Preface

This book is an expanded version of lecture notes for a topics course given by the author at the University of Pennsylvania during the spring of 2004 on the combinatorics of the q,t-Catalan numbers and the space of diagonal harmonics. These subjects are closely related to the study of Macdonald polynomials, which are an important family of multivariable orthogonal polynomials introduced by Macdonald in 1988 with applications to a wide variety of subjects including Hilbert schemes, harmonic analysis, representation theory, mathematical physics, and algebraic combinatorics. Many wonderful results about these polynomials from analytic, algebraic, and geometric viewpoints have been obtained, but the combinatorics behind them had remained rather impenetrable. Toward the end of the spring 2004 semester the author, inspired primarily by new combinatorial identities involving diagonal harmonics discussed in Chapter 6 of this book, was led to a combinatorial formula for Macdonald polynomials. The discovery of this formula, which was proved in subsequent joint work with Mark Haiman and Nick Loehr, has resulted in a volume of broader interest, as in Appendix A we include a discussion of the formula, its proof, and the nice applications it has to the theory of symmetric functions.

Among these applications we might draw the reader's attention to the short, elegant proof in Appendix A of Lascoux and Schützenberger's "cocharge" theorem on Hall-Littlewood polynomials, a fundamental result in the theory of symmetric functions whose original proof was neither short nor elegant. Another application of the combinatorial formula is a way of writing the Macdonald polynomial as a positive sum of LLT polynomials, which are symmetric functions introduced by Lascoux, Leclerc, and Thibon. This decomposition is especially significant in view of two recent preprints, one by Grojnowski and Haiman and another by Sami Assaf, which contain proofs that the coefficients of LLT polynomials, when expanded in terms of Schur functions, are positive. Although Grojnowski and Haiman's proof uses Kazhdan-Lusztig theory and algebraic geometry, Assaf's proof is a self-contained 21 page combinatorial argument. Thus we now have an accessible, combinatorial proof that Macdonald polynomials are Schur positive. (This Macdonald positivity result was first proved in 2000 by Haiman using properties of the Hilbert scheme from algebraic geometry.) The attempt to understand the combinatorics of Macdonald polynomials is what led Garsia and Haiman to study diagonal harmonics and has been the motivation behind quite a bit of research in algebraic combinatorics over the last 20 years.

Chapter 1 contains some well-known introductory material on q-analogues and symmetric functions. Chapter 2 gives some of the historical background and basic theorems involving Macdonald polynomials and diagonal harmonics, including a discussion of how a certain S_n action on the space of diagonal harmonics leads to a number of beautiful and deep combinatorial problems. Chapters $3-6$ deal with the combinatorics of the character induced by this action. The most fundamental object in this subject is the q,t-Catalan numbers, the focus of Chapter 3. From there we move on to a study of the q,t-Schröder numbers in Chapter 4, which are a bigraded version of the multiplicity of a hook shape in the character. Chapter 5 deals with a (conjectured) expression for the bigraded Hilbert series, which has an elegant expression in terms of combinatorial objects called parking functions. In Chapter 6 we study the "shuffle conjecture" of Haiman, Loehr, Remmel, Ulyanov, and the author which gives a combinatorial prediction, parameterized in terms of parking functions, for the expansion of the character into monomials. This conjecture includes all of the results and conjectures from Chapters $3-5$ as special cases. Chapter 7 consists of an exposition of the proof of the broadest special case of this conjecture that we can prove, that of hook shapes. The proof involves the manipulation of technical symmetric function identities involving plethysm and Macdonald polynomials. These identities are rather difficult to learn about from reading journal articles, and it is hoped this chapter will be a useful guide to readers interested in learning these subjects. Appendix B contains a discussion of an amazing extension of the shuffle conjecture recently proposed by Loehr and Warrington.

There are homework exercises interspersed throughout the text, in strategically chosen locations, to help the reader absorb the material. Solutions to all the exercises are given in Appendix C. The book is meant to have value either as a text for a topics course in algebraic combinatorics, a guide for self-study, or a reference book for researchers in this area.

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