} d\overline{z} \qquad (2)$$

with

$$f_z = \tfrac{1}{2}(f_x - i f_y), \quad f_{\overline{z}} = \tfrac{1}{2}(f_x + i f_y). \qquad (3)$$

Geometrically, (1) represents an affine transformation from the (dx, dy) to the (du, dv) plane. It maps circles about the origin into similar ellipses. We wish to compute the ratio between the axes as well as their direction.

In classical notation one writes

$$du^2 + dv^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2 \qquad (4)$$

with

$$E = u_x^2 + v_x^2, \quad F = u_x u_y + v_x v_y, \quad G = u_y^2 + v_y^2.$$

The eigenvalues are determined from

$$\begin{vmatrix} E - \lambda & F \\ F & G - \lambda \end{vmatrix} = 0 \qquad (5)$$

and are

$$\lambda_1, \lambda_2 = \frac{E + G \pm [(E - G)^2 + 4F^2]^{1/2}}{2}. \qquad (6)$$

The ratio $a\colon b$ of the axes is

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} = \frac{E + G + [(E - G)^2 + 4F^2]^{1/2}}{2(EG - F^2)^{1/2}}. \qquad (7)$$

The complex notation is much more convenient. Let us first note that

$$f_z = \tfrac{1}{2}(u_x + v_y) + \tfrac{i}{2}(v_x - u_y) \qquad (8)$$
$$f_{\overline{z}} = \tfrac{1}{2}(u_x - v_y) + \tfrac{i}{2}(v_x + u_y).$$

This gives

$$|f_z|^2 - |f_{\overline{z}}|^2 = u_x v_y - u_y v_x = J \qquad (9)$$

which is the Jacobian. The Jacobian is positive for sense preserving and negative for sense reversing mappings. For the moment we shall consider only the sense preserving case. Then $|f_{\overline{z}}| < |f_z|$.

It now follows immediately from (2) that

$$(|f_z| - |f_{\overline{z}}|)|dz| \le |dw| \le (|f_z| + |f_{\overline{z}}|)|dz| \qquad (10)$$

where both limits can be attained. We conclude that the ratio of the major to the minor axis is

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \ge 1. \qquad (11)$$

This is called the *dilatation* at the point z. It is often more convenient to consider

$$d_f = \frac{|f_{\overline{z}}|}{|f_z|} < 1 \tag{12}$$

related to D_f by

$$D_f = \frac{1+d_f}{1-d_f}, \quad d_f = \frac{D_f - 1}{D_f + 1}. \tag{13}$$

The mapping is conformal at z if and only if $D_f = 1$, $d_f = 0$.

The maximum is attained when the ratio

$$\frac{f_{\overline{z}}d\overline{z}}{f_z dz}$$

is positive, the minimum when it is negative. We introduce now the *complex dilatation*

$$\mu_f = \frac{f_{\overline{z}}}{f_z} \tag{14}$$

with $|\mu_f| = d_f$. The maximum corresponds to the direction

$$\arg dz = \alpha = \tfrac{1}{2} \arg \mu, \tag{15}$$

the minimum to the direction $\alpha \pm \pi/2$. In the dw-plane the direction of the major axis is

$$\arg dw = \beta = \tfrac{1}{2} \arg \nu \tag{16}$$

where we have set

$$\nu_f = \frac{f_{\overline{z}}}{\overline{f}_{\overline{z}}} = \left(\frac{f_z}{|f_z|}\right)^2 \mu_f. \tag{17}$$

The quantity ν_f may be called the *second complex dilatation.*

We will illustrate by the following self-explanatory figure:

Observe that $\beta - \alpha = \arg f_z$.

DEFINITION 1. The mapping f is said to be quasiconformal if D_f is bounded. It is K-quasiconformal if $D_f \leq K$.

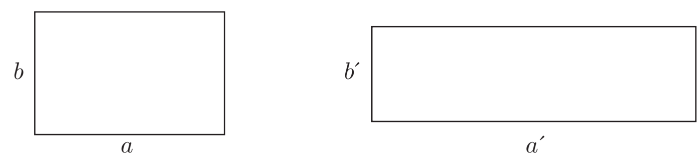
The condition $D_f \leq K$ is equivalent to $d_f \leq k = (K-1)/(K+1)$. A 1-quasiconformal mapping is conformal.

Let it be said at once that the restriction to C^1-mappings is most unnatural. One of our immediate aims is to get rid of this restriction. For the moment, however, we prefer to push this difficulty aside.

B. Solution of Grötzsch's Problem

e pass to Grötzsch's problem and give it a precise meaning by saying that f is most nearly conformal if $\sup D_f$ is as small as possible.

Let R, R' be two rectangles with sides a, b and a', b'. We may assume that $a\colon b \le a'\colon b'$ (otherwise, interchange a and b). The mapping f is supposed to take a-sides into a-sides and b-sides into b-sides.



The computation goes

$$\begin{aligned} a' &\le \int_0^a |df(x+iy)| \le \int_0^a (|f_z| + |f_{\overline{z}}|)dx \\ a'b &\le \int_0^a \int_0^b (|f_z| + |f_{\overline{z}}|)dx\,dy \\ a'^2b^2 &\le \int_0^a \int_0^b \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dx\,dy \int_0^a \int_0^b (|f_z|^2 - |f_{\overline{z}}|^2)dx\,dy \\ &= a'b' \int_0^a \int_0^b D_f dx\,dy \end{aligned}$$

or

$$\frac{a'}{b'} : \frac{a}{b} \le \frac{1}{ab} \iint_R D_f dx\,dy \tag{1}$$

and in particular

$$\frac{a'}{b'} : \frac{a}{b} \le \sup D_f.$$

The minimum is attained for the affine mapping which is given by

$$f(z) = \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) z + \frac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right) \overline{z}.$$

THEOREM 1. *The affine mapping has the least maximal and the least average dilatation.*

The ratios $m = a/b$ and $m' = a'/b'$ are called the modules of R and R' (taken with an orientation). We have proved that there exists a K-q.c. mapping of R on R' if and only if

$$\frac{1}{K} \le \frac{m'}{m} \le K. \tag{2}$$

C. Composed Mappings

We shall determine the complex derivatives and complex dilatations of a composed mapping $g \circ f$. There is the usual trouble with the notation which is most easily resolved by introducing an intermediate variable $\zeta = f(z)$.

The usual rules are applicable and we find

$$(g \circ f)_z = (g_\zeta \circ f) f_z + (g_{\overline{\zeta}} \circ f) \overline{f}_z$$
$$(g \circ f)_{\overline{z}} = (g_\zeta \circ f) f_{\overline{z}} + (g_{\overline{\zeta}} \circ f) \overline{f}_{\overline{z}}. \tag{1}$$

When solved they give

$$g_\zeta \circ f = \frac{1}{J}[(g \circ f)_z \overline{f}_{\overline{z}} - (g \circ f)_{\overline{z}} \overline{f}_z]$$
$$g_{\overline{\zeta}} \circ f = \frac{1}{J}[(g \circ f)_{\overline{z}} f_z - (g \circ f)_z f_{\overline{z}}] \tag{2}$$

where $J = |f_z|^2 - |f_{\overline{z}}|^2$.

For $g = f^{-1}$ the formulas become

$$(f^{-1})_\zeta \circ f = \overline{f}_{\overline{z}}/J, \quad (f^{-1})_{\overline{\zeta}} \circ f = -f_{\overline{z}}/J. \tag{3}$$

One derives, for instance,

$$\mu_{f^{-1}} = -\nu_f \circ f^{-1} \tag{4}$$

and, on passing to the absolute values,

$$d_{f^{-1}} = d_f \circ f^{-1}. \tag{5}$$

In other words, inverse mappings have the same dilatation at corresponding points.

From (2) we obtain

$$\mu_g \circ f = \frac{f_z}{\overline{f}_{\overline{z}}} \frac{\mu_{g \circ f} - \mu_f}{1 - \overline{\mu}_f \mu_{g \circ f}}. \tag{6}$$

If g is conformal, then $\mu_g = 0$ and we find

$$\mu_{g \circ f} = \mu_f. \tag{7}$$

If f is conformal, $\mu_f = 0$ and

$$\mu_g \circ f = \left(\frac{f'}{|f'|} \right)^2 \mu_{g \circ f}, \tag{8}$$

which can also be written as

$$\nu_g \circ f = \nu_{g \circ f}. \tag{9}$$

In any case, the dilatation is invariant with respect to all conformal transformations.

If we set $g \circ f = h$ we find from (6)

$$\mu_{h \circ f^{-1}} \circ f = \frac{f_z}{\overline{f}_{\overline{z}}} \frac{\mu_h - \mu_f}{1 - \overline{\mu}_f \mu_h}. \tag{10}$$

For the dilatation

$$d_{h \circ f^{-1}} \circ f = \left| \frac{\mu_h - \mu_f}{1 - \mu_f \overline{\mu}_h} \right| \tag{11}$$

and

$$\log D_{h \circ f^{-1}} \circ f = [\mu_h, \mu_f], \tag{12}$$

the non-euclidean distance (with respect to the metric $ds = \frac{2|dw|}{1-|w|^2}$ in $|w| < 1$).

We can obviously use $\sup[\mu_h, \mu_f]$ as a distance between the mappings f and h (the Teichmüller distance). It is a metric provided one identifies mappings that differ by a conformal transformation.

The composite of a K_1-q.c. and a K_2-q.c. mapping is K_1K_2-q.c.

D. Extremal Length

Let Γ be a family of curves in the plane. Each $\gamma \in \Gamma$ shall be a countable union of open arcs, closed arcs or closed curves, and every closed subarc shall be rectifiable. We shall introduce a geometric quantity $\lambda(\Gamma)$, called the *extremal length* of Γ, which is a sort of average minimal length. Its importance for our topic lies in the fact that it is invariant under conformal mappings and quasi-invariant under q.c. mappings (the latter means that it is multiplied by a bounded factor).

A function ρ, defined in the whole plane, will be called *allowable* if it satisfies the following conditions:

1. $\rho \geq 0$ and measurable.
2. $A(\rho) = \iint \rho^2 dx\, dy \neq 0, \infty$ (the integral is over the whole plane).

For such a ρ, set

$$L_\gamma(\rho) = \int_\gamma \rho |dz|$$

if ρ is measurable on γ [*] , $L_\gamma(\rho) = \infty$ otherwise. We introduce

$$L(\rho) = \inf_{\gamma \in \Gamma} L_\gamma(\rho)$$

and

Definition.

$$\lambda(\Gamma) = \sup_\rho \frac{L(\rho)^2}{A(\rho)}$$

for all allowable ρ.

We shall say that $\Gamma_1 < \Gamma_2$ if every γ_2 contains a γ_1 (the γ_2 are fewer and longer).

Remark. Observe that $\Gamma_1 \subset \Gamma_2$ implies $\Gamma_2 < \Gamma_1$!

Theorem 2. *If* $\Gamma_1 < \Gamma_2$, *then* $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$.

Proof. If $\gamma_1 \subseteq \gamma_2$, then

$$L_{\gamma_1}(\rho) \leq L_{\gamma_2}(\rho)$$
$$\inf L_{\gamma_1}(\rho) \leq \inf L_{\gamma_2}(\rho)$$

and it follows at once that $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$.

Example 1. Γ is the set of all arcs in a closed rectangle R which joins a pair of opposite sides.

For any ρ

$$\int_0^a \rho(x+iy)dx \geq L(\rho)$$
$$\iint_R \rho dx\, dy \geq bL(\rho)$$
$$b^2 L(\rho)^2 \leq ab \iint_R \rho^2 dx\, dy \leq abA(\rho)$$
$$\frac{L(\rho)^2}{A(\rho)} \leq \frac{a}{b}.$$

[*] (as a function of arc-length)

This proves $\lambda(\Gamma) \leq a/b$.

On the other hand, take $\rho = 1$ in R, $\rho = 0$ outside. Then $L(\rho) = a$, $A(\rho) = ab$, hence $\lambda(\Gamma) \geq a/b$. We have proved

$$\lambda(\Gamma) = \frac{a}{b}.$$

EXAMPLE 2. Γ is the set of all arcs in an annulus $r_1 \leq |z| \leq r_2$ which join the boundary circles.

Computation:

$$\int_{r_1}^{r_2} \rho\, dr \geq L(\rho), \quad \iint \rho\, dr\, d\theta \geq 2\pi L(\rho)$$

$$\begin{aligned} 4\pi^2 L(\rho)^2 &\leq 2\pi \log \frac{r_2}{r_1} \iint \rho^2 r d\, r d\theta \\ \frac{L(\rho)^2}{A(\rho)} &\leq \frac{1}{2\pi} \log \frac{r_2}{r_1}. \end{aligned}$$

Equality for $\rho = 1/r$.

EXAMPLE 3. The module of an annulus.

Let G be a doubly connected region in the finite plane with C_1 the bounded, C_2 the unbounded component of the complement. We say the closed curve γ in G *separates* C_1 and C_2 if γ has non-zero winding number about the points of C_1. Let Γ be the family of closed curves in G which separate C_1 and C_2. The module $M(G) = \lambda(\Gamma)^{-1}$. Consider, for example, the annulus $G = \{r_1 \leq |z| \leq r_2\}$.

$$\begin{aligned} L(\rho) &\leq \int_0^{2\pi} \rho(re^{i\theta}) r\, d\theta \\ \frac{L(\rho)}{r} &\leq \int_0^{2\pi} \rho\, d\theta \\ L(\rho) \log\left(\frac{r_2}{r_1}\right) &\leq \iint \rho\, dr\, d\theta \\ L(\rho)^2 \log^2\left(\frac{r_2}{r_1}\right) &\leq 2\pi \log\left(\frac{r_2}{r_1}\right) \iint \rho^2 r\, dr\, d\theta \\ \frac{L(\rho)^2}{A(\rho)} &\leq \frac{2\pi}{\log(r_2/r_1)}. \end{aligned}$$

Once again $\rho = 1/2\pi r$ gives equality. Indeed, for any $\gamma \in \Gamma$ we have

$$1 \leq |n(\gamma, 0)| = \frac{1}{2\pi} \left| \int_\gamma \frac{dz}{z} \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|dz|}{|z|} = L_\gamma(\rho),$$

so $L(\rho) = 1$ and $A(\rho) = \frac{1}{2\pi} \log(r_2/r_1)$. We conclude that $M(G) = \frac{1}{2\pi} \log(r_2/r_1)$.

Suppose that all $\gamma \in \Gamma$ are contained in a region Ω and let ϕ be a K-quasiconformal mapping of Ω on Ω'. Let Γ' be the image set of Γ.

THEOREM 3. *$K^{-1}\lambda(\Gamma) \leq \lambda(\Gamma') \leq K\lambda(\Gamma)$.*

PROOF. For a given $\rho(z)$ define $\rho'(\zeta) = 0$ outside Ω' and

$$\rho'(\zeta) = \frac{\rho}{|\phi_z| - |\phi_{\overline{z}}|} \circ \phi^{-1}$$

in Ω'. Then

$$\int_{\gamma'} \rho' |d\zeta| \geq \int_\gamma \rho |dz|$$
$$\iint \rho'^2 d\xi\, d\eta = \iint_\Omega \rho^2 \frac{|\phi_z| + |\phi_{\overline{z}}|}{|\phi_z| - |\phi_{\overline{z}}|} dx\, dy \leq K A(\rho).$$

This proves $\lambda' \geq K^{-1}\lambda$, and the other inequality follows by considering the inverse.

COROLLARY. $\lambda(\Gamma)$ *is a conformal invariant.*

There are two important composition principles.

I. $\Gamma_1 + \Gamma_1 = \{\gamma_1 + \gamma_2 | \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}^*$.
II. $\Gamma_1 \cup \Gamma_2$.

THEOREM 4.

a) $\lambda(\Gamma_1 + \Gamma_2) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$*;*
b) $\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geq \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$ *if* Γ_1, Γ_2 *lie in disjoint measurable sets.*

PROOF OF a). We may assume that $0 < \lambda(\Gamma_1,), \lambda(\Gamma_2) < \infty$ for otherwise the inequality is trivial.

We may normalize so that

$$L_1(\rho_1) = A(\rho_1)$$
$$L_2(\rho_2) = A(\rho_2).$$

Choose $\rho = \max(\rho_1, \rho_2)$. Then

$$L(\rho) \geq L_1(\rho_1) + L_2(\rho_2) = A(\rho_1) + A(\rho_2)$$
$$A(\rho) \leq A(\rho_1) + A(\rho_2)$$

$$\lambda = \sup \frac{L(\rho)^2}{A(\rho)} \geq A(\rho_1) + A(\rho_2) = \frac{L_1(\rho_1)^2}{A(\rho_1)} + \frac{L_2(\rho_2)^2}{A(\rho_2)}.$$

It follows that $\lambda \geq \lambda_1 + \lambda_2$.

PROOF OF b). If $\lambda = \lambda(\Gamma_1 \cup \Gamma_2) = 0$ there is nothing to prove. Consider an admissible ρ with $L(\rho) > 0$ and set $\rho_1 = \rho$ on E_1, $\rho_2 = \rho$ on E_2, 0 outside (where E_1 and E_2 are complementary measurable sets with $\Gamma_1 \subseteq E_1$, $\Gamma_2 \subseteq E_2$). Then $L_1(\rho_1) \geq L(\rho)$, $L_2(\rho_2) \geq L(\rho)$, and $A(\rho) = A(\rho_1) + A(\rho_2)$. Thus

$$\frac{A(\rho)}{L(\rho)^2} \geq \frac{A(\rho_1)}{L_1(\rho_1)^2} + \frac{A(\rho_2)}{L_2(\rho_2)^2}$$

and hence

$$\lambda^{-1} \geq \lambda_1^{-1} + \lambda_2^{-1}.$$

□

* $\gamma_1 + \gamma_2$ means "γ_1 followed by γ_2."

E. A Symmetry Principle

For any γ let $\overline{\gamma}$ be its reflection in the real axis, and let γ^+ be obtained by reflecting the part below the real axis and retaining the part above it ($\gamma \cup \overline{\gamma} = \gamma^+ \cup (\gamma^+)^-$).

The notations $\overline{\Gamma}$ and Γ^+ are self-explanatory.

THEOREM 5. *If* $\Gamma = \overline{\Gamma}$ *then* $\lambda(\Gamma) = \frac{1}{2}\lambda(\Gamma^+)$.

PROOF.

1. For a given ρ set $\hat{\rho}(z) = \max(\rho(z), \rho(\overline{z}))$. Then

$$L_\gamma(\hat{\rho}) = L_{\gamma^+}(\hat{\rho}) \geq L_{\gamma^+}(\rho) \geq L^+(\rho)$$

and

$$A(\hat{\rho}) \leq A(\rho) + A(\overline{\rho}) = 2A(\rho).$$

This makes

$$\frac{L^+(\rho)^2}{A(\rho)} \leq 2\frac{L(\hat{\rho})^2}{A(\hat{\rho})} \leq 2\lambda(\Gamma)$$

and hence $\lambda(\Gamma^+) \leq 2\lambda(\Gamma)$.

2. For given ρ set

$$\rho^+(z) = \begin{cases} \rho(z) + \rho(\overline{z}) & \text{in upper halfplane} \\ 0 & \text{in lower halfplane.} \end{cases}$$

Then

$$\begin{aligned} L_{\gamma^+}(\rho^+) &= L_{\gamma^+ + (\gamma^+)^-}(\rho) = L_{\gamma + \gamma^-}(\rho) \\ &= L_\gamma(\rho) + L_{\overline{\gamma}}(\rho) \geq 2L(\rho). \end{aligned}$$

On the other hand

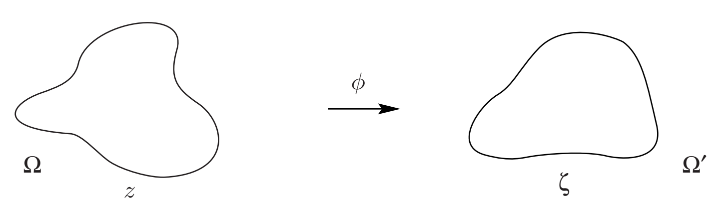
$$A(\rho^+) \leq 2\int \rho^2 + \overline{\rho}^2 = 2A(\rho)$$

and hence

$$\begin{aligned} \frac{L(\rho)^2}{A(\rho)} &\leq \frac{1}{2}\frac{L_{\gamma^+}(\rho^+)^2}{A(\rho^+)} \leq \frac{1}{2}\lambda(\Gamma^+) \\ \lambda(\Gamma) &\leq \frac{1}{2}\lambda(\Gamma^+). \end{aligned}$$

□

F. Dirichlet Integrals



Let ϕ be a K-q.c. mapping from Ω to Ω'. The Dirichlet integral of a C^1 function $u(\zeta)$ is

$$D(u) = \iint_{\Omega'} (u_\xi^2 + u_\eta^2) d\xi\, d\eta = 4 \iint |u_\zeta|^2 d\xi\, d\eta.$$

For the composite $u \circ \phi$ we have

$$(u \circ \phi)_z = (u_\zeta \circ \phi)\phi_z + (u_{\overline{\zeta}} \circ \phi)\overline{\phi}_z$$
$$|(u \circ \phi)_z| \leq (|u_\zeta| \circ \phi)(|\phi_z| + |\phi_{\overline{z}}|)$$

$$\begin{aligned} D(u \circ \phi) &\leq 4 \iint_\Omega (|u_\zeta| \circ \phi)^2 (|\phi_z| + |\phi_{\overline{z}}|)^2 dx\, dy \\ &= 4 \iint_{\Omega'} |u_\zeta|^2 \left(\frac{|\phi_z| + |\phi_{\overline{z}}|}{|\phi_z| - |\phi_{\overline{z}}|} \right) \circ \phi^{-1} d\xi\, d\eta \end{aligned}$$

and thus

(1) $$D(u \circ \phi) \leq K D(u).$$

Dirichlet integrals are quasi-invariant.

There is another formulation of this. We may consider merely corresponding Jordan regions with boundaries γ, γ'. Let v on γ' and $v \circ \phi$ on γ be corresponding boundary values. There is a minimum Dirichlet integral $D_0(v)$ for functions with boundary values v, attained for the harmonic function with these boundary values. Clearly,

(2) $$D_0(v \circ \phi) \leq K D_0(v).$$

One may go a step further and assume that v is given only on part of the boundary. For instance, if $v = 0$ and $v = 1$ on disjoint boundary arcs we get a new proof of the quasi-invariance of the module.

In order to define the Dirichlet integral it is not necessary to assume that u is of class C^1. Suppose that $u(z)$ is continuous with compact support. Thus we can form the Fourier transform

$$\hat{u}(\xi, \eta) = \frac{1}{2\pi} \iint_\Omega e^{i(x\xi + y\eta)} u(x, y) dx\, dy$$

and we know that

$$(u_x)\hat{} = -i\xi\hat{u}$$
$$(u_y)\hat{} = -i\eta\hat{u}.$$

It follows by the Plancherel formula that

$$D(u) = \iint (\xi^2 + \eta^2) |\hat{u}|^2 d\xi\, d\eta$$

and this can be taken as *definition* of $D(u)$.

A Supplement to Ahlfors's Lectures

Clifford J. Earle and Irwin Kra

This article summarizes further developments in some areas related to Ahlfors's book. In general, we shall use his notation; for example, we shall denote the plane and the Riemann sphere by $\mathbb{C}$ and $\widehat{\mathbb{C}}$ respectively. Chapter references will be to chapters of this book unless another book is explicitly cited.

Our first section is a brief supplement to Chapters I through V. Section 2 discusses several approaches to Teichmüller theory, including the two in Chapter VI. The next three sections are about aspects of three important metrics on Teichmüller spaces, and the last is about finitely generated Kleinian groups.

Topics we have not discussed include some generalizations of Teichmüller spaces and connections between Teichmüller spaces and physics (especially string theory). Some generalized Teichmüller spaces are discussed in [**31**], [**41**], [**53**], [**58**], and [**69**]. For a comprehensive survey of geometric function theory, see the two-volume handbook [**103**] and [**104**], edited by R. Kühnau. These volumes are particularly relevant to Chapters I through V.

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1. Quasiconformal mappings and their boundary values

1.1. The metric definition. The class of quasiconformal mappings can be defined in many equivalent ways. The geometric and the two analytic definitions in Chapter II are sufficient for Ahlfors's purposes in this book, but the following additional definition is useful in many settings, for example in the theory of holomorphic motions (see §1.3 below and Shishikura's article [**141**] in this volume).

Definition 1 (The metric definition). Let Ω and Ω' be plane regions and let f be an orientation-preserving homeomorphism of Ω onto Ω'. The *circular dilatation* $H_f(z)$ of f at the point z in Ω is defined by the formula

$$H_f(z) = \limsup_{\epsilon \to 0^+} \frac{\max\{|f(\zeta) - f(z)| : |\zeta - z| = \epsilon\}}{\min\{|f(\zeta) - f(z)| : |\zeta - z| = \epsilon\}}. \tag{1}$$

Obviously, $1 \le H_f(z) \le \infty$ for every z in Ω. According to the metric definition, f is quasiconformal if and only if its circular dilatation function H_f has a finite upper bound in Ω, and a quasiconformal mapping f of Ω onto Ω' is K-quasiconformal (with $1 \le K < \infty$) if and only if $H_f \le K$ almost everywhere in Ω.

THEOREM 1. *The metric definition determines the same class of quasiconformal and K-quasiconformal mappings as the geometric and analytic definitions.*

Theorem 1 is proved in Chapter IV of [**107**] and is generalized to mappings between domains in $\mathbb{R}^n$ in [**157**].

REMARKS. Heinonen and Koskela showed in [**79**] that lim sup can be replaced by lim inf in (1), even in the $\mathbb{R}^n$ setting. In addition, they have extended much of the theory to homeomorphisms with bounded dilatation between metric spaces X and Y with appropriate geometry (see [**80**]). Their dilatation function H_f is defined by

$$H_f(x) := \limsup_{\epsilon \to 0^+} \frac{\sup\{d_Y(f(x), f(x')) : d_X(x, x') \leq \epsilon\}}{\inf\{d_Y(f(x), f(x')) : d_X(x, x') \geq \epsilon\}}, \quad x \text{ in } X,$$

which reduces to (1) when X and Y are plane regions.

1.2. Quasiconformal mappings with given boundary values. Chapter IV of this book introduces the class of homeomorphisms of the real axis that satisfy the Beurling–Ahlfors M-condition, and it describes the "Beurling–Ahlfors extension" of such a homeomorphism to a quasiconformal self-mapping of the upper half-plane (and, by symmetry, to the entire plane).

The Beurling–Ahlfors extension has a useful invariance property. Let φ be the extension of h, and let $f(z) = az + b$ and $g(z) = cz + d$ be conformal maps of the upper half-plane onto itself. Then $f \circ \varphi \circ g$ is the Beurling–Ahlfors extension of $f \circ h \circ g$. That property simplifies the proof that the extension of h is quasiconformal when h satisfies the M-condition, but its general utility is limited by the special role of the point at infinity.

In [**45**], Douady and Earle showed how to extend each homeomorphism h of the unit circle to a homeomorphism $ex(h)$ of the closed unit disk so that

$$ex(f \circ h \circ g) = f \circ ex(h) \circ g \tag{2}$$

whenever h is a homeomorphism of the unit circle and f and g are Möbius transformations that map the closed unit disk onto itself.

The homeomorphism $ex(h)$ is often called the barycentric (or Douady–Earle) extension of h. It is equal to h on the unit circle and is real analytic with nonzero Jacobian at every point of the open unit disk. In addition, if h has a quasiconformal extension to the unit disk, then $ex(h)$ is bi-Lipschitz with respect to the Poincaré metric. In fact, for each $K \geq 1$ there is a number $C(K) \geq 1$ such that if h has a K-quasiconformal extension to the unit disk, then

$$\frac{1}{C(K)} d(z, z') \leq d(ex(h)(z), ex(h)(z')) \leq C(K) d(z, z') \tag{3}$$

for all z and z' in the open unit disk, where d is the Poincaré metric. (See the proof of Theorem 2 in [**45**].)

Properties (2) and (3) have led to a number of applications. For example, if Ω is a Jordan region in the sphere and L is its boundary, barycentric extensions can be used to define a sense-reversing reflection j_Ω in L that is real analytic in the complement of L and commutes with every Möbius transformation that maps Ω onto itself. In addition, if L contains the point at infinity and admits a quasiconformal reflection, then j_Ω is Lipschitz continuous. This observation strengthens Lemma 3 in Chapter IV D, which relied on the Beurling–Ahlfors extension. It was made independently in the book [**106**] and the paper [**60**]. That paper examines j_Ω in detail and applies it to the study of Teichmüller spaces.

For more applications of the barycentric extension, see [**59**].

1.3. Holomorphic motions and the λ-lemma. Let f be a quasiconformal mapping of the plane that fixes the points 0, 1, and ∞. We can write its complex dilatation in the form $k\mu$, where $0 \leq k < 1$ and the L^∞ function μ has norm one. By Theorem 3 of Chapter V, f belongs to the one-parameter family of quasiconformal mappings $f^{\lambda\mu}$, where the complex parameter λ ranges over the open unit disk D.

The map $(\lambda, z) \mapsto f^{\lambda\mu}(z)$ from $D\times\mathbb{C}$ to $\mathbb{C}$ is a holomorphic motion of the plane. More generally, if E is any subset of the Riemann sphere, the function $\phi\colon D\times E \to \widehat{\mathbb{C}}$ is called a *holomorphic motion of E* if it satisfies the three conditions:

(a) $\phi(0, z) = z$ for all z in E,
(b) $\phi(\cdot, z)$ is a holomorphic ($\widehat{\mathbb{C}}$-valued) function on D for each z in E, and
(c) $\phi(\lambda, \cdot)$ is injective on E for each λ in D.

The λ-lemma of Mañé, Sad, and Sullivan states three surprising properties of holomorphic motions. (See [**109**].)

LEMMA 1 (the λ-lemma). *Let ϕ be a holomorphic motion of E. Then*

(i) *ϕ is a continuous map from $D \times E$ to $\widehat{\mathbb{C}}$,*
(ii) *the map $z \mapsto \phi(\lambda, z)$ is quasiconformal on E for each λ in D, and*
(iii) *ϕ extends to a holomorphic motion $\widetilde{\phi}$ of the closure $\widetilde{E}$ of E.*

The extended holomorphic motion $\widetilde{\phi}$ of course has the properties (i) and (ii) with $\widetilde{E}$ replacing E.

Questions might be raised about the meaning of (ii) when the set E is not open. They are answered conclusively in the papers [**149**] and [**30**], which give quite independent proofs that each map $z \mapsto \phi(\lambda, z)$ is the restriction to E of a quasiconformal mapping of the sphere. An even more decisive result is Slodkowski's theorem that every holomorphic motion of E is the restriction to $D \times E$ of a holomorphic motion of the sphere. See his papers [**142**] and, for later developments, [**143**] and [**144**]. A simpler proof of Slodkowski's theorem is given in Chirka [**39**]; for a self-contained presentation of that proof, see Hubbard's book [**84**].

Slodkowski's theorem was applied to Teichmüller spaces by the authors and S. L. Krushkal′ in [**56**] (see the remarks in §4.1 and §4.2 below).

Together, the λ-lemma and the mapping theorem in Chapter V imply that a homeomorphism of the sphere is quasiconformal if and only if it can be written in the form $z \mapsto \phi(\lambda, z)$ for some λ in D and some holomorphic motion ϕ of $\widehat{\mathbb{C}}$. That fact plays an important role in Astala's groundbreaking paper [**12**], which obtains sharp versions of important analytic and geometric inequalities about quasiconformal mappings. Further sharp results are given in Eremenko and Hamilton [**63**], where holomorphic motions are replaced by the flow $t \mapsto f^{t\mu}$ with $0 \leq t < 1$ and $\|\mu\|_\infty = 1$. See Hamilton's article about area distortion in [**103**].

1.4. More about the Beltrami equation. Because the Beltrami equation

$$w_{\bar{z}} = \mu w_z \tag{4}$$

is so important, we shall make a few historical remarks about the mapping theorems proved in Chapter V. Let $\mu(z)$ be any measurable function on $\mathbb{C}$ with L^∞ norm less than one. Classically, when μ is sufficiently regular, local solutions of (4) with nonzero Jacobians provide isothermal parameters (coordinates) for the Riemannian metric

$$ds^2 = |dz + \mu(z)d\bar{z}|^2 .$$

(See, for example, the introduction to Chern [**38**].) Classical results of Korn and Lichtenstein prove local existence if μ is Hölder continuous (see their papers cited in [**4**], [**38**], and [**11**]). The local solutions are charts for a new Riemann surface structure on $\mathbb{C}$, so the uniformization theorem implies existence of a global solution.

Morrey's 1938 paper [**128**] removed the regularity conditions of Korn and Lichtenstein and proved the existence of homeomorphic solutions of (4) for all measurable functions μ with L^∞ norm less than one. That result is often called the measurable Riemann mapping theorem.

The use of complex derivatives to study the Beltrami equation was widespread by 1955. Chern wrote the paper [**38**] to advertise their use, and the papers Ahlfors [**4**], Bojarski [**32**], and Vekua [**162**] appeared the same year as Chern's. All use some form of the integral operator T from Chapter V, as do Bers's lecture notes [**14**] from the same era.

The decisive step of using the Calderón–Zygmund machinery to interpret T as an operator on $L^p(\mathbb{C})$ with $p > 2$ was taken in Bojarski [**32**], where the proof of Theorem 1 presented in Chapter V B first appeared. The famous paper [**11**] of Ahlfors and Bers used this L^p theory for a systematic study of the dependence of normalized homeomorphic solutions of (4) on μ. That paper is self-contained modulo the Calderón-Zygmund theory. The treatment in this book is completely self-contained.

Vekua's book [**163**] contains some of the material in Chapter V and also discusses the case when μ is of class $C^{n,\alpha}$ with $0 < \alpha < 1$. That case is also studied in Bers [**14**]. Both discussions show that f^μ is a C^∞ diffeomorphism when μ is of class C^∞, a fact that also follows from general regularity theorems for solutions of elliptic partial differential equations. It is also proved in Douady [**44**], where in addition the existence of a homeomorphic solution of (4) for every measurable function μ with $\|\mu\|_\infty < 1$ is proved using only classical L^2 and Fourier transform techniques.

Remark. The article by Srebro and Yakubov in [**104**] describes solutions of (4) when the condition $\|\mu\|_\infty < 1$ is weakened. David's paper [**42**] on that topic has applications to complex dynamics (see Shishikura [**141**]).

1.5. Quasiconformal and quasiregular mappings in $\mathbb{R}^n$. Let μ be an L^∞ function in a plane region Ω. If $k := \|\mu\|_\infty$ is less than one, then the most general solution of (4) in Ω has the form $F \circ w$, where w is a quasiconformal map with complex dilatation μ and F is a holomorphic function in the region $w(\Omega)$. The distributional derivatives of the map $f := F \circ w$ are locally integrable and satisfy the inequality $|f_{\bar{z}}| \le k|f_z|$ in Ω. Such mappings f were once called *quasiconformal functions* (see [**107**]) but are now called *quasiregular mappings*. Topologically, f has the same well-understood covering properties as the holomorphic function F.

The inequality $|f_{\bar{z}}| \le k|f_z|$ occurs in Definitions B and B′ in Chapter II, and both definitions have n-dimensional versions that characterize quasiconformal mappings in $\mathbb{R}^n$ (see Väisälä [**157**]). Dropping the requirement that f be a homeomorphism from the first of them, we get the class of quasiregular mappings in $\mathbb{R}^n$ (see Reshetnyak [**134**] and Rickman [**135**]).

The theories of quasiconformal and quasiregular mappings in $\mathbb{R}^n$ have been extensively developed and are very active today. Pioneers in these theories include Lavrentiev, Gehring, Reshetnyak, and many others. As guides to the literature, we recommend the above-cited books, the Gehring birthday volume [**49**], and Gehring's article in [**104**].

REMARKS. From the point of view of geometric function theory, quasiregular mappings are the natural generalization of holomorphic functions to the $\mathbb{R}^n$ setting. For example, Rickman's book contains a generalization of Nevanlinna's value distribution theory to the quasiregular case.

Quasiconformal methods have also become an indispensable tool in the value distribution theory of meromorphic functions in the plane, as Drasin's papers [**46**] and [**47**] illustrate. For more about that topic, see the article by Drasin, Gol'dberg, and Poggi-Corradini in [**104**]. Readers of this book will also enjoy Drasin's survey article [**48**].

2. Definitions and basic properties of Teichmüller space

Chapter VI of this book contains one of the first published treatments of infinite-dimensional Teichmüller spaces. It was written before the definitions had been standardized, and his choice of definition forces Ahlfors to exclude many Riemann surfaces from consideration.

In this section we mention some current approaches to Teichmüller theory and compare them with the ones in Chapter VI. We consider only Riemann surfaces whose universal covering surface is conformally isomorphic to the upper half-plane H. We call these surfaces hyperbolic, as they all admit a complete Poincaré metric of constant negative curvature.

2.1. The Teichmüller space $T(S_0)$. We begin with the now-standard quasiconformal mapper's definition, due to Lipman Bers. Let S_0 be any hyperbolic Riemann surface, and let $\pi_0\colon H \to S_0$ be a holomorphic universal covering of S_0. We say that a quasiconformal mapping f of S_0 onto itself is *Teichmüller trivial* if it has a lift $\widetilde{f}\colon H \to H$ that fixes the extended real axis pointwise.

Now consider all pairs (S, f), where S is a Riemann surface and f is a quasiconformal mapping of S_0 onto S. We say that (S_1, f_1) and (S_2, f_2) are *equivalent* if there is a conformal map g of S_1 onto S_2 such that the map $f_2^{-1} \circ g \circ f_1$ of S_0 onto itself is Teichmüller trivial. The space of equivalence classes of these pairs is, by definition, the *Teichmüller space* $T(S_0)$.

The *Teichmüller metric* d_T on $T(S_0)$ is defined by setting

$$d_T(t_1, t_2) = \frac{1}{2}\log K, \qquad t_1 \text{ and } t_2 \text{ in } T(S_0), \tag{5}$$

where K is the smallest number such that $f_1 \circ f_2^{-1}$ is K-quasiconformal for some pairs (S_1, f_1) in t_1 and (S_2, f_2) in t_2. The compactness properties of quasiconformal mappings imply that the metric d_T is complete.

With these definitions, the discussion in Sections B and C of Chapter VI applies to every Teichmüller space $T(S_0)$. As in VI C, let Γ be the group of cover transformations of $\pi_0\colon H \to S_0$, and let $B_1(\Gamma)$ be the open unit ball in the Banach space $B(\Gamma)$ of Beltrami differentials. It is shown in VI B and C that the map $\mu \mapsto \Phi(\mu) := \phi_\mu$, μ in $B_1(G)$, induces a homeomorphism of $T(S_0)$ onto a bounded open subset of a complex Banach space $Q(\Gamma)$ (see also Earle [**50**], where Lemma 2 of VI C first appeared). The space $T(S_0)$ is often identified with its homeomorphic image $\Phi(B_1(\Gamma))$ in $Q(\Gamma)$. This realization of $T(S_0)$ is known as the *Bers embedding.*

Bers proved even more about the map Φ in [**19**].

THEOREM 2 (Bers). *The map $\Phi\colon B_1(\Gamma) \to Q(\Gamma)$ is holomorphic, and its derivative at any point of $B_1(\Gamma)$ has a right inverse.*

That result is a cornerstone of the theory of infinite-dimensional Teichmüller spaces. It implies, among other things, that the complex manifold structure induced on $T(S_0)$ by the Bers embedding is precisely the quotient structure induced by the natural complex manifold structure of the open ball $B_1(\Gamma)$. See the book [**129**] for more details and further results.

REMARKS. Our definition of Teichmüller's metric differs in two respects from the definition in Chapter VI A. One is that the definition of $T(S_0)$ itself has been changed. The other is the scaling factor 1/2 in equation (5). Ahlfors uses Teichmüller's original scaling (see [**150**]). We use Royden's rescaling (see [**136**]) for reasons that will become clear in §4.1.

Dragomir Saric has pointed out to us that a quasiconformal mapping f of S_0 onto itself is Teichmüller trivial if and only if there exist a number C and a homotopy f_t from f to the identity such that the Poincaré distance from x to $f_t(x)$ is less than C for all x and t. In fact, a theorem of Earle and McMullen (see [**59**]) implies that every Teichmüller trivial f has this bounded homotopy property, and the converse is easy to prove.

The difference between the definitions of $T(S_0)$ given here and in Chapter VI lies, of course, in how equivalence of pairs is defined. Here, following Bers, we have required $f := f_2^{-1} \circ g \circ f_1 \colon S_0 \to S_0$ to be Teichmüller trivial. In Chapter VI, Ahlfors requires only that f be homotopic to the identity. That condition is the same as Teichmüller triviality only when the group Γ of cover transformations is of the first kind. When Γ is not of the first kind, Ahlfors's equivalence relation produces the so-called reduced Teichmüller space $T^\sharp(S_0)$. See Chapter 5 of the book [**68**] for more details.

There is an important class of Riemann surfaces S_0 for which Γ is of the first kind. We say that S_0 has *finite conformal type* if and only if there are a compact Riemann surface S and a (possibly empty) finite subset E of S such that S_0 is conformally equivalent to $S \setminus E$. The genus p of S and the number n of points in E are uniquely determined, and (p, n) is called the *type* of S_0. A Riemann surface of type (p, n) is hyperbolic if and only if $2p - 2 + n > 0$.

If S_0 is hyperbolic and has type (p, n), then Γ is of the first kind, and $Q(\Gamma)$ and $T(S_0)$ have dimension $3p - 3 + n$. If S_0 does not have finite conformal type, then $Q(\Gamma)$ and $T(S_0)$ are infinite dimensional.

2.2. $T(S_0)$ as an orbit space. Let $QC(S_0)$ be the group of quasiconformal maps of S_0 onto itself, and let $QC_0(S_0)$ be the normal subgroup of Teichmüller trivial mappings. These groups act on the open ball $B_1(\Gamma)$ in the following way.

Each f in $QC(S_0)$ lifts to a quasiconformal map $\widetilde{f}\colon H \to H$ such that the groups Γ and $\widetilde{f}\Gamma\widetilde{f}^{-1}$ are equal. We define $\sigma_f\colon B_1(\Gamma) \to B_1(\Gamma)$ to be the map that sends μ in $B_1(\Gamma)$ to the *Beltrami coefficient* (i.e., the complex dilatation) of the quasiconformal mapping $f^\mu \circ \widetilde{f}^{-1}$.

The map σ_f is well defined because every lift of f has the form $\gamma \circ \widetilde{f}$ for some γ in Γ, and it maps $B_1(\Gamma)$ into itself because $\widetilde{f}\Gamma\widetilde{f}^{-1} = \Gamma$. The proofs of these observations use the chain rule formulas (6) and (8) in Chapter I C. Formula (6) shows that each σ_f is a holomorphic map of $B_1(\Gamma)$ into itself. Since $\sigma_{f^{-1}}$ and σ_f are inverse maps, each σ_f is bijective and biholomorphic.

It follows readily from Lemma 2 in Chapter VI B that μ and ν in $B_1(\Gamma)$ determine the same point in $T(S_0)$ if and only if $\nu = \sigma_f(\mu)$ for some f in $QC_0(S_0)$, so $T(S_0)$ is the space of $QC_0(S_0)$-orbits in $B_1(\Gamma)$.

2.3. The Teichmüller modular groups. Let $\Phi\colon B_1(\Gamma) \to T(S_0)$ be the holomorphic map produced by the Bers embedding. The maps σ_f in §2.2 induce maps $\rho_f\colon T(S_0) \to T(S_0)$ such that $\rho_f \circ \Phi = \Phi \circ \sigma_f$ for all f. Theorem 2 implies that the maps ρ_f are biholomorphic.

The definition in §2.1 of $T(S_0)$ as the space of equivalence classes of pairs yields a simple description of ρ_f: it maps the class of (S, g) to the class of $(S, g \circ f^{-1})$. If f is a sense-reversing quasiconformal self-mapping of S_0, we define ρ_f similarly. It maps the class of (S, g) to the class of $(S^*, j_S \circ g \circ f^{-1})$, where S^* is the conjugate surface of S, and $j_S\colon S \to S^*$ is the canonical conjugation. In this case, ρ_f is conjugate holomorphic. All the maps ρ_f preserve Teichmüller distances.

The group $QC_0(S_0)$ is a normal subgroup of both $QC(S_0)$ and the extended group $QC^*(S_0)$ of all quasiconformal self-mappings of S_0, including the sense-reversing ones. Since ρ_f is the identity map for all f in $QC_0(S_0)$, the maps ρ_f produce an action of the quotient groups $QC(S_0)/QC_0(S_0)$ and $QC^*(S_0)/QC_0(S_0)$ on $T(S_0)$. These groups are the *Teichmüller modular group Mod*(S_0) and the *extended modular group Mod*$^*(S_0)$, respectively.

Remarks. Let *Diff*(S_0) be the group of smooth (C^∞) diffeomorphisms of S_0, and let *Diff*$^+(S_0)$ and *Diff*$_0(S_0)$ be the normal subgroups of diffeomorphisms that are orientation-preserving and homotopic to the identity, respectively. If S_0 has finite conformal type, then every homeomorphism of S_0 onto itself is homotopic to a (possibly sense-reversing) quasiconformal diffeomorphism. Therefore *Mod*(S_0) and *Mod*$^*(S_0)$ are isomorphic to *Diff*$^+(S_0)/$*Diff*$_0(S_0)$ and *Diff*$(S_0)/$*Diff*$_0(S_0)$, respectively. These are often called the mapping class groups.

If f is a quasiconformal map of S_0 onto S_1, the *change of basepoint map* ρ_f from $T(S_0)$ to $T(S_1)$ maps the class of (S, g) to the class of $(S, g \circ f^{-1})$. It also is biholomorphic and preserves Teichmüller distances.

2.4. The compact case: a fiber bundle approach. When S_0 is compact, *Diff*(S_0) is a subgroup of $QC^*(S_0)$. The action of $QC(S_0)$ on $B_1(\Gamma)$ defined in §2.2 restricts to an action of *Diff*$^+(S_0)$ on the set $\mathcal{M}(\Gamma)$ of smooth functions μ in $B_1(\Gamma)$, and the orbit space $\mathcal{M}(\Gamma)/$*Diff*$_0(S_0)$ is the Teichmüller space $T(S_0)$. (The proof of the last statement depends on the fact that f^μ is a smooth diffeomorphism of H when μ is smooth in H (see §1.4).)

When $\mathcal{M}(\Gamma)$ and *Diff*(S_0) are given the C^∞ topology of locally uniform convergence of partial derivatives of all orders, *Diff*(S_0) becomes a topological group, and the quotient map from $\mathcal{M}(\Gamma)$ to $T(S_0)$ defines a principal fiber bundle whose structure group is *Diff*$_0(S_0)$. Earle and Eells used that fact in [**51**] to show that *Diff*$_0(S_0)$ is contractible. Tromba uses it systematically in the book [**155**] to develop many aspects of Teichmüller theory by purely differential geometric methods.

2.5. The compact case: hyperbolic metrics. Each μ in $\mathcal{M}(\Gamma)$ determines a smooth Riemannian metric $ds = |dz + \mu(z)d\bar{z}|$ on H. That metric does not descend to S_0 because it is not Γ-invariant. However, it is conformally equivalent to the smooth constant curvature metric

$$ds = \frac{|f^\mu_z(z)|}{\left|f^\mu(z) - \overline{f^\mu(z)}\right|}|dz + \mu(z)d\bar{z}|,$$

which is Γ-invariant. We can in this way identify $\mathcal{M}(\Gamma)$ with the space of smooth hyperbolic metrics on S_0 (now viewed as an oriented smooth surface).

If $f \in \mathit{Diff}(S_0)$, let σ_f be the map that sends each smooth hyperbolic metric to its pullback by f^{-1}. If $f \in \mathit{Diff}^+(S_0)$, this agrees with the definition of σ_f in §2.2. For all f in $\mathit{Diff}(S_0)$, σ_f induces the map $\rho_f \colon T(S_0) \to T(S_0)$ defined in §2.3.

The interpretation of $T(S_0)$ as the space of smooth hyperbolic metrics on S_0 modulo the action of $\mathit{Diff}_0(S_0)$ has been very fruitful. It is the starting point for Thurston's analysis of surface diffeomorphisms (see §2.6). Classically (see [**66**] or the more recent books [**65**] and [**140**]), it allows compact hyperbolic Riemann surfaces to be described by discrete groups of hyperbolic isometries of H, and $T(S_0)$ becomes a set of isomorphisms from the fundamental group of S_0 onto such discrete groups.

Ahlfors uses that viewpoint in Chapter VI D, which is drawn from his paper [**5**]. The reader should observe that the sets V and T defined in VI D are not connected. The four components of T are determined by whether the fixed points of A_1 and A_2 at 0 and 1 are attracting or repelling. Each component of T is a model for $T(S_0)$, with the natural complex structure described in VI D.

2.6. Thurston's compactification of $T(S_0)$ and classification of diffeomorphisms. We continue to assume that S_0 is compact, and we identify the Teichmüller modular group $\mathit{Mod}(S_0)$ with the quotient group $\mathit{Diff}^+(S_0)/\mathit{Diff}_0(S_0)$.

The elements of that quotient group are the homotopy classes of (orientation-preserving) diffeomorphisms of S_0. They were intensively studied by Nielsen (see [**130**], [**131**], and [**132**]), using methods of hyperbolic geometry. The modern theory began with revolutionary work of Thurston, who defined a natural geometric compactification of $T(S_0)$ on which $\mathit{Mod}(S_0)$ acts as a group of homeomorphisms (see [**33**], [**64**], and [**153**]). We shall denote this compactification by $\overline{T}(S_0)$. It is homeomorphic to a closed Euclidean ball, whose interior consists of the points of $T(S_0)$ and whose boundary sphere is known as the *Thurston boundary*.

As $\overline{T}(S_0)$ is a closed ball, every γ in $\mathit{Mod}(S_0)$ has at least one fixed point. Typically, γ has exactly two fixed points, both on the Thurston boundary. By Thurston's constructions, these boundary points correspond to transverse "measured foliations" on S_0, and γ has a "pseudo-Anosov" representative f that preserves both of them, expanding one of them by a factor K, and contracting the other by the same factor.

We refer to [**64**] and [**153**] for a discussion of these matters. Here we remark only that these pseudo-Anosov mappings have exactly the same geometry and dynamics as the extremal Teichmüller mappings that we discuss in §3.2 below. That fact inspired Bers to find a new proof of Thurston's theorem (see §3.5).

Remarks. The construction of Thurston's boundary from currents in Bonahon [**33**] has been generalized in Saric [**139**], producing a "Thurston boundary" for every $T(S_0)$. The boundary is not compact unless $T(S_0)$ is finite dimensional. When S_0 is compact, the theory of harmonic maps provides another natural construction of Thurston's compactification (see Wolf [**166**]).

In [**75**] and [**76**], W. J. Harvey introduced a simplicial complex $\mathcal{C}(S_0)$, called the *curve complex* of the surface S_0. Its vertices are the isotopy classes of simple closed curves on S_0, and $Mod(S_0)$ acts on $\mathcal{C}(S_0)$ in a natural way. This leads to another proof of Thurston's classification (see Harvey [**77**]), though it does not capture the behavior of the pseudo-Anosov mappings. The curve complex reappears in the study of other aspects of Teichmüller theory (see §4.4 and §5.4).

2.7. A functorial approach to $T(S_0)$. Let S_0 be compact. At the end of Chapter VI C, Ahlfors defines a complex manifold $V(S_0)$ and a holomorphic map π from $V(S_0)$ to $T(S_0)$ such that for each t in $T(S_0)$ the fiber $\pi^{-1}(t)$ is a complex submanifold of $V(S_0)$ isomorphic to the Riemann surface that t represents. The map $\pi\colon V(S_0) \to T(S_0)$ defines a holomorphic family of compact Riemann surfaces, which is known as the *Teichmüller curve*.

The same construction produces a Teichmüller curve over any Teichmüller space $T(S_0)$. (That was well known long before the details were written down in [**52**].) Each Teichmüller curve has a universal property that determines $T(S_0)$ as a complex manifold (see [**71**], [**62**], and [**52**]). The finite-dimensional Teichmüller spaces can therefore be constucted by the methods of algebraic geometry. Grothendieck [**71**] does this when S_0 is compact, and Engber [**62**] does the general finite-dimensional case. We discuss sections of the map $\pi\colon V(S_0) \to T(S_0)$ in §4.5.

3. Extremal quasiconformal mappings and Teichmüller's metric

3.1. Extremal quasiconformal mappings and quadratic differentials. By definition, the *dilatation* $K(f)$ of a quasiconformal mapping f is the smallest number K such that f is K-quasiconformal. If $\mathcal{F}$ is a set of quasiconformal mappings, we call f_0 in $\mathcal{F}$ *extremal* if $K(f_0) \leq K(f)$ for all f in $\mathcal{F}$. Finding the extremal mappings in a given class $\mathcal{F}$ is often an interesting problem.

Ahlfors presents solutions of two such problems in this book. Chapter I opens with Grötzsch's problem, in which $\mathcal{F}$ is the set of quasiconformal diffeomorphisms of a square Q onto a rectangle R that map vertices to vertices in a given order. Grötzsch's solution to this extremal problem in [**72**] is generally regarded as the beginning of the theory of quasiconformal mappings.

The second extremal problem, in Chapter III D, comes from §27 of Teichmüller [**150**]. Here $\mathcal{F}$ is a topological class of quasiconformal self-mappings of $\widehat{\mathbb{C}}$ that carry one given four-point set to another. In both problems, the family $\mathcal{F}$ contains a unique extremal mapping. That mapping is either conformal or an affine map $z \mapsto az + b\bar{z} + c$, when it is viewed in the right local coordinates.

Teichmüller's theory of extremal mappings between compact Riemann surfaces is based on the insight that these mappings will also be affine, with respect to the local coordinates that come from appropriate quadratic differentials. To describe his use of these objects, we must first mention some of their analytic and geometric properties. For more details, see [**68**] or [**146**].

By definition, a quadratic differential on the Riemann surface S is a tensor φ with the local form $f dz^2$, where z is a coordinate chart and f is a holomorphic function on the domain of z. For each such φ, we define a differential 2-form $|\varphi|$ on S by setting $|\varphi| := |f| dx \wedge dy$ in the domain of any chart $z = x + iy$ where $\varphi = f dz^2$.

We say that the quadratic differential φ is *integrable* if $\iint_S |\varphi|$ is finite, and we define $Q^1(S)$ to be the Banach space of integrable quadratic differentials, with the norm

$$\|\varphi\|_1 = \iint_S |\varphi|, \qquad \varphi \text{ in } Q^1(S).$$

Every quadratic differential φ has a well-defined zero set, and the zeros of a nontrivial φ are isolated. If $\varphi = dz^2$ in the domain of the coordinate chart z, we call z a *φ-chart*. When the domains of two φ-charts overlap, the transition functions have the local form $z \mapsto \pm z + c$. Every point outside the zero set of φ is in the domain of some φ-chart.

A curve γ in S is called *horizontal* (resp. *vertical*) if y (resp. x) is locally constant along γ for every φ-chart $z = x + iy$. If z is a φ-chart, then iz is a $(-\varphi)$-chart, so replacing φ by $-\varphi$ converts horizontal curves to vertical curves and vice versa.

REMARK. When the hyperbolic Riemann surface S is compact, the horizontal and vertical curves with respect to a nontrivial quadratic differential φ form the leaves of transverse measured foliations in the sense of Thurston (see Hubbard and Masur [**85**]). The relationship between quadratic differentials and measured foliations is thoroughly explored in that paper and in Kerckhoff [**91**].

3.2. Teichmüller's extremal problem. Let $h : S_1 \to S_2$ be an orientation-preserving homeomorphism between two compact hyperbolic Riemann surfaces. Teichmüller studied the extremal mappings in the set $\mathcal{F}$ of all quasiconformal mappings of S_1 onto S_2 that are homotopic to h.

As $\mathcal{F}$ is not empty (it contains any diffeomorphism homotopic to f), the compactness properties of quasiconformal mappings imply that $\mathcal{F}$ contains at least one extremal mapping. Teichmüller proved much more.

DEFINITION 2. Let S_1 and S_2 be arbitrary hyperbolic Riemann surfaces. A quasiconformal mapping f of S_1 onto S_2 is called a *Teichmüller mapping* if and only if there are a number $K > 1$ and quadratic differentials φ in $Q^1(S_1)$ and ψ in $Q^1(S_2)$ such that

(a) $\|\varphi\|_1 = \|\psi\|_1 = 1$,
(b) if p in S_1 is a zero of φ, then $f(p)$ is a zero of ψ, and
(c) if U is the domain of a φ-chart $z = x + iy$ on S_1, then $f(U)$ is the domain of a ψ-chart $w = u + iv$ on S_2 such that

$$u(f(p)) = \sqrt{K}\, x(p) \ \text{ and } \ v(f(p)) = y(p)/\sqrt{K} \quad \text{for all } p \text{ in } U. \tag{6}$$

We call φ and ψ the *initial and terminal quadratic differentials of f*.

Observe that f^{-1} is a Teichmüller mapping with initial and terminal quadratic differentials $-\psi$ and $-\varphi$. Teichmüller mappings stretch horizontal curves and compress vertical curves by the same factor. They are a beautiful generalization of Grötzch's extremal mappings, and they have the same extremal properties. Teichmüller dealt with the compact case.

THEOREM 3 (Teichmüller). *Let $h : S_1 \to S_2$ be an orientation-preserving homeomorphism between compact hyperbolic Riemann surfaces, and let $\mathcal{F}$ be the set of all quasiconformal mappings of S_1 onto S_2 that are homotopic to h. Then*

(a) *if f_0 in $\mathcal{F}$ is either conformal or a Teichmüller mapping, then f_0 is extremal and is the unique extremal map in $\mathcal{F}$, and*

(b) *conversely, any extremal map in $\mathcal{F}$ is either conformal or a Teichmüller mapping.*

COROLLARY 1 (Teichmüller). *For any compact hyperbolic Riemann surface S_0, the spaces $T(S_0)$ and $Q^1(S_0)$ are homeomorphic.*

To derive the corollary from Theorem 3, Teichmüller maps $T(S_0)$ to the open unit ball of $Q^1(S_0)$ in the following way. Let (S, f) represent a point of $T(S_0)$, and let f_0 be the unique extremal map homotopic to f. If f_0 is conformal, map (S, f) to zero. If f_0 is a Teichmüller mapping, let φ and ψ be its initial and terminal quadratic differentials, let $K > 1$ be the number such that (6) holds, and map (S, f) to the quadratic differential $\frac{K-1}{K+1}\varphi$. Theorem 3 implies readily that this map is a well-defined bijection. Teichmüller shows in [**150**] that it is a homeomorphism. Today's theory of quasiconformal mappings makes that step relatively easy. For details, we recommend the article [**15**] and books [**1**], [**106**] and [**87**] cited in the following remarks.

REMARKS. Teichmüller announced Theorem 3 in [**150**] and proved part (a) there. His proof of (b), based on the method of continuity, is given in [**151**] (see also Bers [**28**]). Teichmüller used a smaller competing class of quasiconformal maps than we use today. Ahlfors's paper [**3**] codified today's geometric definition of quasiconformal mappings and presented proofs of (a) and (b) in the new setting. His proof of (a) is based on Teichmüller's, but his proof of (b) is quite different, relying on a direct variational method.

Bers [**15**] contains a detailed and very readable version of Teichmüller's proof of (a), adapted to the general setting. Later versions of Bers's proof, in Abikoff [**1**], Lehto [**106**], and Imayoshi–Taniguchi [**87**], include a minor amplification from the appendix to Bers [**26**].

Once (a) is given, (b) reduces to the statement that $\mathcal{F}$ contains either a conformal map or a Teichmüller mapping. Bers [**15**] deduces (b) from (a) by a transparent connectedness argument in the Teichmüller space $T(S_1)$; that argument is also used in [**1**] and [**87**]. The proof of (b) in [**106**] uses methods of Krushkal′ and Hamilton, who independently found direct proofs that use functional analysis (see [**101**], [**73**], and §3.6 below).

Theorem 3 and Corollary 1 hold verbatim when S_1 has finite conformal type. Teichmüller [**150**] and Ahlfors [**3**] describe how to reduce the theorem to the compact case by using appropriate branched coverings. See Abikoff [**1**] for more details.

3.3. Geometry of Teichmüller's metric. From now until the end of §3.4, we require S_0 to have finite conformal type. In this case the Teichmüller metric d_T has a rich geometric structure. Using Theorem 3, Kravetz [**100**] showed that $T(S_0)$ with the metric d_T is *straight* in the sense of Busemann. This means that any two distinct points can be joined by a unique geodesic segment, and that segment extends uniquely to an isometric image of the real line. We call that extension a *Teichmüller line* or *geodesic*.

Any two distinct points of $T(S_0)$ lie on a unique *Teichmüller disk*, which is by definition a closed one-dimensional complex submanifold of $T(S_0)$ that is isometric to the unit disk with its Poincaré metric (defined as in §4.1 below). A Teichmüller disk contains the geodesic through any two of its points.

Let L be a one-dimensional real (resp. complex) subspace of $Q^1(S_0)$. Consider the pairs (S, f) such that f is either conformal or a Teichmüller mapping whose initial quadratic differential belongs to L. The set of their equivalence classes is a Teichmüller geodesic (resp. disk) in $T(S_0)$. Every Teichmüller geodesic or disk that contains the basepoint has that form.

The metric d_T was once thought to have negative curvature (see [**100**]), but that is not so. Given any point in $T(S_0)$, Masur found distinct rays that start from that point and do not diverge (see [**116**]). Later, Masur and Wolf showed that $T(S_0)$ with the metric d_T is not Gromov hyperbolic (see [**122**]).

Since the maps $\rho_f\colon T(S_0) \to T(S_0)$ defined in §2.3 preserve Teichmüller distances, they map geodesics to geodesics. Since they are biholomorphic or conjugate holomorphic diffeomorphisms, they also map Teichmüller disks to Teichmüller disks. If ρ_f fixes two distinct points, it will map the Teichmüller disk that contains them onto itself.

3.4. Dynamics of Teichmüller's metric. The classical geodesic and horocyclic flows in the unit disk with its Poincaré metric (see [**81**] and [**78**]) have counterparts in $T(S_0)$, involving the quadratic differentials associated with Teichmüller geodesics. Masur introduced them in [**119**] and [**120**] and continued to study them in such papers as [**118**] and [**121**]. The latter paper uses properties of the geodesic flow to solve a seemingly unrelated problem about interval exchange maps. Veech solved the same problem simultaneously and independently without explicitly using quadratic differentials (see [**158**]), but he soon turned his attention to them. See for example [**159**] and [**161**].

In [**90**], Kerckhoff, Masur, and Smillie opened a new area by using the geodesic and horocyclic flows on Teichmüller disks to study the dynamics of quadratic differentials. This had applications to the dynamics of rational billiards. Further progress was stimulated by Veech's discovery of a large family of Teichmüller disks, related to triangular billiards, such that the quotient of the disk by the subgroup of $\mathit{Mod}(S_0)$ that maps it onto itself is a Riemann surface of finite conformal type (see Veech [**160**]). This field is too lively for us to summarize. We refer the reader to [**86**] and the papers in its extensive bibliography.

3.5. Bers's extremal problem. Teichmüller's theorem and Thurston's classification of surface diffeomorphisms inspired Bers to pose a new extremal problem whose solution gives a new proof of the classification. We describe it here.

In this subsection, all homeomorphisms between Riemann surfaces will be orientation-preserving. If a homeomorphism h is not quasiconformal, we define $K(h)$ to be $+\infty$.

Let S_0 be a hyperbolic Riemann surface of finite conformal type, and let f be a homeomorphism of S_0 onto itself. Bers's problem, posed in [**25**], is to minimize

$$K(\sigma \circ \widetilde{f} \circ \sigma^{-1}) \tag{7}$$

as $\widetilde{f}$ varies over the homotopy class of f and σ varies over all homeomorphisms of S_0 onto (variable) Riemann surfaces S.

Bers shows in §3 of [**25**] that the infimum of the numbers (7) is unchanged if the target Riemann surfaces S are required to have finite conformal type. The mappings f, σ, and $\widetilde{f}$ can then be required to be quasiconformal, and the extremal problem takes the following equivalent form (see §4 of [**25**]).

Let S_0 be a hyperbolic Riemann surface of finite conformal type, and let f be a quasiconformal map of S_0 onto itself. Minimize the function

$$t \mapsto d_T(t, \rho_f(t)), \qquad t \text{ in } T(S_0). \tag{8}$$

It is clear that the function defined by (8) is continuous on $T(S_0)$. Its infimum is a nonnegative real number, which we shall denote by $\alpha(f)$. We consider three cases.

Case 1. Suppose $t_0 \in T(S_0)$ and $d_T(t_0, \rho_f(t_0)) = \alpha(f) = 0$. Then ρ_f fixes t_0. By a change of basepoint (see §2.3), we may assume t_0 is the basepoint of $T(S_0)$. Then f is homotopic to a conformal map f_0 of S_0 onto itself. Since f_0 has finite order, so does $\rho_{f_0} = \rho_f$. Conversely, if ρ_f has finite order, then it has a fixed point in $T(S_0)$, and the infimum $\alpha(f)$ is achieved at that point. (Kravetz proves this on pages 26 and 27 of [**100**] without using his false result about the negative curvature of d_T.)

Case 2. Suppose $t_0 \in T(S_0)$ and $d_T(t_0, \rho_f(t_0)) = \alpha(f) > 0$. Again we assume t_0 is the basepoint. In this case, ρ_f has infinite order. Bers shows that ρ_f maps the Teichmüller geodesic determined by t_0 and $\rho_f(t_0)$ onto itself. In addition, if φ is the initial quadratic differential of the Teichmüller mapping $f_0 \colon S_0 \to S_0$ homotopic to f, then $-\varphi$ is the terminal quadratic differential. That means f_0 expands in the horizontal directions of φ and contracts in the vertical directions by the same factor (see (6)), so f_0 is a pseudo-Anosov mapping in Thurston's sense (see §2.6).

Conversely, Bers shows that if ρ_f maps a Teichmüller geodesic onto itself and has infinite order, then Case 2 occurs and $d_T(t, \rho_f(t)) = \alpha(f)$ for all t on this invariant geodesic. The invariant geodesic is unique (see [**112**]) and is called the *axis* of ρ_f.

Case 3. Suppose $d_T(t, \rho_f(t)) > \alpha(f)$ for all t in $T(S_0)$. In this case, ρ_f has infinite order and is *reducible*, which means that there exist an f_0 homotopic to f and a nonempty set $\mathcal{C}$ of simple closed geodesics on S_0 such that $f_0(\mathcal{C}) = \mathcal{C}$. To solve his extremal problem in this situation, Bers augments $T(S_0)$ by adding "noded Riemann surfaces" (intuitively, the curves in $\mathcal{C}$ are pinched to points). We refer to [**25**] or [**1**] for the details, which are too technical for this article. The augmented Teichmüller space $\widehat{T}(S_0)$ described in [**1**] will reappear in §5.2.

Bers calls ρ_f *elliptic* in Case 1, *hyperbolic* in Case 2, and *parabolic* or *pseudo-hyperbolic* in Case 3, according as $\alpha(f)$ is zero or positive.

Remark. If f is homotopic to a pseudo-Anosov mapping, then the axis of ρ_f determines a Teichmüller disk that is mapped onto itself by ρ_f. Conversely, suppose ρ_f maps a Teichmüller disk $\mathbb{D}$ to itself. Its restriction to $\mathbb{D}$ can be regarded as a Möbius transformation A on the unit disk D. If A is hyperbolic, its axis is a Poincaré geodesic in D. Viewed in $\mathbb{D}$, it is a ρ_f-invariant Teichmüller geodesic, so f is homotopic to a pseudo-Anosov mapping. The special Teichmüller disks discussed at the end of §3.4 therefore produce an abundance of pseudo-Anosov mappings.

3.6. Extremal mappings in the general case. The natural generalization of Teichmüller's extremal problem to Riemann surfaces of nonfinite conformal type is to study the extremal mappings in the Teichmüller equivalence class of a given quasiconformal map $f \colon S_0 \to S_1$. Their existence is guaranteed by the compactness properties of quasiconformal mappings. The problem is to characterize them.

It was solved by work of Hamilton, Krushkal′, Reich, and Strebel. Their solution applies to all hyperbolic Riemann surfaces. Once more, it involves quadratic differentials. To describe it, we again choose a holomorphic universal covering $\pi_0\colon H \to S_0$ of S_0, and we denote the group of cover transformations by Γ.

The quadratic differentials on S_0 lift to Γ-invariant quadratic differentials on H. These have the form $\phi(z)dz^2$, where ϕ is a holomorphic function on H such that $(\phi \circ \gamma)(\gamma')^2 = \phi$ for all γ in Γ. As in Chapter VI, we call these functions ϕ quadratic differentials. To avoid ambiguity, we shall use the notation $Q^1(\Gamma)$ for the Banach space that Ahlfors denotes by $Q(\Gamma)$ in VI D. It consists of the quadratic differentials ϕ with finite L^1 norm

$$\|\phi\|_1 = \iint_{S_0} |\phi(z)|\, dx dy. \tag{9}$$

Integrals such as the one in (9) are taken over some (possibly infinite-sided) fundamental polygon for Γ in H.

There is an obvious isometric isomorphism of $Q^1(\Gamma)$ onto the space $Q^1(S_0)$ of integrable quadratic differentials on S_0: send ϕ in $Q^1(\Gamma)$ to the quadratic differential on S_0 whose lift to H is $\phi(z)dz^2$.

Now let $f\colon S_0 \to S_1$ be quasiconformal, let $\pi_1\colon H \to S_1$ be a holomorphic universal covering, and let $\widetilde{f}\colon H \to H$ be a lift of f. The Beltrami coefficient μ of $\widetilde{f}$ belongs to the open unit ball $B_1(\Gamma)$ and is independent of the choices of π_1 and $\widetilde{f}$. We call μ the *Beltrami coefficient of f*.

Theorem 4 (Hamilton–Krushkal′–Reich–Strebel). *Let S_0 and S_1 be hyperbolic Riemann surfaces. The quasiconformal map f of S_0 onto S_1 is extremal in its Teichmüller class if and only if its Beltrami coefficient μ satisfies*

$$\|\mu\|_\infty = \sup\left\{\left|\iint_{S_0} \mu(z)\phi(z)\, dx dy\right| : \phi \textit{ in } Q^1(\Gamma) \textit{ and } \|\phi\|_1 \le 1\right\}. \tag{10}$$

Remarks. Equation (10) is known as the Hamilton–Krushkal′ condition. Its necessity for extremality was proved independently by Hamilton and Krushkal′ (see [**101**], [**73**], and [**102**]). Reich and Strebel proved its sufficiency in [**133**]. That paper considers only the open unit disk, but its methods extend to all hyperbolic Riemann surfaces (see [**145**] or [**68**]).

If f is extremal and the supremum in (10) is attained at some ϕ, then f is either conformal or a Teichmüller mapping, and it is the unique extremal mapping in its Teichmüller class. If f is extremal and the supremem in (10) is not attained, then f may or may not be uniquely extremal.

For many years, all known uniquely extremal maps had the general form of Teichmüller mappings, but with the quadratic differentials φ and ψ in Definition 2 not necessarily integrable. The paper [**34**] by Bozin, Lakic, Markovic, and Mateljevic changed the situation dramatically. They give necessary and sufficient conditions for unique extremality, and they give examples of uniquely extremal maps whose Beltrami coefficients do not even have constant absolute value. For a wealth of information on these and related matters, see Reich's article in [**103**].

Even though Theorem 3 fails in the general case, every Teichmüller space $T(S_0)$ is contractible. See [**45**] and Tukia [**156**] for two quite different proofs.

4. Royden's theorems

Let S_0 be any hyperbolic Riemann surface. We saw in §2.3 that the Teichmüller modular group $Mod(S_0)$ acts on the Teichmüller space $T(S_0)$ as a group of biholomorphic mappings. In this section, we discuss two important theorems of Royden, which together imply the following remarkable result.

THEOREM 5 (Royden). *Let S_0 be a compact hyperbolic Riemann surface. Every biholomorphic map of $T(S_0)$ onto itself is induced by some element of $Mod(S_0)$.*

We shall also discuss the noncompact case.

4.1. The Teichmüller and Kobayashi metrics. Every complex manifold carries a canonical pseudometric generally called the Kobayashi metric. Royden proved in [**136**] that if S_0 is compact and hyperbolic, then the Kobayashi metric of $T(S_0)$ is the Teichmüller metric.

Kobayashi's metric is defined as follows. First, following Kobayashi [**93**] and [**94**], we define the Poincaré metric d_D on the open unit disk D as the arc length metric determined by the infinitesimal metric $ds = (1-|z|^2)^{-1}|dz|$. (This differs from the Poincaré metric in Chapters I–VI by a factor of two.)

For any complex manifolds X and Y, let $\mathcal{O}(X,Y)$ be the set of holomorphic maps of X into Y. A pseudometric d on X is called a *Schwarz–Pick metric* if

$$d(f(z), f(z')) \leq d_D(z, z') \quad \text{for all } f \text{ in } \mathcal{O}(D,X) \text{ and all } z \text{ and } z' \text{ in } D.$$

By definition, the *Kobayashi metric* d_X is the largest Schwarz–Pick metric on X. (That is, for any x and x', $d_X(x,x')$ is the supremum of the numbers $d(x,x')$ over all Schwarz–Pick metrics d on X.) For example, the Kobayashi metric of D is the Poincaré metric, and the Kobayashi metric of $\mathbb{C}$ is the zero pseudometric.

For any complex manifolds X and Y, it is obvious that

$$d_Y(f(x), f(x')) \leq d_X(x, x') \quad \text{for all } f \text{ in } \mathcal{O}(X,Y) \text{ and all } x \text{ and } x' \text{ in } X. \tag{11}$$

In particular, biholomorphic maps preserve Kobayashi distances.

THEOREM 6 (Royden–Gardiner). *For any hyperbolic Riemann surface S_0, the Kobayashi metric of $T(S_0)$ equals the Teichmüller metric.*

COROLLARY 2. *If S_0 and S_1 are hyperbolic Riemann surfaces, then every biholomorphic map of $T(S_0)$ onto $T(S_1)$ preserves Teichmüller distances.*

REMARKS. We have been using the symbol d_T for Teichmüller's metric on $T(S_0)$, and now we have introduced the symbol d_X for the Kobayashi metric on any X. Theorem 6 resolves the potential conflict of notation.

Royden's proof of Theorem 6 for S_0 compact carries over readily to S_0 of finite conformal type. The proof for S_0 of nonfinite conformal type follows by an approximation argument due to Gardiner. See [**68**] for details.

In [**56**], Slodkowski's theorem about extending holomorphic motions is used to prove that, for any $T(S_0)$, every f in $\mathcal{O}(D, T(S_0))$ has the form $f = \Phi \circ g$, where $g \in \mathcal{O}(D, B_1(\Gamma))$ and $\Phi \colon B_1(\Gamma) \to T(S_0)$ is the Bers map of Theorem 2. Theorem 6 follows readily because the Teichmüller metric is the quotient metric on $T(S_0)$ induced by the Kobayashi metric on $B_1(\Gamma)$ and the Bers map (see [**56**]).

4.2. The infinitesimal metric. Corollary 2 reduces the study of biholomorphic self-mappings of $T(S_0)$ to the study of biholomorphic Teichmüller isometries. Royden's analysis in [**136**] is based on the observation that the Teichmüller metric is an arc-length metric. Biholomorphic Teichmüller isometries preserve the infinitesimal Teichmüller metric, which we shall now describe.

Let Γ be the group of cover transformations for a holomorphic universal covering of S_0 by H. As in §3.6, let $Q^1(\Gamma)$ be the Banach space of quadratic differentials ϕ on H with finite L^1 norm (9). As in §2.1 and VI C, let $Q(\Gamma)$ be the Banach space of holomorphic functions ψ on H^* that satisfy $(\psi \circ \gamma)(\gamma')^2 = \psi$ for all γ in Γ and have finite norm

$$\|\psi\|_\infty = \sup\{y^2|\psi(z)| : z = x + iy \text{ in } H^*\}. \tag{12}$$

The Bers embedding of $T(S_0)$ as a domain in $Q(\Gamma)$ allows us to regard $Q(\Gamma)$ as the tangent space to $T(S_0)$ at its basepoint $\Phi(0)$, but the norm (12) is not the infinitesimal Teichmüller metric.

By another theorem of Bers (see [**18**]), the formula

$$L(\psi)\phi = \iint_{S_0} \psi(\bar{z})\phi(z)y^2\, dxdy, \quad \psi \text{ in } Q(\Gamma) \text{ and } \phi \text{ in } Q^1(\Gamma),$$

defines an isomorphism L of $Q(\Gamma)$ onto the dual space $Q^1(\Gamma)^*$ of $Q^1(\Gamma)$. We shall therefore identify $Q^1(\Gamma)^*$ (or, equivalently, $Q^1(S_0)^*$) with the tangent space to $T(S_0)$ at its basepoint. That has two advantages. First, it identifies the cotangent space with $Q^1(S_0)$ in the finite-dimensional case. Second, the standard norm

$$\|\ell\| = \sup\{|\ell(\varphi)| : \varphi \text{ in } Q^1(S_0) \text{ and } \|\varphi\|_1 \le 1\}, \quad \ell \text{ in } Q^1(S_0)^*,$$

on $Q^1(S_0)^*$ is precisely the infinitesimal Teichmüller metric for tangent vectors at the basepoint of $T(S_0)$.

Remark. Every complex manifold has an infinitesimal Kobayashi metric. If X is a domain in a complex Banach space V, the *Kobayashi length* of a tangent vector v in V at a point x in X is the number

$$F_X(x, v) := \inf\{|t| : \text{some } f \text{ in } \mathcal{O}(D, X) \text{ satisfies } f(0) = x \text{ and } f'(0)t = v\}.$$

The arc-length pseudometric determined by the function F_X is the Kobayashi metric, and biholomorphic mappings preserve the Kobayashi lengths of all tangent vectors. (See Kobayashi [**94**], Harris [**74**], or Dineen [**43**].)

The lifting theorem in [**56**] implies that the infinitesimal Kobayashi and Teichmüller metrics on $T(S_0)$ are equal. This provides another proof that biholomorphic maps preserve the infinitesimal Teichmüller metric.

4.3. Isometries between the spaces $Q^1(S_0)$. Let S_0 and S_1 be hyperbolic Riemann surfaces, and let $f : T(S_0) \to T(S_1)$ be a biholomorphic map that maps basepoint to basepoint. The previous discussion shows that the derivative of f at the basepoint of $T(S_0)$ is a $\mathbb{C}$-linear isometry of $Q^1(S_0)^*$ onto $Q^1(S_1)^*$. In the finite-dimensional case, it follows at once that the adjoint of that derivative is a $\mathbb{C}$-linear isometry of $Q^1(S_1)$ onto $Q^1(S_0)$.

In this paragraph, all isometries will be surjective and $\mathbb{C}$-linear. There are two obvious types of such isometries between certain spaces $Q^1(S_1)$ and $Q^1(S_0)$. The map $\varphi \mapsto c\varphi$ is an isometry of $Q^1(S_0)$ onto itself whenever c is a complex number of absolute value one. If f is a conformal map of S_0 onto S_1, each φ in $Q^1(S_1)$ can be pulled back to a quadratic differential $f^*(\varphi)$ on $Q^1(S_0)$, and the map $\varphi \mapsto f^*(\varphi)$ is an isometry.

We say a Riemann surface has *exceptional type* if it has finite conformal type (p, n) and $2p + n < 5$. All nonhyperbolic Riemann surfaces have exceptional type.

THEOREM 7 (Royden–Lakic–Markovic). *Let S_0 and S_1 be Riemann surfaces, and let $L\colon Q^1(S_1) \to Q^1(S_0)$ be a surjective $\mathbb{C}$-linear isometry. If neither S_0 nor S_1 has exceptional type, there are a complex number c and a conformal map f of S_0 onto S_1 such that $|c| = 1$ and $L(\varphi) = cf^*(\varphi)$ for all φ in $Q^1(S_1)$.*

REMARKS. Theorem 7 holds whenever S_0 and S_1 are compact and hyperbolic, even though compact Riemann surfaces of genus two have the exceptional type $(2, 0)$.

Royden proved Theorem 7 in [**136**] for S_0 and S_1 compact and hyperbolic by studying the shape of the unit sphere in $Q^1(S_i)$. Its shape at a point φ is determined by the orders of the zeros of φ.

That method extends readily to Riemann surfaces of nonexceptional finite conformal type, even if S_0 and S_1 are not assumed to be homeomorphic. (See [**54**], where it is also shown that surjective $\mathbb{R}$-linear isometries are either $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear for surfaces of nonexceptional finite conformal type.)

Riemann surfaces of infinite conformal type present greater difficulties. Royden's methods can be refined to handle surfaces of infinite conformal type and finite genus (see Lakic [**105**]), but the infinite genus case was solved by quite different means.

In [**113**], Markovic proved Theorem 7 for all Riemann surfaces of infinite conformal type. Using considerable technical power, he obtained the required conformal map from a general theorem of Rudin [**138**]. When applied to surfaces of finite conformal type, his method becomes quite transparent; Rudin's theorem produces the desired conformal map very readily in that case (see [**57**]).

4.4. The proof of Theorem 5. Royden [**136**] gives the following proof of Theorem 5. Together, Theorems 6 and 7 show that for each τ in $T(S_0)$ there is a g_τ in $Mod(S_0)$ such that $f(\tau) = \rho_{g_\tau}(\tau)$. As $Mod(S_0)$ acts properly discontinuously on $T(S_0)$ when S_0 is compact (see, e.g., [**51**], [**68**], or [**100**]), ρ_{g_τ} is independent of τ.

REMARKS. Royden's argument applies verbatim if S_0 is not compact but has nonexceptional finite conformal type. If the conformal type of S_0 is not finite, the path from Theorem 7 to the classification of biholomorphic Teichmüller isometries is more intricate, but the outcome is the same. Markovic's general version of Theorem 7 implies that every biholomorphic map between two Teichmüller spaces $T(S_0)$ and $T(S_1)$ is induced by a quasiconformal map between S_0 and S_1 unless at least one of them has exceptional type (see [**113**]).

Ivanov [**88**] gives a quite different approach to Theorem 5, based on properties of Harvey's curve complex $\mathcal{C}(S_0)$. First, he shows that every automorphism of $\mathcal{C}(S)$ comes from the extended modular group $Mod^*(S_0)$. Next he shows that every Teichmüller isometry of $T(S_0)$ onto itself induces an automorphism of $\mathcal{C}(S_0)$. For that, he uses properties of Teichmüller geodesic rays that follow from methods and results of Kerckhoff [**91**] and Masur [**116**] and [**118**]. When combined with Theorem 6, Ivanov's results give a new proof of Theorem 5.

4.5. An application of Theorem 6 to the Teichmüller curves. Let S_0 be any hyperbolic Riemann surface, and let $S_0' := S_0 \setminus \{a\}$ for some point a in S_0.

The Bers isomorphism theorem (see [**23**]) and the description of the Teichmüller curve $\pi\colon V(S_0) \to T(S_0)$ in [**52**] together imply that the universal covering space of $V(S_0)$ is biholomorphically equivalent to $T(S_0')$. Every holomorphic section of the map π therefore determines a holomorphic map from $T(S_0)$ to $T(S_0')$, to which Theorem 6 and the inequality (11) apply. Using that fact, Hubbard showed in [**82**] that if S_0 has finite conformal type (p,n) with $p \geq 2$, then $\pi\colon V(S_0) \to T(S_0)$ has no holomorphic sections except for the obvious "Weierstrass sections" in the $(2,0)$ case. Some generalizations of Hubbard's results are proved in [**55**].

5. Weil–Petersson geometry

André Weil suggested using the Petersson inner product of quadratic differentials to define a Kähler metric on the Teichmüller space of a compact Riemann surface (see [**164**] and [**165**]). This idea has been very productive. In this section we shall report some basic facts about the Weil–Petersson metric and give references for further results in this active area.

5.1. The Weil–Petersson metric. Let $\pi\colon H \to S_0$ be a holomorphic universal covering of the hyperbolic Riemann surface S_0, and let Γ be the group of cover transformations. The Hilbert space $Q^2(\Gamma)$ of square integrable quadratic differentials consists of the holomorphic functions φ on H such that $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ for all γ in Γ and the L^2 norm $\|\varphi\|_2$ defined by

$$\|\varphi\|_2^2 = \iint_{S_0} |\varphi(z)|^2 y^2 \, dx\,dy \qquad (z = x + iy)$$

is finite. The associated inner product

$$\langle \varphi, \psi \rangle = \iint_{S_0} \varphi(z)\overline{\psi(z)} y^2 \, dx\,dy, \quad \varphi \text{ and } \psi \text{ in } Q^2(\Gamma),$$

is the *Petersson inner product* on $Q^2(\Gamma)$.

In general, the vector spaces $Q^2(\Gamma)$ and $Q^1(\Gamma)$ are not equal, though their intersection is dense in both of them. However, when S_0 has finite conformal type, $Q^2(\Gamma)$ and $Q^1(\Gamma)$ are the same finite-dimensional space, so the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are equivalent. The Petersson inner product therefore defines an Hermitian inner product on the cotangent space to $T(S_0)$ at its basepoint. Duality defines an inner product on the tangent space at the basepoint, and changes of basepoint produce an inner product on the tangent space at every point of $T(S_0)$. These inner products define the *Weil–Petersson* (*or WP*) *metric* on $T(S_0)$ whenever S_0 has finite conformal type. We shall discuss only the compact case.

Ahlfors proved in [**5**] that the Weil–Petersson metric is a real analytic Kähler metric on $T(S_0)$. See also Ahlfors [**6**], Royden [**137**], Tromba [**155**], and Wolpert [**169**]. The computations in [**5**], [**137**], [**155**], and [**169**] all use special local coordinates at the basepoint. These are precisely the coordinates given by the Bers embedding.

Weil never published his proof of the Kähler property.

Remark. For any Beltrami differential ν in $B(\Gamma)$, define the linear functional ℓ_ν on $Q^1(\Gamma)$ by the formula

$$\ell_\nu(\psi) = \iint_{S_0} \nu(z)\psi(z) \, dx\,dy, \quad \psi \text{ in } Q^1(\Gamma).$$

All linear functionals on $Q^1(\Gamma)$, hence all tangent vectors to $T(S_0)$ at its basepoint, have that form, and the WP norm of ℓ_ν is

$$\|\ell_\nu\|_{\mathrm{WP}}^2 = \sup\{|\ell_\nu(\psi)|^2 : \psi \text{ in } Q^1(\Gamma) \text{ and } \|\psi\|_2 = 1\}.$$

As in Chapter VI D, let $N(\Gamma)$ be the kernel of the map $\nu \mapsto \ell_\nu$. Define $\phi[\nu]$ in $Q^1(\Gamma)$ by equation (4) in VI D. Ahlfors proves in VI D that the operators Λ and Λ^* defined there satisfy $\nu - \Lambda^*\Lambda\nu \in N(\Gamma)$ for all ν in $B(\Gamma)$. Therefore

$$\ell_\nu(\psi) = -\iint_{S_0} \psi(z)\overline{\phi[\nu](z)}\,dx\,dy = -\langle \psi, \phi[\nu]\rangle \text{ for all } \psi \text{ in } Q^1(\Gamma),$$

so

$$\|\ell_\nu\|_{\mathrm{WP}} = \|\phi[\nu]\|_2 \text{ and } \langle \ell_\nu, \ell_\mu\rangle_{\mathrm{WP}} = \langle \phi[\mu], \phi[\nu]\rangle, \quad \mu \text{ and } \nu \text{ in } B(\Gamma).$$

These observations are the starting points for Ahlfors's calculations in [**5**].

5.2. Completion of the WP metric and compactification of the moduli space. The WP metric on $T(S_0)$ is not complete (see Chu [**40**] and Wolpert [**167**]). Masur showed in [**117**] that it extends to a complete metric on the augmented Teichmüller space $\widehat{T}(S_0)$. The action of $Mod^*(S_0)$ on $T(S_0)$ also extends to $\widehat{T}(S_0)$.

Let $g \geq 2$ be the genus of S_0. The quotient space $T(S_0)/Mod(S_0)$ is the moduli space $\mathfrak{M}_g$ of compact Riemann surfaces of genus g, and $\overline{\mathfrak{M}}_g := \widehat{T}(S_0)/Mod(S_0)$ is the moduli space of noded Riemann surfaces of genus g (see Bers [**27**]). Both $\mathfrak{M}_g$ and $\overline{\mathfrak{M}}_g$ are V-manifolds, $\overline{\mathfrak{M}}_g$ is compact, and $\overline{\mathfrak{M}}_g \setminus \mathfrak{M}_g$ is a finite union of compact V-manifolds of codimension one (see [**27**] and Wolpert [**168**]).

Since $Mod^*(S_0)$ acts on $T(S_0)$, hence on $\widehat{T}(S_0)$, as a group of WP isometries, the WP metric descends to $\overline{\mathfrak{M}}_g$. In an important series of papers, Wolpert studies the WP Kähler form on $\mathfrak{M}_g$ and $\overline{\mathfrak{M}}_g$ and shows how its properties lead to an embedding of $\overline{\mathfrak{M}}_g$ in complex projective space (see [**168**] and the papers cited there).

5.3. Curvature of the WP metric and the Nielsen realization problem. Ahlfors proved in [**6**] that the scalar, Ricci, and holomorphic sectional curvatures of the WP metric are all negative, and Royden [**137**] gives a negative upper bound for the last of these. Tromba, Wolpert, and Royden (unpublished) proved that all sectional curvatures are negative (see [**155**] and [**169**]). See also Jost [**89**].

Although it is not complete, the WP metric on $T(S_0)$ resembles a complete negatively curved metric. By studying "geodesic length functions" along WP geodesics, Wolpert showed in [**170**] that every pair of points in $T(S_0)$ is joined by a unique WP geodesic, the WP exponential map is a homeomorphism, and every finite group of WP isometries has a fixed point in $T(S_0)$. In particular, every finite subgroup of $Mod^*(S_0)$ has a fixed point. That fixed point theorem was first proved by Kerckhoff, using the behavior of geodesic length functions along Thurston's earthquake paths (see [**92**]). Later, Tromba gave a third proof, using a combination of his own and Wolf's approaches to the Teichmüller theory of harmonic maps (see Tromba [**155**] and Wolf [**166**]).

The fixed point result solves Nielsen's realization problem (see for instance [**100**]).

THEOREM 8 (Nielsen Realization Theorem). *Every finite subgroup of $Mod^*(S_0)$ is the isomorphic image of a finite subgroup of $Diff(S_0)$ under the natural homomorphism $\theta\colon Diff(S_0) \to Diff(S_0)/Diff_0(S_0)$.*

REMARKS. A quite different proof of Theorem 8 is given in Gabai [**67**].

Tromba [**154**] and Wolf [**166**] reveal close connections between harmonic maps and the WP metric. Jost used formulas from these papers in Chapter 6 of [**89**], where he derives much of the theory of $T(S_0)$ and its WP metric from the theory of harmonic maps.

5.4. The WP isometry group. As the WP metric is Hermitian, its isometry group cannot be studied by the infinitesimal approach that works for the Teichmüller metric. Ivanov's results about Harvey's curve complex $\mathcal{C}(S_0)$ are much more relevant.

The connection between $\mathcal{C}(S_0)$ and the WP metric is that the incompleteness of the WP metric on $T(S_0)$ is caused by the pinching of geodesics on Riemann surfaces (see [**40**] and [**167**]). In fact, $\widehat{T}(S_0) \setminus T(S_0)$ is the union of strata, each described by a set of pinched geodesics, and each stratum is WP convex. (See Wolpert's comprehensive survey [**171**] for a precise discussion of these and related facts.) It is plausible that a given WP isometry will permute the strata, inducing an automorphism of $\mathcal{C}(S_0)$. That automorphism comes from an element of $Mod^*(S_0)$, which induces a WP isometry. The induced isometry should be the same as the one we started with.

In their paper [**123**], Masur and Wolf accomplish the difficult task of turning the plausibility argument above into a proof of the following remarkable theorem. For another proof, following the same outline but with different details, see Wolpert [**171**].

THEOREM 9. *Every WP isometry of $T(S_0)$ is induced by an element of the extended modular group $Mod^*(S_0)$.*

REMARKS. Although we have been requiring S_0 to be compact, many of the stated results about WP geometry hold for S_0 of finite conformal type. In particular, the proofs of Theorem 9 in [**123**] and [**171**] apply to $T(S_0)$ whenever S_0 has type $(p, n) \neq (1, 2)$ with $3p-3+n > 1$. The missing cases $(p, n) = (0, 4), (1, 1),$ and $(1, 2)$ are covered by Brock and Margalit in [**35**]. They use the "pants graph" instead of $\mathcal{C}(S_0)$ (see [**35**]).

Wolpert's survey article [**172**] is an update of [**171**]. Among other things, it calls attention to Mirzakhani's new methods in WP geometry (see [**126**] and [**127**]) and to results about related metrics (see [**125**] and [**108**]). For other aspects of WP geometry, see [**152**] and its extensive bibliography.

6. Finitely generated Kleinian groups

Sections B, C, and D of Chapter VI invite a discussion of finitely generated Kleinian groups. The "Fuchsoid" groups used in VI B and C are important examples of such groups (see §6.5), and Lemma 1 in VI D contains ideas that led to the proof of Ahlfors's finiteness theorem (see §6.2).

6.1. Definitions. Until we move to three dimensions in §6.6, a *Kleinian group* in our terminology will be a group Γ of Möbius transformations that acts properly discontinuously on a nonempty open subset of the Riemann sphere. Its *ordinary set* or *region of discontinuity* Ω is the maximal open set on which it acts properly discontinuously. Its *limit set* Λ is the complement of Ω in $\widehat{\mathbb{C}}$. If Λ contains at most two points, we say that Γ is *elementary.*

We shall assume that Γ is finitely generated and is not elementary. This implies that the set Λ is perfect and nowhere dense.

The complex analytic theory concentrates attention on the action of Γ on Ω. The quotient Ω/Γ is a union of Riemann surfaces, and the natural projection of Ω onto Ω/Γ is a branched covering map.

6.2. The Ahlfors finiteness theorem. First we recall part of Chapter VI D.

Let S_0 be a compact hyperbolic Riemann surface, $\pi_0\colon H \to S_0$ a holomorphic universal covering, and Γ the Fuchsian group of cover transformations. Given a Beltrami differential ν in $B(\Gamma)$, Ahlfors defines for each A in Γ a real polynomial P_A of degree at most two so that

$$P_{AB} = \frac{P_A \circ B}{B'} + P_B \qquad \text{for all } A \text{ and } B \text{ in } \Gamma. \tag{13}$$

In Lemma 1 of VI D, he shows that for a given ν the polynomials P_A are all zero if and only if

$$\iint_{S_0} \nu(z)\varphi(z)\,dxdy = 0 \qquad \text{for all } \varphi \text{ in } Q^1(\Gamma).$$

That fact has an interesting consequence. For any $n \geq 0$, let Π_n be the $(n+1)$–dimensional vector space of polynomial functions $P(z)$ of degree at most n. As Γ is finitely generated, the vector space of maps $A \mapsto P_A$ from Γ to Π_2 that satisfy (13) is finite dimensional. By the lemma, its dimension is an upper bound for the dimension of $Q^1(\Gamma)$.

Ahlfors does not pursue these ideas in VI D or the paper [**5**] on which VI D is based. They reappear in his groundbreaking paper [**7**], where he initiated the modern theory of Kleinian groups by proving

THEOREM 10 (The Ahlfors Finiteness Theorem). *If Γ is a finitely generated nonelementary Kleinian group, then Ω/Γ is a finite union of Riemann surfaces of finite conformal type and the natural projection $\Omega \to \Omega/\Gamma$ is ramified over finitely many points.*

To prove Theorem 10, Ahlfors used a powerful extension of the techniques by which he proved Lemma 1 in VI D. His proof has two minor defects. First, as it considers only the space $Q^1(\Gamma)$, it does not exclude the presence of infinitely many thrice-punctured spheres in Ω/Γ. Second, formula (7.4) in [**7**] ignores the boundary term in Stokes's theorem. These gaps were soon filled. The boundary term in (7.4) is in fact zero (see [**17**]). The number of thrice-punctured spheres was shown to be finite in Greenberg [**70**] and, by quite different methods, in Bers [**21**]. Bers's approach leads to sharper results that we shall now discuss.

6.3. Eichler cohomology and the Bers area theorem. Equation (13) is a cocycle condition for the case $q = 2$ of a more general cohomology theory, which Eichler introduced for number-theoretic purposes (see [**61**]). We shall describe how Bers [**20**] uses Eichler cohomology and certain spaces of automorphic forms to strengthen the Ahlfors finiteness theorem.

Fix any integer $q \geq 2$. The group $PSL(2,\mathbb{C})$ of Möbius transformations acts on the vector space Π_{2q-2} of polynomials of degree at most $2q-2$ by the *Eichler action*

$$v\gamma := (v \circ \gamma)(\gamma')^{1-q} \qquad \text{for } v \text{ in } \Pi_{2q-2} \text{ and } \gamma \text{ in } PSL(2,\mathbb{C}).$$

For any subgroup Γ of $PSL(2,\mathbb{C})$, a *cocycle* is a map $\chi\colon \Gamma \to \Pi_{2q-2}$ such that

$$\chi(\gamma_1 \circ \gamma_2) = \chi(\gamma_1)\gamma_2 + \chi(\gamma_2) \quad \text{for all } \gamma_1 \text{ and } \gamma_2 \text{ in } \Gamma.$$

The *coboundary* of a polynomial v in Π_{2q-2} is the cocycle $\gamma \mapsto v\gamma - v$, γ in Γ.

The cocycles form a vector space of maps from Γ to Π_{2q-2}, and the coboundaries form a subspace. By definition, the quotient vector space is the *Eichler cohomology group* $H^1(\Gamma, \Pi_{2q-2})$.

Now let Γ be a nonelementary finitely generated Kleinian group. We shall assume that Ω is a subset of $\mathbb{C}$. (That can be achieved by replacing Γ by a conjugate subgroup of $PSL(2,\mathbb{C})$.)

If Γ is generated by N elements, it is obvious that for each $q \geq 2$ the vector space of cocycles has dimension at most N times the dimension $2q-1$ of Π_{2q-2}. Its dimension is exactly $N(2q-1)$ if Γ is the free group on these generators. As Γ is not elementary, it is easy to verify that the space of coboundaries has dimension $2q-1$ (see [**20**]). That proves

LEMMA 2 (Bers [**20**]). *If Γ is nonelementary and generated by N elements, then*

$$\dim H^1(\Gamma, \Pi_{2q-2}) \leq (2q-1)(N-1),$$

with equality if Γ is free on N generators.

Now we observe that each component of Ω has a Poincaré metric. We write the infinitesimal metric as $ds = \lambda(z)|dz|$, z in Ω. Every γ in Γ maps Ω conformally onto itself and preserves the infinitesimal metric. Therefore the quotient space has a well-defined Poincaré area, given by

$$\operatorname{Area}(\Omega/\Gamma) = \iint_{\Omega/\Gamma} \lambda(z)^2\, dx\, dy,$$

the integral being taken over any fundamental set whose boundary has zero area.

As in Bers [**20**], we denote by $A_q(\Omega,\Gamma)$ the space of *cusp forms of weight* $(-2q)$ *for* Γ *in* Ω. It consists of the holomorphic functions ψ in Ω satisfying

$$\sup\{|\psi(z)|\lambda(z)^{-q} : z \in \Omega\} < \infty \text{ and } (\psi \circ \gamma)(\gamma')^q = \psi \text{ for all } \gamma \text{ in } \Gamma. \tag{14}$$

Bers maps $A_q(\Omega,\Gamma)$ to $H^1(\Gamma, \Pi_{2q-2})$ in the following way. (See [**20**] for the details.)

Given ψ in $A_q(\Omega,\Gamma)$, set $\mu(z) = \overline{\psi(z)}\lambda(z)^{2-2q}$, z in Ω, and $\mu(z) = 0$, z in $\mathbb{C} \setminus \Omega$. Let F be a continuous function on $\mathbb{C}$ such that $|F(z)|$ is $O(|z|^{2q-2})$ as $z \to \infty$ and $F_{\bar{z}} = \mu$ in the sense of distributions. Set

$$\chi(\gamma) = (F \circ \gamma)(\gamma')^{1-q} - F, \qquad \gamma \text{ in } \Gamma.$$

Then $\chi(\gamma) \in \Pi_{2q-2}$ for all γ in Γ, the map $\chi\colon \Gamma \to \Pi_{2q-2}$ is a cocycle, and its cohomology class in $H^1(\Gamma, \Pi_{2q-2})$ depends only on ψ. In honor of Bers, we denote it by $\beta_q(\psi)$.

Bers proves in [**20**] that β_q is an injective map, so Lemma 2 implies that

$$\dim A_q(\Omega,\Gamma) \leq (2q-1)(N-1) \tag{15}$$

when $q \geq 2$ and Γ is nonelementary and generated by N elements. Using classical formulas for the dimension of $A_q(\Omega,\Gamma)$ (see [**20**]), dividing both sides of (15) by $2q-1$, and letting q go to ∞, Bers obtains

THEOREM 11 (The Bers Area Theorem). *If Γ is a nonelementary Kleinian group generated by N elements, then*

$$\operatorname{Area}(\Omega/\Gamma) \leq 4\pi(N-1). \tag{16}$$

REMARKS. The constant 4π in (16) is obtained when the Poincaré metric is scaled to have curvature -1 as in Bers [**20**] and Chapters I through VI of this book. The Kobayashi scaling that we used in §4 would produce the constant π.

There has been a continuing interest in studying the structure of the Eichler cohomology groups; among the papers on this subject are [**10**] and [**95**]. Applications of the structure theorems are explored in [**9**], [**24**], [**97**], and [**147**].

6.4. The Teichmüller space of a finitely generated Kleinian group. The machinery of Chapter VI applies readily to Kleinian groups, but the topology of the regular set produces some complications. These were sorted out in the papers Bers [**22**], Maskit [**114**], and Kra [**96**]. See also Kra's chapter in [**29**]. We shall summarize the results here, making one simplifying assumption and taking advantage of some recent developments. (See §6.6 and the remarks at the end of this section.)

Let Γ be a Kleinian group, and let Ω and Λ be its regular set and limit set. By our standing assumptions, Γ is finitely generated and not elementary. As we consider conjugate subgroups of $PSL(2,\mathbb{C})$ to be essentially the same, we require 0, 1, and ∞ to belong to Λ. For simplicity, we require Γ to contain no elliptic transformations.

Consider the isomorphisms $\gamma \mapsto \theta(\gamma) := f \circ \gamma \circ f^{-1}$ of Γ onto Kleinian groups, where f is a quasiconformal self-mapping of $\widehat{\mathbb{C}}$. We say that θ and θ' are *equivalent* when they are conjugate by some Möbius transformation A, i.e., when there is A in $PSL(2,\mathbb{C})$ such that $\theta'(\gamma) = A \circ \theta(\gamma) \circ A^{-1}$ for all γ in Γ. By definition, the *Teichmüller space* $T(\Gamma)$ is the space of equivalence classes of these isomorphisms.

Each equivalence class has a unique "normalized" representative θ satisfying

$$\theta(\gamma) = \theta^\mu(\gamma) := f^\mu \circ \gamma \circ (f^\mu)^{-1}, \qquad \gamma \text{ in } \Gamma, \tag{17}$$

for some (in fact many) μ in $L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$. We identify $T(\Gamma)$ with the set of these normalized isomorphisms.

The mappings $f^\mu \circ \gamma \circ (f^\mu)^{-1}$ in (17) are all required to be Möbius transformations. That requirement is satisfied if and only if

$$\mu = (\mu \circ \gamma)\overline{\gamma'}/\gamma' \quad \text{ for all } \gamma \text{ in } \Gamma. \tag{18}$$

Let $B(\Gamma,\mathbb{C})$ be the space of functions μ in $L^\infty(\mathbb{C})$ that satisfy (18), and let $B_1(\Gamma,\mathbb{C})$ be its open unit ball. Let Φ be the surjective map $\mu \mapsto \theta^\mu$ from $B_1(\Gamma,\mathbb{C})$ to $T(\Gamma)$. We shall describe how to write it as a composite of simpler maps.

As Γ contains no elliptic transformations, the quotient map $\pi\colon \Omega \to \Omega/\Gamma$ is an unbranched covering. By Theorem 10, Ω/Γ is the disjoint union of Riemann surfaces $S_1, \dots, S_n$, each of finite conformal type. For each S_j, we choose a component Ω_j of $\pi^{-1}(S_j)$. Let $\varpi_j\colon H \to \Omega_j$ be a holomorphic universal covering map, and set $\widetilde{\pi}_j := \pi \circ \varpi_j$. Then $\widetilde{\pi}_j\colon H \to S_j$ is a universal covering. Let $\widetilde{\Gamma_j}$ be the (Fuchsian) group of cover transformations, and let $B(\widetilde{\Gamma_j})$ be its space of Beltrami differentials. (These are defined only in H, as in Chapter VI.) Let $\Phi_j\colon B_1(\widetilde{\Gamma_j}) \to T(S_j)$ be the usual holomorphic quotient map.

As Λ has zero area (see §6.6), there is an obvious isometric map of $B(\Gamma,\mathbb{C})$ onto $B(\widetilde{\Gamma_1}) \times \cdots \times B(\widetilde{\Gamma_n})$, where the latter space has the L^∞ norm

$$\|(\mu_1, \dots, \mu_n)\| = \max\{\|\mu_j\| : 1 \le j \le n\}.$$

Restricting that isometry to $B_1(\Gamma, \mathbb{C})$ and composing it with the map $\Phi_1 \times \cdots \times \Phi_n$, we obtain a surjective map $\widetilde{\Phi}\colon B_1(\Gamma, \mathbb{C}) \to T(S_1) \times \cdots \times T(S_n)$. By Theorem 2, $\widetilde{\Phi}$ is a holomorphic split submersion. The results of [**22**], [**114**], and [**96**] yield

THEOREM 12. *The map $\Phi : B_1(\Gamma, \mathbb{C}) \to T(\Gamma)$ has the form $F \circ \widetilde{\Phi}$, where $\widetilde{\Phi}$ is the above-defined holomorphic split submersion and $F\colon T(S_1) \times \cdots \times T(S_n) \to T(\Gamma)$ is a holomorphic covering map. In addition, $T(\Gamma)$ is isomorphic to a product $T_1 \times \cdots \times T_n$ so that $F = F_1 \times \cdots \times F_n$ and each $F_j\colon T(S_j) \to T_j$ is a holomorphic covering map.*

If Ω_j is simply connected, then F_j is biholomorphic; if each component of Ω is simply connected, then $T(\Gamma)$ is isomorphic to the product $T(S_1) \times \cdots \times T(S_n)$.

REMARKS. Because of Theorem 12, finitely generated Kleinian groups such that all components of Ω are simply connected have been used in some studies of the strata in augmented Teichmüller spaces. See [**27**], [**98**], and [**111**].

If Γ contains elliptic transformations, the results are the same except that the images of the branch points of $\pi\colon \Omega \to \Omega/\Gamma$ must be deleted from the S_j. See [**22**], [**29**], [**96**], and [**114**] for details.

Those sources explicitly excluded from $B_1(\Gamma, \mathbb{C})$ all Beltrami differentials that do not vanish on Λ, as it was not then known whether Λ could have positive area. A special case of the "Ahlfors conjecture" that Λ has zero area for all finitely generated Kleinian groups is proved in Ahlfors [**8**]. The general case was proved only recently (see §6.6). Before that question was settled, Sullivan showed in [**148**] that if Γ is finitely generated, then every μ in $B(\Gamma, \mathbb{C})$ equals zero almost everywhere in Λ.

6.5. Quasi-Fuchsian groups and simultaneous uniformization. By definition, a Kleinian group G is *quasi-Fuchsian* if there are a quasiconformal map f of $\widehat{\mathbb{C}}$ onto itself and a Fuchsian group Γ of the first kind such that $G = f\Gamma f^{-1}$. Such groups G were called "Fuchsoid" in VI B and C, but that terminology has not survived. As no normalization was imposed on the map f, we may and shall assume that the Fuchsian group Γ maps the upper and lower half-planes onto themselves.

If, as we shall now assume, the Fuchsian group Γ is finitely generated and contains no elliptic transformations, we can apply Theorem 12 to the Teichmüller space $T(\Gamma)$, regarding Γ as a Kleinian group. In this case Λ is the extended real axis, Ω_1 and Ω_2 are the upper and lower half-planes, $S_1 = H/\Gamma$, and S_2 is the conjugate surface S_1^*. Theorem 12 says that the mapping $\Phi(\mu) \mapsto \theta^\mu$, μ in $B_1(\Gamma, \mathbb{C})$, from $T(S_1) \times T(S_1^*)$ to the set of normalized isomorphisms is a well-defined bijection. This is the famous simultaneous uniformization theorem of Bers (see Bers [**16**]).

REMARK. According to a remarkable theorem of Maskit, a finitely generated Kleinian group Γ is quasi-Fuchsian if and only if its ordinary set has exactly two connected components, each of which is Γ-invariant. For proofs using two-dimensional methods, see Maskit's authoritative book [**115**] or the paper [**99**]. Of course, the quasi-Fuchsian groups in Maskit's theorem are allowed to contain elliptic transformations.

6.6. Kleinian groups and hyperbolic 3-manifolds. We shall now use the map $z = x_1 + ix_2 \mapsto (x_1, x_2, 0)$ to identify $\mathbb{C}$ with the plane $\{(x_1, x_2, x_3) : x_3 = 0\}$ in $\mathbb{R}^3$. That map obviously extends to an embedding of $\widehat{\mathbb{C}}$ in $\widehat{\mathbb{R}}^3 := \mathbb{R}^3 \cup \{\infty\}$.

Each Möbius transformation in $PSL(2, \mathbb{C})$ extends uniquely to a Möbius transformation of $\widehat{\mathbb{R}}^3$ that maps the upper half-space $H^3 := \{(x_1, x_2, x_3) : x_3 > 0\}$ onto itself (see Beardon [**13**]). With the metric $ds^2 := x_3^{-2}(dx_1^2 + dx_2^2 + dx_3^2)$, H^3 becomes a model for hyperbolic 3-space, and the Möbius transformations in $PSL(2,\mathbb{C})$ become hyperbolic isometries.

Every discrete subgroup Γ of $PSL(2,\mathbb{C})$ acts properly discontinuously on H^3, so in this setting it is natural to call all such groups Kleinian. We shall still assume that Γ is finitely generated and not elementary. We shall also require Γ to have no elliptic elements. In this case, Γ acts freely on H^3, and the quotient space H^3/Γ is a hyperbolic 3-manifold with fundamental group Γ.

Their action on H^3 is a powerful tool for understanding finitely generated Kleinian groups. In [**8**], Ahlfors showed that if Γ has a finite-sided fundamental polyhedron in H^3, then either $\Lambda = \widehat{\mathbb{C}}$ or Λ has zero area. Marden's paper [**110**] initiated the modern systematic study of the structure of H^3/Γ. It includes a 3-dimensional interpretation and proof of Maskit's theorem about quasi-Fuchsian groups (with no elliptic elements). Since then, Thurston's far-reaching theory of geometric structures on 3-manifolds has revolutionized both the field of 3-dimensional topology and the study of hyperbolic 3-manifolds. We can make only brief comments.

First, two major conjectures have been solved recently. One is Marden's conjecture in [**110**] that H^3/Γ is homeomorphic to the interior of a compact 3-manifold; among other things, this implies Ahlfors's conjecture about the area of Λ (see [**2**] and [**37**]). The other is Thurston's "ending lamination conjecture", a sweeping generalization of Bers's simultaneous uniformization theorem. This is proved in [**36**]. Properties of the curve complex (see [**124**]) play a role in the proof. The solutions of both problems are the culmination of work of many people (see the bibliographies and historical comments in the cited papers).

Finally, we refer to Hubbard's article [**83**] in this book for an instructive illustration of the use of 2-dimensional quasiconformal mappings in Thurston's theory of 3-manifolds.

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