CHAPTER 1

Introduction to q-Analogues and Symmetric Functions

Permutation Statistics and Gaussian Polynomials

In combinatorics a *q-analogue* of a counting function is typically a polynomial in q which reduces to the function in question when $q = 1$, and furthermore satisfies versions of some or all of the algebraic properties, such as recursions, of the function. We sometimes regard q as a real parameter satisfying $0 < q < 1$. We define the q-analogue of the real number x, denoted $[x]$ as

$$[x] = \frac{1 - q^x}{1 - q}.$$

By l'Hôpital's rule, $[x] \to x$ as $q \to 1^-$. Let $\mathbb{N}$ denote the nonnegative integers. For $n \in \mathbb{N}$, we define the q-analogue of $n!$, denoted $[n]!$ as

$$[n]! = \prod_{i=1}^{n} [i] = (1 + q)(1 + q + q^2) \cdots (1 + q + \ldots + q^{n-1}).$$

We let $|S|$ denote the cardinality of a finite set S¿ By a *statistic* on a set S we mean a combinatorial rule which associates an element of $\mathbb{N}$ to each element of S. A permutation statistic is a statistic on the symmetric group S_n. We use the one-line notation $\sigma_1\sigma_2 \cdots \sigma_n$ for the element $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma_1 & \sigma_2 & \ldots & \sigma_n \end{pmatrix}$ of S_n. More generally, a word (or multiset permutation) $\sigma_1\sigma_2 \cdots \sigma_n$ is a linear list of the elements of some multiset of nonnegative integers. (The reader may wish to consult [**Sta86**, Chapter 1] for more background on multiset permutations.) An *inversion* of a word σ is a pair (i, j), $1 \le i < j \le n$ such that $\sigma_i > \sigma_j$. A *descent* of σ is an integer i, $1 \le i \le n - 1$, for which $\sigma_i > \sigma_{i+1}$. The set of such i is known as the descent set, denoted $\mathrm{Des}(\sigma)$. We define the inversion statistic $\mathrm{inv}(\sigma)$ as the number of inversions of σ and the major index statistic $\mathrm{maj}(\sigma)$ as the sum of the descents of σ, i.e.

$$\mathrm{inv}(\sigma) = \sum_{\substack{i<j \\ \sigma_i > \sigma_j}} 1, \quad \mathrm{maj}(\sigma) = \sum_{\substack{i \\ \sigma_i > \sigma_{i+1}}} i.$$

A permutation statistic is said to be *Mahonian* if its distribution over S_n is $[n]!$.

THEOREM 1.1. *Both inv and maj are Mahonian, i.e.*

$$\sum_{\sigma \in S_n} q^{inv(\sigma)} = [n]! = \sum_{\sigma \in S_n} q^{maj(\sigma)}. \tag{1.1}$$

PROOF. Given $\beta \in S_{n-1}$, let $\beta(k)$ denote the permutation in S_n obtained by inserting n between the $(k-1)$st and kth elements of β. For example, $2143(3) =$

21543. Clearly $\mathrm{inv}(\beta(k)) = \mathrm{inv}(\beta) + n - k$, so

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\beta \in S_{n-1}} (1 + q + q^2 + \ldots + q^{n-1}) q^{\mathrm{inv}(\beta)} \tag{1.2}$$

and thus by induction inv is Mahonian.

A modified version of this idea works for maj. Say the descents of $\beta \in S_{n-1}$ are at places $i_1 < i_2 < \cdots < i_s$. Then

$$\mathrm{maj}(\beta(n)) = \mathrm{maj}(\beta), \quad \mathrm{maj}(\beta(i_s + 1)) = \mathrm{maj}(\beta) + 1,$$
$$\ldots, \mathrm{maj}(\beta(i_1 + 1)) = \mathrm{maj}(\beta) + s, \quad \mathrm{maj}(\beta(1)) = s + 1.$$

If the non-descents less than $n-1$ of β are at places $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-2-s}$, then

$$\mathrm{maj}(\beta(\alpha_1 + 1)) = \mathrm{maj}(\beta) + s - (\alpha_1 - 1) + \alpha_1 + 1 = \mathrm{maj}(\beta) + s + 2.$$

To see why, note that $s - (\alpha_1 - 1)$ is the number of descents of β to the right of α_1, each of which will be shifted one place to the right by the insertion of n just after β_{α_1}. Also, we have a new descent at $\alpha_1 + 1$. By similar reasoning,

$$\begin{aligned}\mathrm{maj}(\beta(\alpha_2)) &= \mathrm{maj}(\beta) + s - (\alpha_2 - 2) + \alpha_2 + 1 = \mathrm{maj}(\beta) + s + 3,\\ &\vdots\\ \mathrm{maj}(\beta(\alpha_{n-2-s})) &= \mathrm{maj}(\beta) + s - (\alpha_{n-2-s} - n - 2 - s) + \alpha_{n-2-s} + 1\\ &= \mathrm{maj}(\beta) + n - 1.\end{aligned}$$

Thus

$$\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = \sum_{\beta \in S_{n-1}} (1 + q + \ldots + q^s + q^{s+1} + \ldots + q^{n-1}) q^{\mathrm{maj}(\beta)} \tag{1.3}$$

and again by induction maj is Mahonian. □

Major P. MacMahon introduced the major-index statistic and proved it is Mahonian [**Mac60**]. Foata [**Foa68**] found a map ϕ which sends a permutation with a given major index to another with the same value for inv. Furthermore, if we denote the descent set of σ^{-1} by $\mathrm{Ides}(\sigma)$, then ϕ fixes $\mathrm{Ides}(\sigma)$. The map ϕ can be described as follows. If $n \le 2$, $\phi(\sigma) = \sigma$. If $n > 2$, we add a number to ϕ one at a time; begin by setting $\phi^{(1)} = \sigma_1$, $\phi^{(2)} = \sigma_1\sigma_2$ and $\phi^{(3)} = \sigma_1\sigma_2\sigma_3$. Then if $\sigma_3 > \sigma_2$, draw a bar after each element of $\phi^{(3)}$ which is greater than σ_3, while if $\sigma_3 < \sigma_2$, draw a bar after each element of $\phi^{(3)}$ which is less than σ_3. Also add a bar before $\phi_1^{(3)}$. For example, if $\sigma = 4137562$ we now have $\phi^{(3)} = |41|3$. Now regard the numbers between two consecutive bars as "blocks", and in each block, move the last element to the beginning, and finally remove all bars. We end up with $\phi^{(3)} = 143$.

Proceeding inductively, we begin by letting $\phi^{(i)}$ be the result of adding σ_i to the end of $\phi^{(i-1)}$. Then if if $\sigma_i > \sigma_{i-1}$, draw a bar after each element of $\phi^{(i)}$ which is greater than σ_i, while if $\sigma_i < \sigma_{i-1}$, draw a bar after each element of $\phi^{(i)}$ which is less than σ_i. Also draw a bar before $\phi_1^{(i)}$. Then in each block, move the last element to the beginning, and finally remove all bars. If $\sigma = 4137562$ the successive stages of the algorithm yield

$$\phi^{(3)} = 143$$
$$\phi^{(4)} = |1|4|3|7 \to 1437$$
$$\phi^{(5)} = |1437|5 \to 71435$$
$$\phi^{(6)} = |71|4|3|5|6 \to 174356$$
$$\phi^{(7)} = |17|4|3|5|6|2 \to 7143562$$

and so $\phi(4137562) = 7143562$.

PROPOSITION 1.1.1. *We have maj*(σ) = *inv*$(\phi(\sigma))$. *Furthermore, Ides*(σ) = *Ides*$(\phi(\sigma))$, *and* ϕ *fixes* σ_n.

PROOF. We claim $\mathrm{inv}(\phi^{(k)}) = \mathrm{maj}(\sigma_1 \cdots \sigma_k)$ for $1 \leq k \leq n$. Clearly this is true for $k \leq 2$. Assume it is true for $k < j$, where $2 < j \leq n$. If $\sigma_j > \sigma_{j-1}$, $\mathrm{maj}(\sigma_1 \cdots \sigma_j) = \mathrm{maj}(\sigma_1 \cdots \sigma_{j-1}) + j - 1$. One the other hand, for each block arising in the procedure creating $\phi^{(j)}$, the last element is greater than σ_j, which creates a new inversion, and when it is moved to the beginning of the block, it also creates a new inversion with each element in its block. It follows that $\mathrm{inv}(\phi^{(j-1)}) = \mathrm{inv}(\phi^{(j)}) + j - 1$. Similar remarks hold if $\sigma_j < \sigma_{j-1}$. In this case $\mathrm{maj}(\sigma_1 \cdots \sigma_{j-1}) = \mathrm{maj}(\sigma_1 \cdots \sigma_j)$. Also, each element of ϕ which is not the last element in its block is larger than σ_j, which creates a new inversion, but a corresponding inversion between this element and the last element in its block is lost when we cycle the last element to the beginning. Hence $\mathrm{inv}(\phi^{(j-1)}) = \mathrm{inv}(\phi^{(j)})$ and the first part of the claim follows.

Note that $\mathrm{Ides}(\sigma)$ equals the set of all i, $1 \leq i < n$ such that $i+1$ occurs before i in σ. In order for the ϕ map to change this set, at some stage, say when creating $\phi^{(j)}$, we must move i from the end of a block to the beginning, passing $i-1$ or $i+1$ along the way. But this could only happen if σ_j is strictly between i and either $i-1$ or $i+1$, an impossibility. □

Let $\beta = \phi^{-1}$, and begin by setting $\beta^{(1)} = \sigma$. Then if $\sigma_n > \sigma_1$, draw a bar *before* each number in $\beta^{(1)}$ which is less than σ_n, and also before σ_n. If $\sigma_n < \sigma_1$, draw a bar before each number in $\beta^{(1)}$ which is greater than σ_n, and also before σ_n. Next move each number at the beginning of a block to the end of the block.

The last letter of β is now fixed. Next set $\beta^{(2)} = \beta^{(1)}$, and compare the $n-1$st letter with the first, creating blocks as above, and draw an extra bar before the $n-1$st letter. For example, if $\sigma = 7143562$ the successive stages of the β algorithm yield

$$\beta^{(1)} = |71|4|3|5|6|2 \to 1743562$$
$$\beta^{(2)} = |17|4|3|5|62 \to 7143562$$
$$\beta^{(3)} = |7143|562 \to 1437562$$
$$\beta^{(4)} = |1|4|3|7562 \to 1437562$$
$$\beta^{(5)} = |14|37562 \to 4137562$$
$$\beta^{(6)} = \beta^{(7)} = 4137562$$

and so $\phi^{-1}(7143562) = 4137562$. Notice that at each stage we are reversing the steps of the ϕ algorithm, and it is easy to see this holds in general.

An *involution* on a set S is a bijective map from S to S whose square is the identity. Foata and Schützenberger [**FS78**] showed that the map $i\phi i\phi^{-1}i$, where i is the inverse map on permutations, is an involution on S_n which interchanges inv and maj.

For $n, k \in \mathbb{N}$, let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)} \tag{1.4}$$

denote the Gaussian polynomial. These are special cases of more general objects known as q-binomial coefficients, which are defined for $x \in \mathbb{R}$ as

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{(q^{x-k+1};q)_k}{(q;q)_k}, \tag{1.5}$$

where $(a;q)_k = (a)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a)$ is the "q-rising factorial".

A partition λ is a nonincreasing finite sequence $\lambda_1 \geq \lambda_2 \geq \ldots$ of positive integers. λ_i is called the ith part of λ. We let $\ell(\lambda)$ denote the number of parts, and $|\lambda| = \sum_i \lambda_i$ the sum of the parts. For various formulas it will be convenient to assume $\lambda_j = 0$ for $j > \ell(\lambda)$. The *Ferrers graph* of λ is an array of unit squares, called cells, with λ_i cells in the ith row, with the first cell in each row left-justified. We often use λ to refer to its Ferrers graph, We define the conjugate partition, λ' as the partition of whose Ferrers graph is obtained from λ by reflecting across the diagonal $x = y$. See Figure 1. for example $(i,j) \in \lambda$ refers to a cell with (column, row) coordinates (i,j), with the lower left-hand-cell of λ having coordinates $(1,1)$. The notation $x \in \lambda$ means x is a cell in λ. For technical reasons we say that 0 has one partition, the emptyset $\emptyset$, with $\ell(\emptyset) = 0 = |\emptyset|$.

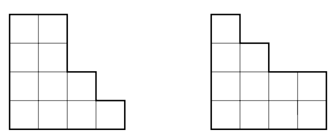

FIGURE 1. On the left, the Ferrers graph of the partition $(4,3,2,2)$, and on the right, that of its conjugate $(4,3,2,2)' = (4,4,2,1)$.

The following result shows the Gaussian polynomials are in fact polynomials in q, which is not obvious from their definition.

THEOREM 1.2. *For $n, k \in \mathbb{N}$,*

$$\begin{bmatrix} n+k \\ k \end{bmatrix} = \sum_{\lambda \subseteq n^k} q^{|\lambda|}, \tag{1.6}$$

where the sum is over all partitions λ whose Ferrers graph fits inside a $k \times n$ rectangle, i.e. for which $\lambda_1 \leq n$ and $\ell(\lambda) \leq k$.

PROOF. Let

$$P(n,k) = \sum_{\lambda \subseteq n^k} q^{|\lambda|}.$$

Clearly

$$P(n,k) = \sum_{\substack{\lambda \subseteq n^k \\ \lambda_1 = n}} q^{|\lambda|} + \sum_{\substack{\lambda \subseteq n^k \\ \lambda_1 \leq n-1}} q^{|\lambda|} = q^n P(n,k-1) + P(n-1,k). \tag{1.7}$$

On the other hand

$$\begin{aligned} q^n \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1+k \\ k \end{bmatrix} &= q^n \frac{[n+k-1]!}{[k-1]![n]!} + \frac{[n-1+k]!}{[k]![n-1]!} \\ &= \frac{q^n [k][n+k-1]! + [n-1+k]![n]}{[k]![n]!} \\ &= \frac{[n+k-1]!}{[k]![n]!}(q^n(1+q+\ldots+q^{k-1}) + 1 + q + \ldots + q^{n-1}) \\ &= \frac{[n+k]!}{[k]![n]!}. \end{aligned}$$

Since $P(n,0) = P(0,k) = 1$, both sides of (1.6) thus satisfy the same recurrence and initial conditions. □

Given $\alpha = (\alpha_0, \ldots, \alpha_s) \in \mathbb{N}^{s+1}$, let

$$\{0^{\alpha_0} 1^{\alpha_1} \cdots s^{\alpha_s}\}$$

denote the multiset with α_i copies of i, where $\alpha_0 + \ldots + \alpha_s = n$. We let M_α denote the set of all permutations of this multiset and refer to α as the *weight* of any given one of these words. We extend the definitions of inv and maj as follows

$$\operatorname{inv}(\sigma) = \sum_{\substack{i<j \\ \sigma_i > \sigma_j}} 1, \quad \operatorname{maj}(\sigma) = \sum_{\substack{i \\ \sigma_i > \sigma_{i+1}}} i. \tag{1.8}$$

Also let

$$\begin{bmatrix} n \\ \alpha_0, \ldots, \alpha_s \end{bmatrix} = \frac{[n]!}{[\alpha_0]! \cdots [\alpha_s]!} \tag{1.9}$$

denote the "q-multinomial coefficient".

The following result is due to MacMahon [**Mac60**].

Theorem 1.3. *Both inv and maj are multiset Mahonian, i.e. given $\alpha \in \mathbb{N}^{s+1}$,*

$$\sum_{\sigma \in M_\alpha} q^{inv(\sigma)} = \begin{bmatrix} n \\ \alpha_0, \ldots, \alpha_s \end{bmatrix} = \sum_{\sigma \in M_\alpha} q^{maj(\sigma)}, \tag{1.10}$$

where the sums are over all elements $\sigma \in M_\alpha$.

Remark 1.4. Foata's map also proves Theorem 1.3 bijectively. To see why, given a multiset permutation σ of $M(\beta)$ let σ' denote the *standardization* of σ, defined to be the permutation obtained by replacing the β_1's by the numbers 1 through β_1, in increasing order as we move left to right in σ, then replacing the β_2's by the numbers $\beta_1 + 1$ through $\beta_1 + \beta_2$, again in increasing order as we move left to right in σ, etc. For example, the standardization of 31344221 is 51678342. Note that

$$\operatorname{Ides}(\sigma') \subseteq \{\beta_1, \beta_1 + \beta_2, \ldots\} \tag{1.11}$$

and in fact standardization gives a bijection between elements of $M(\beta)$ and permutations satisfying (1.11).

EXERCISE 1.5. If σ is a word of length n define the co-major index of σ as follows.

$$\operatorname{comaj}(\sigma) = \sum_{\substack{\sigma_i > \sigma_{i+1} \\ 1 \le i < n}} n - i. \tag{1.12}$$

Show that Foata's map ϕ implies there is a bijective map coϕ on words of fixed weight such that

$$\operatorname{comaj}(\sigma) = \operatorname{inv}(\operatorname{co}\phi(\sigma)). \tag{1.13}$$

The Catalan Numbers and Dyck Paths

A lattice path is a sequence of North $N(0,1)$ and East $E(1,0)$ steps in the first quadrant of the xy-plane, starting at the origin $(0,0)$ and ending at say (n,m). We let $L_{n,m}$ denote the set of all such paths, and $L^+_{n,m}$ the subset of $L_{n,m}$ consisting of paths which never go below the line $y = \frac{m}{n}x$. A Dyck path is an element of $L^+_{n,n}$ for some n.

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the nth Catalan number, so

$$C_0, C_1, \ldots = 1, 1, 2, 5, 14, 42, \ldots.$$

See [**Sta99**, Ex. 6.19, p. 219] for a list of over 66 objects counted by the Catalan numbers. One of these is the number of elements of $L^+_{n,n}$. For $1 \le k \le n$, the number of Dyck paths from $(0,0)$ to (k,k) which touch the line $y = x$ only at $(0,0)$ and (k,k) is C_{k-1}, since such a path must begin with a N step, end with an E step, and never go below the line $y = x+1$ as it goes from $(0,1)$ to $(k-1,k)$. The number of ways to extend such a path to (n,n) and still remain a Dyck path is clearly C_{n-k}. It follows that

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k} \quad n \ge 1. \tag{1.14}$$

There are two natural q-analogues of C_n. Given $\pi \in L_{n,m}$, let $\sigma(\pi)$ be the element of $M_{(m,n)}$ resulting from the following algorithm. First initialize σ to the empty string. Next start at $(0,0)$, move along π and add a 0 to the end of $\sigma(\pi)$ every time a N step is encountered, and add a 1 to the end of $\sigma(\pi)$ every time an E step is encountered. Similarly, given $\sigma \in M_{(m,n)}$, let $\pi(\sigma)$ be the element of $L_{n,m}$ obtained by inverting the above algorithm. We call the transformation of π to σ or its inverse the *coding* of π or σ. Let $a_i(\pi)$ denote the number of complete squares, in the ith row from the bottom of π, which are to the right of π and to the left of the line $y = x$. We refer to $a_i(\pi)$ as the *length* of the ith row of π. Furthermore call $(a_1(\pi), a_2(\pi), \ldots, a_n(\pi))$ the *area vector* of π, and set $\operatorname{area}(\pi) = \sum_i a_i(\pi)$. For example, the path in Figure 2 has area vector $(0,1,1,2,1,2,0)$, and $\sigma(\pi) = 00100110011101$. By convention we say $L^+_{0,0}$ contains one path, the empty path $\emptyset$, with $\operatorname{area}(\emptyset) = 0$.

Let $M^+_{(m,n)}$ denote the elements σ of $M_{(m,n)}$ corresponding to paths in $L^+_{n,m}$. Paths in $M^+_{n,n}$ are thus characterized by the property that in any initial segment there are at least as many 0's as 1's. The first q-analogue of C_n is given by the following.

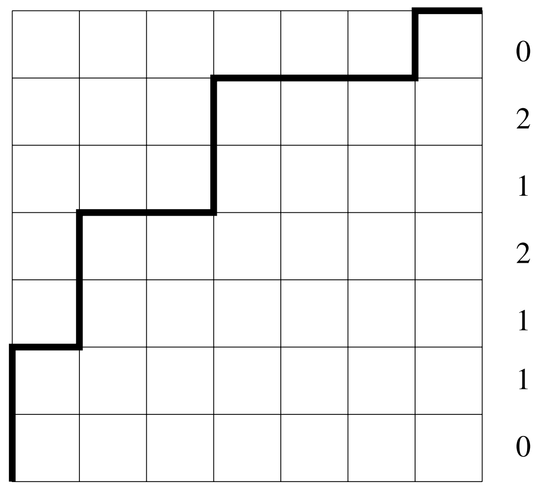

FIGURE 2. A Dyck path, with row lengths on the right. The area statistic is $1+1+2+1+2=7$.

THEOREM 1.6. *(MacMahon* [**Mac60**, p. 214]*)*

$$\sum_{\pi \in L^+_{n,n}} q^{maj(\sigma(\pi))} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}. \tag{1.15}$$

PROOF. We give a *bijective* proof, taken from [**FH85**]. Let $M^-_{(m,n)} = M_{(m,n)} \setminus M^+_{(m,n)}$, and let $L^-_{n,m} = L_{n,m} \setminus L^+_{n,m}$ be the corresponding set of lattice paths. Given a path $\pi \in L^-_{n,n}$, let P be the point with smallest x-coordinate among those lattice points (i,j) in π for which $i-j$ is maximal, i.e. whose distance from the line $y=x$ in a SE direction is maximal. (Since $\pi \in L^-_{n,n}$, this maximal value of $i-j$ is positive.) Let P' be the lattice point on π before P. There must be a E step connecting P' to P. Change this into a N step and shift the remainder of the path after P up one unit and left one unit. We now have a path $\phi(\pi)$ from $(0,0)$ to $(n-1,n+1)$, and moreover $\text{maj}(\sigma(\phi(\pi))) = \text{maj}(\sigma(\pi)) - 1$.

It is easy to see that this map is invertible. Given a lattice path π' from $(0,0)$ to $(n-1,n+1)$, let P' be the point with maximal x-coordinate among those lattice points (i,j) in π' for which $j-i$ is maximal. Thus

$$\sum_{\sigma \in M^-_{(n,n)}} q^{\text{maj}(\sigma)} = \sum_{\sigma' \in M_{(n+1,n-1)}} q^{\text{maj}(\sigma')+1} = q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}, \tag{1.16}$$

using (1.10). Hence

$$\sum_{\pi \in L^+_{n,n}} q^{\text{maj}(\sigma(\pi))} = \sum_{\sigma \in M_{(n,n)}} q^{\text{maj}(\sigma)} - \sum_{\sigma \in M^-_{(n,n)}} q^{\text{maj}(\sigma)} \tag{1.17}$$

$$= \begin{bmatrix} 2n \\ n \end{bmatrix} - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}. \tag{1.18}$$

□

The second natural q-analogue of C_n was studied by Carlitz and Riordan [**CR64**]. They define

$$C_n(q) = \sum_{\pi \in L^+_{n,n}} q^{\text{area}(\pi)}. \tag{1.19}$$

PROPOSITION 1.6.1.

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_k(q) C_{n-k}(q), \quad n \geq 1. \tag{1.20}$$

PROOF. As in the proof of (1.14), we break up our path π according to the "point of first return" to the line $y = x$. If this occurs at (k, k), then the area of the part of π from $(0, 1)$ to $(k-1, k)$, when viewed as an element of $L^+_{k-1,k-1}$, is $k-1$ less than the area of this portion of π when viewed as a path in $L^+_{n,n}$. □

EXERCISE 1.7. Define a *co-inversion* of σ to be a pair (i, j) with $i < j$ and $\sigma_i < \sigma_j$. Show

$$C_n(q) = \sum_{\pi \in L^+_{n,n}} q^{\text{coinv}(\sigma(\pi)) - \binom{n+1}{2}}, \tag{1.21}$$

where $\text{coinv}(\sigma)$ is the number of co-inversions of σ.

The q-Vandermonde Convolution

Let

$$_{p+1}\phi_p \left(\begin{matrix} a_1, & a_2, & \ldots, & a_{p+1} \\ & b_1, & \ldots, & b_p \end{matrix} ; q; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{(q)_n (b_1)_n \cdots (b_p)_n} z^n \tag{1.22}$$

denote the basic hypergeometric series. A good general reference for this subject is [**GR04**]. The following result is known as Cauchy's q-binomial series.

THEOREM 1.8.

$$_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix} ; q; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}, \quad |z| < 1, \, |q| < 1, \tag{1.23}$$

where $(a; q)_\infty = (a)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)$.

PROOF. The following proof is based on the proof in [**GR04**, Chap. 1]. Let

$$F(a, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n.$$

Then

$$F(a, z) - F(a, qz) = (1 - a) z F(aq, z) \tag{1.24}$$

and

$$F(a, z) - F(aq, z) = -az F(aq, z). \tag{1.25}$$

Eliminating $F(aq, z)$ from (1.24) and (1.25) we get

$$F(a, z) = \frac{(1 - az)}{(1 - z)} F(a, qz).$$

Iterating this n times then taking the limit as $n \to \infty$ we get

$$
\begin{aligned}
F(a,z) &= \lim_{n\to\infty} \frac{(az;q)_n}{(z;q)_n} F(a,q^n z) \\
&= \frac{(az;q)_\infty}{(z;q)_\infty} F(a,0) = \frac{(az;q)_\infty}{(z;q)_\infty}. \qquad (1.26)
\end{aligned}
$$

□

COROLLARY 1.8.1. *The "q-binomial theorem".*

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k = (-z;q)_n \tag{1.27}$$

and

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} z^k = \frac{1}{(z;q)_{n+1}}. \tag{1.28}$$

PROOF. To prove (1.27), set $a = q^{-n}$ and $z = -zq^n$ in (1.23) and simplify. To prove (1.28), let $a = q^{n+1}$ in (1.23) and simplify. □

For any function $f(z)$, let $f(z)|_{z^k}$ denote the coefficient of z^k in the Maclaurin series for $f(z)$.

COROLLARY 1.8.2.

$$\sum_{k=0}^{h} q^{(n-k)(h-k)} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} = \begin{bmatrix} m+n \\ h \end{bmatrix}. \tag{1.29}$$

PROOF. By (1.27),

$$
\begin{aligned}
q^{\binom{h}{2}} \begin{bmatrix} m+n \\ h \end{bmatrix} &= \prod_{k=0}^{m+n-1} (1+zq^k)|_{z^h} \\
&= \prod_{k=0}^{n-1} (1+zq^k) \prod_{j=0}^{m-1} (1+zq^n q^j)|_{z^h} \\
&= (\sum_{k=0}^{n-1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k)(\sum_{j=0}^{m-1} q^{\binom{j}{2}} \begin{bmatrix} m \\ j \end{bmatrix} (zq^n)^j)|_{z^h} \\
&= \sum_{k=0}^{h} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{h-k}{2}} \begin{bmatrix} m \\ h-k \end{bmatrix} (q^n)^{h-k}.
\end{aligned}
$$

The result now reduces to the identity

$$\binom{k}{2} + \binom{h-k}{2} + n(h-k) - \binom{h}{2} = (n-k)(h-k).$$

□

COROLLARY 1.8.3.

$$\sum_{k=0}^{h} q^{(m+1)k} \begin{bmatrix} n-1+k \\ k \end{bmatrix} \begin{bmatrix} m+h-k \\ h-k \end{bmatrix} = \begin{bmatrix} m+n+h \\ h \end{bmatrix}. \tag{1.30}$$

Proof. By (1.28),

$$\begin{aligned}\begin{bmatrix} m+n+h \\ h \end{bmatrix} &= \frac{1}{(z)_{m+n+1}}|_{z^h} \\ &= \frac{1}{(z)_{m+1}}\frac{1}{(zq^{m+1})_n}|_{z^h} \\ &= \left(\sum_{j=0}^{h} z^j \begin{bmatrix} m+j \\ j \end{bmatrix}\right)\left(\sum_{k=0}^{h}(zq^{m+1})^k \begin{bmatrix} n-1+k \\ k \end{bmatrix}\right)|_{z^h} \\ &= \sum_{k=0}^{h} q^{(m+1)k}\begin{bmatrix} n-1+k \\ k \end{bmatrix}\begin{bmatrix} m+h-k \\ h-k \end{bmatrix}.\end{aligned}$$

□

Both (1.29) and (1.30) are special cases of the following result, known as the q-Vandermonde convolution. For a proof see [**GR04**, Chap. 1].

Theorem 1.9. *Let $n \in \mathbb{N}$. Then*

$$ {}_2\phi_1\left(\begin{matrix} a, & q^{-n} \\ & c \end{matrix}; q; q\right) = \frac{(c/a)_n}{(c)_n} a^n. \tag{1.31}$$

Exercise 1.10. By reversing summation in (1.31), show that

$$ {}_2\phi_1\left(\begin{matrix} a, & q^{-n} \\ & c \end{matrix}; q; cq^n/a\right) = \frac{(c/a)_n}{(c)_n}. \tag{1.32}$$

Exercise 1.11. Show Newton's binomial series

$$\sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{n!} z^n = \frac{1}{(1-z)^a}, \quad |z|<1,\ a \in \mathbb{R} \tag{1.33}$$

can be derived from (1.23) by replacing a by q^a and letting $q \to 1^-$. For simplicity you can assume $a, z \in \mathbb{R}$.

Symmetric Functions

The Basics. Given $f(x_1, \ldots, x_n) \in K[x_1, x_2, \ldots, x_n]$ for some field K, and $\sigma \in S_n$, let

$$\sigma f = f(x_{\sigma_1}, \ldots, x_{\sigma_n}). \tag{1.34}$$

We say f is a *symmetric function* if $\sigma f = f, \quad \forall \sigma \in S_n$. It will be convenient to work with more general functions f depending on countably many indeterminates $x_1, x_2, \ldots$, indicated by $f(x_1, x_2, \ldots)$, in which case we view f as a formal power series in the x_i, and say it is a symmetric function if it is invariant under any permutation of the variables. The standard references on this topic are [**Sta99**, Chap. 7] and [**Mac95**]. We will often let X_n and X stand for the set of variables $\{x_1, \ldots, x_n\}$ and $\{x_1, x_2, \ldots\}$, respectively.

We let Λ denote the ring of symmetric functions in $x_1, x_2, \ldots$ and Λ^n the sub vector space consisting of symmetric functions of homogeneous degree n. The most basic symmetric functions are the monomial symmetric functions, which depend on a partition λ in addition to a set of variables. They are denoted by $m_\lambda(X) = m_\lambda(x_1, x_2, \ldots)$. In a symmetric function it is typical to leave off explicit mention of the variables, with a set of variables being understood from context, so $m_\lambda =$

$m_\lambda(X)$. We illustrate these first by means of examples. We let $\mathrm{Par}(n)$ denote the set of partitions of n, and use the notation $\lambda \vdash n$ as an abbreviation for $\lambda \in \mathrm{Par}(n)$.

EXAMPLE 1.12.

$$\begin{aligned} m_{1,1} &= \sum_{i<j} x_i x_j \\ m_{2,1,1}(X_3) &= x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 \\ m_2(X) &= \sum_i x_i^2. \end{aligned}$$

In general, $m_\lambda(X)$ is the sum of all distinct monomials in the x_i whose multiset of exponents equals the multiset of parts of λ. Any element of Λ can be expressed uniquely as a linear combination of the m_λ.

We let 1^n stand for the partition consisting of n parts of size 1. The function m_{1^n} is called the nth *elementary symmetric function*, which we denote by e_n. Then

$$\prod_{i=1}^{\infty}(1 + z x_i) = \sum_{n=0}^{\infty} z^n e_n, \quad e_0 = 1. \tag{1.35}$$

Another important special case is $m_n = \sum_i x_i^n$, known as the *power-sum symmetric functions*, denoted by p_n. We also define the *complete homogeneous symmetric functions* h_n, by $h_n = \sum_{\lambda \vdash n} m_\lambda$, or equivalently

$$\frac{1}{\prod_{i=1}^{\infty}(1 - z x_i)} = \sum_{n=0}^{\infty} z^n h_n, \quad h_0 = 1. \tag{1.36}$$

For $\lambda \vdash n$, we define $e_\lambda = \prod_i e_{\lambda_i}$, $p_\lambda = \prod_i p_{\lambda_i}$, and $h_\lambda = \prod_i h_{\lambda_i}$. For example,

$$\begin{aligned} e_{2,1} &= \sum_{i<j} x_i x_j \sum_k x_k = m_{2,1} + 3m_{1,1,1} \\ p_{2,1} &= \sum_i x_i^2 \sum_j x_j = m_3 + m_{2,1} \\ h_{2,1} &= (\sum_i x_i^2 + \sum_{i<j} x_i x_j) \sum_k x_k = m_3 + 2m_{2,1} + 3m_{1,1,1}. \end{aligned}$$

It is known that $\{e_\lambda, \lambda \vdash n\}$ forms a basis for Λ^n, and so do $\{p_\lambda, \lambda \vdash n\}$ and $\{h_\lambda, \lambda \vdash n\}$.

DEFINITION 1.13. Two simple functions on partitions we will often use are

$$\begin{aligned} n(\lambda) &= \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2} \\ z_\lambda &= \prod_i i^{n_i} n_i!, \end{aligned}$$

where $n_i = n_i(\lambda)$ is the number of parts of λ equal to i.

EXERCISE 1.14. Use (1.35) and (1.36) to show that

$$\begin{aligned} e_n &= \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)} p_\lambda}{z_\lambda}, \\ h_n &= \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}. \end{aligned}$$

We let ω denote the standard endomorphism $\omega : \Lambda \to \Lambda$ defined by

$$\omega(p_k) = (-1)^{k-1} p_k. \tag{1.37}$$

Thus ω is an involution with $\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda$, and by Exercise 1.14, $\omega(e_n) = h_n$, and more generally $\omega(e_\lambda) = h_\lambda$.

For $f \in \Lambda$, the special value $f(1, q, q^2, \ldots, q^{n-1})$ is known as the principal specialization (of order n) of f.

THEOREM 1.15.

$$e_m(1, q, \ldots, q^{n-1}) = q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} \tag{1.38}$$

$$h_m(1, q, \ldots, q^{n-1}) = \begin{bmatrix} n-1+m \\ m \end{bmatrix} \tag{1.39}$$

$$p_m(1, q, \ldots, q^{n-1}) = \frac{1-q^{nm}}{1-q^m}. \tag{1.40}$$

PROOF. The principal specializations for e_m and h_m follow directly from (1.27), (1.28), (1.35) and (1.36). □

REMARK 1.16. The principal specialization of m_λ doesn't have a particularly simple description, although if ps_n^1 denotes the set of n variables, each equal to 1, then [**Sta99**, p. 303]

$$m_\lambda(ps_n^1) = \binom{n}{n_1, n_2, n_3, \ldots}, \tag{1.41}$$

where again n_i is the multiplicity of the number i in the multiset of parts of λ.

REMARK 1.17. Identities like

$$h_{2,1} = m_3 + m_{2,1} + m_{1,1,1}$$

appear at first to depend on a set of variables, but it is customary to view them as polynomial identities in the p_λ. Since the p_k are algebraically independent, we can specialize them to whatever we please, forgetting about the original set of variables X.

EXERCISE 1.18. (1) Show that

$$\prod_{i=1}^{\infty}(1 + x_i t_1 + x_i^2 t_2 + x_i^3 t_3 + \ldots) = \sum_{\lambda \in \mathrm{Par}} m_\lambda(X) \prod_i t_{\lambda_i}. \tag{1.42}$$

Use this to prove (1.41).

(2) If $f \in \Lambda$ and $x \in \mathbb{R}$, let $f(ps_x^1)$ denote the polynomial in x obtained by first expanding f in terms of the p_λ, then replacing each p_k by x. Use Theorem 1.15 and 1.41 to show that

$$e_m(ps_x^1) = \binom{x}{m}$$

$$h_m(ps_x^1) = \binom{x+m-1}{m}$$

$$m_\lambda(ps_x^1) = \binom{x}{n_1, n_2, n_3, \ldots}.$$

We define the Hall scalar product, a bilinear form from $\Lambda \times \Lambda \to \mathbb{Q}$, by

$$\langle p_\lambda, p_\beta \rangle = z_\lambda \chi(\lambda = \beta), \tag{1.43}$$

where for any logical statement L

$$\chi(L) = \begin{cases} 1 & \text{if } L \text{ is true} \\ 0 & \text{if } L \text{ is false.} \end{cases} \tag{1.44}$$

Clearly $< f, g >=< g, f >$. Also, $< \omega f, \omega g >=< f, g >$, which follows from the definition if $f = p_\lambda, g = p_\beta$, and by bilinearity for general f, g since the p_λ form a basis for Λ.

Theorem 1.19. *The h_λ and the m_β are dual with respect to the Hall scalar product, i.e.*

$$\langle h_\lambda, m_\beta \rangle = \chi(\lambda = \beta). \tag{1.45}$$

Proof. See [**Mac95**] or [**Sta99**]. □

For any $f \in \Lambda$, and any basis $\{b_\lambda, \lambda \in \text{Par}\}$ of Λ, let $f|_{b_\lambda}$ denote the coefficient of b_λ when f is expressed in terms of the b_λ. Then (1.45) implies

Corollary 1.19.1.

$$\langle f, h_\lambda \rangle = f|_{m_\lambda}. \tag{1.46}$$

Tableaux and Schur Functions. Given $\lambda, \mu \in \text{Par}(n)$, a semi-standard Young tableaux (or SSYT) of shape λ and weight μ is a filling of the cells of the Ferrers graph of λ with the elements of the multiset $\{1^{\mu_1} 2^{\mu_2} \cdots\}$, so that the numbers weakly increase across rows and strictly increase up columns. Let $SSYT(\lambda, \mu)$ denote the set of these fillings, and $K_{\lambda,\mu}$ the cardinality of this set. The $K_{\lambda,\mu}$ are known as the Kostka numbers. Our definition also makes sense if our weight is a weak composition of n , i.e. any finite sequence of nonnegative integers whose sum is n. For example, $K_{(3,2),(2,2,1)} = K_{(3,2),(2,1,2)} = K_{(3,2),(1,2,2)} = 2$ as in Figure 3.

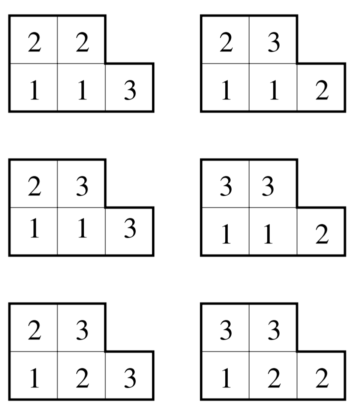

Figure 3. Some SSYT of shape $(3, 2)$.

If the Ferrers graph of a partition β is contained in the Ferrers graph of λ, denoted $\beta \subseteq \lambda$, let λ/β refer to the subset of cells of λ which are not in β. This is referred to as a *skew shape*. Define a SSYT of shape λ/β and weight ν, where

$|\nu| = |\lambda| - |\beta|$, to be a filling of the cells of λ/β with elements of $\{1^{\nu_1}2^{\nu_2}\cdots\}$, again with weak increase across rows and strict increase up columns. The number of such tableaux is denoted $K_{\lambda/\beta,\nu}$.

Let wcomp(μ) denote the set of all weak compositions whose multiset of nonzero parts equals the multiset of parts of μ. It follows easily from Figure 3 that $K_{(3,2),\alpha} = 2$ for all $\alpha \in \text{wcomp}(2,2,1)$. Hence

$$\sum_{\alpha,T}\prod_i x_i^{\alpha_i} = 2m_{(2,2,1)}, \tag{1.47}$$

where the sum is over all tableaux T of shape $(3,2)$ and weight some element of wcomp$(2,2,1)$.

This is a special case of a more general phenomenon. For $\lambda \in \text{Par}(n)$, define

$$s_\lambda = \sum_{\alpha,T}\prod_i x_i^{\alpha_i}, \tag{1.48}$$

where the sum is over all weak compositions α of n, and all possible tableaux T of shape λ and weight α. Then

$$s_\lambda = \sum_{\mu\vdash n} K_{\lambda,\mu} m_\mu. \tag{1.49}$$

The s_λ are called *Schur functions*, and are fundamental to the theory of symmetric functions. Two special cases of (1.49) are $s_n = h_n$ (since $K_{n,\mu} = 1$ for all $\mu \in \text{Par}(n)$) and $s_{1^n} = e_n$ (since $K_{1^n,\mu} = \chi(\mu = 1^n)$).

A SSYT of weight 1^n is called *standard*, or a SYT. The set of SYT of shape λ is denoted $SYT(\lambda)$. For $(i,j) \in \lambda$, let the "content" of (i,j), denoted $c(i,j)$, be $j-i$. Also, let $h(i,j)$ denote the "hook length" of (i,j), defined as the number of cells to the right of (i,j) in row i plus the number of cells above (i,j) in column j plus 1. For example, if $\lambda = (5,5,3,3,1)$, $h(2,2) = 6$. It is customary to let f^λ denote the number of SYT of shape λ, i.e. $f^\lambda = K_{\lambda,1^n}$. There is a beautiful formula for f^λ, namely

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}. \tag{1.50}$$

Below we list some of the important properties of Schur functions.

THEOREM 1.20. *Let $\lambda, \mu \in Par$. Then*

(1) *The Schur functions are orthonormal with respect to the Hall scalar product, i.e.*

$$< s_\lambda, s_\mu >= \chi(\lambda = \mu).$$

Thus for any $f \in \Lambda$,

$$< f, s_\lambda >= f|_{s_\lambda}.$$

(2) *Action by ω:*

$$\omega(s_\lambda) = s_{\lambda'}.$$

(3) *(The Jacobi-Trudi identity)*

$$s_\lambda = det(h_{\lambda_i - i + j})_{i,j=1}^{\ell(\lambda)},$$

where we set $h_0 = 1$ *and* $h_k = 0$ *for* $k < 0$. *For example,*

$$\begin{aligned} s_{2,2,1} &= \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} \\ &= h_{2,2,1} - h_{3,2} - h_{3,1,1} + h_{4,1}. \end{aligned}$$

(4) *(The Pieri rule). Let* $k \in \mathbb{N}$. *Then*

$$s_\lambda h_k = \sum_\gamma s_\gamma,$$

where the sum is over all γ *whose Ferrers graph contains* λ *with* $|\gamma/\lambda| = k$ *and such that* γ/λ *is a "horizontal strip", i.e. has no two cells in any column. Also,*

$$s_\lambda e_k = \sum_\gamma s_\gamma,$$

where the sum is over all partitions γ *whose Ferrers graph contains* λ *with* $|\gamma/\lambda| = k$ *and such that* γ/λ *is a "vertical strip", i.e. has no two cells in any row. For example,*

$$\begin{aligned} s_{2,1}h_2 &= s_{4,1} + s_{3,2} + s_{3,1,1} + s_{2,2,1} \\ s_{3,1}e_2 &= s_{4,2} + s_{4,1,1} + s_{3,2,1} + s_{3,1,1,1}. \end{aligned}$$

(5) *(Principal Specialization).*

$$s_\lambda(1, q, q^2, \ldots, q^{n-1}) = q^{n(\lambda)} \prod_{(i,j)\in\lambda} \frac{[n + c(i,j)]}{[h(i,j)]},$$

and taking the limit as $n \to \infty$ *we get*

$$s_\lambda(1, q, q^2, \ldots) = q^{n(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1 - q^{h(i,j)}}.$$

(6) *For any two alphabets* Z, W,

$$s_\lambda(Z + W) = \sum_{\beta\subseteq\lambda} s_\beta(Z)s_{\lambda'/\beta'}(W). \tag{1.51}$$

(7) *(The Littlewood-Richardson rule) For all partitions* λ, β,

$$s_\lambda s_\beta = \sum_\gamma s_\gamma c^\gamma_{\lambda\beta},$$

where $c^\gamma_{\lambda,\beta}$ *is the number of SSYT* T *of skew shape* γ/β *and weight* λ *such that if we form the word* $\sigma_1\sigma_2\cdots$ *by reading the entries of* T *across rows from right to left, and from bottom to top, then in any initial segment* $\sigma_1\cdots\sigma_j$ *there are at least as many* i*'s as* $i+1$*'s, for each* $1 \le i$. *(Such words are known as lattice permutations, and the corresponding SSYT are called Yamanouchi). Note that this rule contains the Pieri rules above as special cases. For a proof of the Littlewood-Richardson rule see* [**Mac88**, Chap. 1].

EXERCISE 1.21. Let $0 \le 2k \le n$. Show

$$\sum_{\lambda} f^{\lambda} = \binom{n}{k},$$

where the sum is over all $\lambda \in \mathrm{Par}(n)$ with $\ell(\lambda) \le 2$ and $\lambda_1 \ge n-k$.

Statistics on Tableaux. There is a q-analogue of the Kostka numbers, denoted by $K_{\lambda,\mu}(q)$, which has many applications in representation theory and the combinatorics of tableaux. Originally defined algebraically in an indirect fashion, the $K_{\lambda,\mu}(q)$ are polynomials in q which satisfy $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$. Foulkes [**Fou74**] conjectured that there should be a statistic $\mathrm{stat}(T)$ on SSYT T of shape λ and weight μ such that

$$K_{\lambda,\mu}(q) = \sum_{T \in SSYT(\lambda)} q^{\mathrm{stat}(T)}. \tag{1.52}$$

This conjecture was resolved by Lascoux and Schützenberger [**LS78**], who found a statistic *charge* to generate these polynomials. Butler [**But94**] provided a detailed account of their proof, filling in a lot of missing details. A short proof, based on the new combinatorial formula for Macdonald polynomials, is contained in Appendix A.

Assume we have a tableau $T \in SSYT(\lambda,\mu)$ where $\mu \in \mathrm{Par}$. It will be more convenient for us to describe a slight modification of $\mathrm{charge}(T)$, called $\mathrm{cocharge}(T)$, which is defined as $n(\mu) - \mathrm{charge}$. The *reading word* $\mathrm{read}(T)$ of T is obtained by reading the entries in T from left to right in the top row of T, then continuing left to right in the second row from the top of T, etc. For example, the tableau in the upper-left of Figure 3 has reading word 22113. To calculate $\mathrm{cocharge}(T)$, perform the following algorithm on $\mathrm{read}(T)$.

ALGORITHM 1.22. (1) *Start at the end of read*(T) *and scan left until you encounter a* 1 *- say this occurs at spot* i_1, *so read*$(T)_{i_1} = 1$. *Then start there and scan left until you encounter a* 2. *If you hit the end of read*(T) *before finding a* 2, *loop around and continue searching left, starting at the end of read*(T). *Say the first* 2 *you find equals read*$(T)_{i_2}$. *Now iterate, start at* i_2 *and search left until you find a* 3, *etc. Continue in this way until you have found* $4, 5, \ldots, \mu_1$, *with* μ_1 *occurring at spot* i_{μ_1}. *Then the first subword of textread*(T) *is defined to be the elements of the set* $\{\mathrm{read}(T)_{i_1}, \ldots, \mathrm{read}(T)_{i_{\mu_1}}\}$, *listed in the order in which they occur in read*(T) *if we start at the beginning of read*(T) *and move left to right. For example, if read*$(T) = 21613244153$ *then the first subword equals* 632415, *corresponding to places* $3, 5, 6, 8, 9, 10$ *of read*(T).

Next remove the elements of the first subword from read(T) *and find the first subword of what's left. Call this the second subword. Remove this and find the first subword in what's left and call this the third subword of read*(T), *etc. For the word* 21613244153, *the subwords are* 632415, 2143, 1.

(2) *The value of charge*(T) *will be the sum of the values of charge on each of the subwords of* $rw(T)$. *Thus it suffices to assume* $rw(T) \in S_m$ *for some* m, *in which case we set*

$$cocharge(rw(T)) = comaj(rw(T)^{-1}),$$

where $read(T)^{-1}$ *is the usual inverse in* S_m*, with comaj as in (1.12). (Another way of describing cocharge(read(T)) is the sum of* $m-i$ *over all* i *for which* $i+1$ *occurs before* i *in read(T).) For example, if* $\sigma = 632415$*, then* $\sigma^{-1} = 532461$ *and cocharge*$(\sigma) = 5+4+1 = 10$*, and finally*

$$\mathit{cocharge}(21613244153) = 10 + 4 + 0 = 14.$$

Note that to compute charge, we could create subwords in the same manner, and count $m-i$ for each i with $i+1$ occurring to the right of i instead of to the left. For $\lambda, \mu \in \mathrm{Par}(n)$ we set

$$\tilde{K}_{\lambda,\mu}(q) = q^{n(\lambda)} K_{\lambda,\mu}(1/q) = \sum_{T \in SSYT(\lambda,\mu)} q^{\mathrm{cocharge}(T)}. \tag{1.53}$$

These polynomials have an interpretation in terms of representation theory which we describe in Chapter 2.

In addition to the cocharge statistic, there is a major index statistic on SYT which is often useful. Given a SYT tableau T of shape λ , define a descent of T to be a value of i, $1 \le i < |\lambda|$, for which $i+1$ occurs in a row above i in T. Let

$$\mathrm{maj}(T) = \sum i \tag{1.54}$$

$$\mathrm{comaj}(T) = \sum |\lambda| - i, \tag{1.55}$$

where the sums are over the descents of T. Then [**Sta99**, p.363]

$$s_\lambda(1, q, q^2, \ldots,) = \frac{1}{(q)_n} \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)} = \frac{1}{(q)_n} \sum_{T \in SYT(\lambda)} q^{\mathrm{comaj}(T)}. \tag{1.56}$$

The RSK Algorithm

In this subsection we overview some of the basic properties of the famous Robinson-Schensted-Knuth (RSK) algorithm, which gives a bijection between certain two-line arrays of positive integers and pairs of SSYT (P,Q) of the same shape. A more detailed discussion of the RSK algorithm can be found in [**Sta99**].

One of the key components of the algorithm is *row insertion*, which takes as input an arbitrary positive integer k and a SSYT P and outputs a SSYT denoted by $P \leftarrow k$. Row insertion begins by comparing k with the numbers in the bottom row (i.e. row 1) of P. If there are no entries in row 1 which are larger than k, then we form $P \leftarrow k$ by adjoining k to the end of row 1 of P and we are done. Otherwise, we replace (or "bump") the leftmost number in row 1 which is larger than k (say this number is m) by k, and then insert m into row 2 in the same way. That is, if there are no numbers in row 2 larger than m we adjoin m to the end of row 2, otherwise we replace the leftmost number larger than m by m and then insert this number into row 3, etc. It is fairly easy to see that we eventually end up adjoining some number to the end of a row, and so row insertion always terminates, We call this row the terminal row. The output $P \leftarrow k$ is always a SSYT. See Figure 4 for an example.

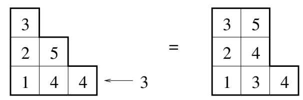

FIGURE 4. Row insertion of 3 into the tableau on the left results in the 3 bumping the 4 in square $(2,1)$, which bumps the 5 in square $(2,2)$, which is adjoined to the end of row 3 (the terminal row) resulting in the tableau on the right.

Now start with a $2 \times n$ matrix A of positive integers, where the top row is monotone nondecreasing, i.e. $a_{1,i} \leq a_{1,i+1}$. Also assume that below equal entries in the top row, the bottom row is nondecreasing, i.e. that

$$a_{1,i} = a_{1,i+1} \implies a_{2,i} \leq a_{2,i+1}. \tag{1.57}$$

We call such a matrix an *ordered two-line array*, and let A_1 denote the word $a_{11}a_{12}\cdots a_{1n}$ and A_2 the word $a_{21}a_{22}\cdots a_{2n}$. Given such an A, form a sequence of SSYT $P^{(j)}$ by initializing $P^{(0)} = \emptyset$, then for $1 \leq j \leq n$, let

$$P^{(j)} = P^{(j-1)} \leftarrow a_{2,j}. \tag{1.58}$$

As we form the $P^{(j)}$, we also form a complementary sequence of SSYT $Q^{(j)}$ by initializing $Q^{(0)} = \emptyset$, and for $1 \leq j \leq n$, we create $Q^{(j)}$ by adjoining $a_{1,j}$ to the end of row r_j of $Q^{(j-1)}$, where r_j is the terminal row of $P^{(j-1)} \leftarrow a_{2,j}$. Finally, we set $P = P^{(n)}$ (the "insertion tableau") and $Q = Q^{(n)}$ (the "recording tableau"). If $A_1 = 112445$ and $A_2 = 331251$ the corresponding P and Q tableau are given in Figure 5.

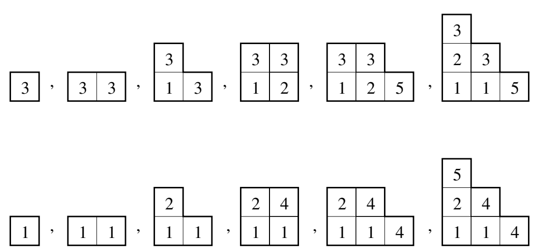

FIGURE 5. On the top, the sequence $P^{(j)}$ and on the bottom, the sequence $Q^{(j)}$, for the ordered two-line array A with $A_1 = 112445$ and $A_2 = 331251$.

THEOREM 1.23. [**Rob38**], [**Sch61**], [**Knu70**] *There is a bijection between ordered two-line arrays A and pairs of SSYT (P,Q) of the same shape. Under this correspondence, the weight of A_1 equals the weight of Q, the weight of A_2 is the weight of P, and $Des(Q) = Des(A_2)$. If A_1 is the identity permutation $12\cdots n$, then we get a bijection between words w and pairs (P_w, Q_w) of tableau of the same shape, with P_w a SSYT of the same weight as w, and Q a SYT. If the top row is the identity permutation and the bottom row is a permutation β, then we get a bijection between permutations β and pairs (P_β, Q_β) of SYT of the same shape.*

REMARK 1.24. It is an interesting fact that for $\beta \in S_n$,

$$P_\beta = Q_{\beta^{-1}}.$$

Plethystic Notation. Many of the theorems later in the course involving symmetric functions will be expressed in plethystic notation. In this subsection we define this and give several examples in order to acclimate the reader.

Let $E(t_1, t_2, t_3 \ldots)$ be a formal series of rational functions in the parameters $t_1, t_2, \ldots$. We define the plethystic substitution of E into p_k, denoted $p_k[E]$, by

$$p_k[E] = E(t_1^k, t_2^k, \ldots). \tag{1.59}$$

Note the square "plethystic" brackets around E - this is to distinguish $p_k[E]$ from the ordinary kth power sum in a set of variables E, which we have already defined as $p_k(E)$. One thing we need to emphasize is that any minus signs occurring in the definition of E are left as is when replacing the t_i by t_i^k.

EXAMPLE 1.25.

(1) $p_k[X] = p_k(X)$. As it is, $p_k[X]$ makes no sense since X indicates a set of variables, not a formal series of rational functions. However, it is traditional to adopt the convention that inside plethystic brackets a capital letter, say Z, stands for $p_1(Z) = z_1 + z_2 + \ldots$.

(2) For z a real parameter,
$$p_k[zX] = z^k p_k[X]$$

(3)
$$p_k[X(1-t)] = \sum_i x_i^k (1-t^k)$$

(4)
$$p_k[X-Y] = \sum_i x_i^k - y_i^k = p_k[X] - p_k[Y]$$

(5)
$$p_k\left[\frac{X}{1-q}\right] = \sum_i \frac{x_i^k}{1-q^k}$$

(6)
$$p_k\left[\frac{X(1-z)}{1-q}\right] = \sum_i \frac{x_i^k(1-z^k)}{1-q^k}.$$

Note that (4) and (6) are consistent in that

$$p_k\left[\frac{X(1-z)}{1-q}\right] = p_k\left[\frac{X-Xz}{1-q}\right] =$$
$$p_k\left[\frac{X}{1-q} - \frac{Xz}{1-q}\right] = \sum_i \frac{x_i^k}{1-q^k} - \sum_i \frac{z^k x_i^k}{1-q^k}$$

which reduces to $p_k[X]$ when $z = q$, as it should.

Let $Z = (-x_1, -x_2, \ldots)$. Note that $p_k(Z) = \sum_i (-1)^k x_i^k$, which is different from $p_k[-X]$. Thus we need a special notation for the case where we wish to replace variables by their negatives inside plethystic brackets. We use the ϵ symbol to denote this, i.e.

$$p_k[\epsilon X] = \sum_i (-1)^k x_i^k. \tag{1.60}$$

We now extend the definition of plethystic substitution of E into f for an arbitrary $f \in \Lambda$ by first expressing f as a polynomial in the p_k, say $f = \sum_\lambda c_\lambda p_\lambda$ for constants c_λ, then defining $f[E]$ as

$$f[E] = \sum_\lambda c_\lambda \prod_i p_{\lambda_i}[E]. \tag{1.61}$$

EXAMPLE 1.26. For any $f \in \Lambda$,

$$\omega(f(X)) = f[-\epsilon X]. \tag{1.62}$$

We now derive some very useful identities involving plethysm.

THEOREM 1.27. *The "addition formulas". Let $E = E(t_1, t_2, \ldots)$ and $F = F(w_1, w_2, \ldots)$ be two formal series of rational terms in their indeterminates. Then*

$$e_n[E - F] = \sum_{k=0}^{n} e_k[E] e_{n-k}[-F] \tag{1.63}$$

$$e_n[E + F] = \sum_{k=0}^{n} e_k[E] e_{n-k}[F] \tag{1.64}$$

$$h_n[E - F] = \sum_{k=0}^{n} h_k[E] h_{n-k}[-F] \tag{1.65}$$

$$h_n[E + F] = \sum_{k=0}^{n} h_k[E] h_{n-k}[F]. \tag{1.66}$$

We also have the Cauchy identities

$$e_n[EF] = \sum_{\lambda \in Par(n)} s_\lambda[E] s_{\lambda'}[F] \tag{1.67}$$

$$h_n[EF] = \sum_{\lambda \in Par(n)} s_\lambda[E] s_\lambda[F]. \tag{1.68}$$

PROOF. By definition of $p_k[E-F]$ and Exercise 1.14,

$$
\begin{aligned}
e_n[E-F] &= \sum_{\lambda\in\mathrm{Par}(n)} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda}\prod_m (E(t_1^{\lambda_m},\ldots)-F(w_1^{\lambda_m},\ldots)) \\
&= \sum_{\lambda\in\mathrm{Par}(n)} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda}\prod_i (E(t_1^{i},\ldots)-F(w_1^{i},\ldots))^{n_i} \\
&= \sum_{\lambda\in\mathrm{Par}(n)} \frac{(-1)^{n-\ell(\lambda)}}{\prod_i i^{n_i} n_i!}\prod_i \sum_{r_i=0}^{n_i}\frac{n_i!}{r_i!(n_i-r_i)!}(E(t_1^i,\ldots))^{r_i}(-F(w_1^i,\ldots))^{n_i-r_i} \\
&= \sum_{\lambda\in\mathrm{Par}(n)} (-1)^{n-\ell(\lambda)} \\
&\quad\times \prod_i \sum_{r_i=0}^{n_i}\frac{1}{i^{r_i}r_i!i^{n_i-r_i}(n_i-r_i)!}(E(t_1^i,\ldots))^{r_i}(-F(w_1^i,\ldots))^{n_i-r_i} \\
&= \sum_{k=0}^{n}\sum_{\substack{\beta\in\mathrm{Par}(k)\\ (n_i(\beta)=r_i)}} \frac{(-1)^{k-\ell(\beta)}}{z_\beta}\prod_m E(t_1^{\beta_m},\ldots) \\
&\quad\times \sum_{\substack{\zeta\in\mathrm{Par}(n-k)\\ (n_i(\zeta)=n_i-r_i)}} \frac{(-1)^{n-k-\ell(\zeta)}}{z_\zeta}\prod_m (-F(w_1^{\zeta_m},\ldots)) \\
&= \sum_{k=0}^{n} e_k[E]e_{n-k}[-F].
\end{aligned}
$$

Minor modifications to the above sequence of steps also proves (1.64), (1.65) and (1.66).

By definition,

$$
p_k[EF] = E(t_1^k,\ldots)F(w_1^k,\ldots) \tag{1.69}
$$

so for any $\lambda\in\mathrm{Par}$,

$$
p_\lambda[EF] = \prod_i E(t_1^{\lambda_i},\ldots)F(w_1^{\lambda_i},\ldots) = p_\lambda[E]p_\lambda[F]. \tag{1.70}
$$

Using this both sides of (1.67) and (1.68) can be viewed as polynomial identities in the $p_k[E]$ and $p_k[F]$. Thus they are true if they are true when we replace $p_k[E]$ by $p_k(X)$ and $p_k[F]$ by $p_k(Y)$, in which case they reduce to the well-known non-plethystic Cauchy identities. □

EXAMPLE 1.28. For $\mu\vdash n$, the Hall-Littlewood symmetric function $P_\mu(X;q)$ can be defined as

$$
P_\mu(X;q) = \sum_{\lambda\vdash n}\frac{K_{\lambda,\mu}(q)}{\prod_i (q)_{n_i(\lambda)}} s_\lambda[X(1-q)]. \tag{1.71}
$$

(It is well-known that the $s_\lambda[X(1-q)]$ form a basis for the ring of symmetric functions with coefficients in $\mathbb{Q}(q)$, the family of rational functions in q, and so do the $P_\mu(X;q)$.) Let $\lambda\vdash n$, and z a real parameter. Then $s_\lambda[1-z]=0$ if λ is not a "hook" (a hook shape is where $\lambda_2 \le 1$), in fact

$$s_\lambda[1-z] = \begin{cases} (-z)^r(1-z) & \text{if } \lambda = (n-r, 1^r), \quad 0 \le r \le n-1 \\ 0 & \text{else.} \end{cases} \tag{1.72}$$

Representation Theory

Let G be a finite group. A representation of G is a set of square matrices $\{M(g),\, g \in G\}$ with the property that

$$M(g)M(h) = M(gh) \quad \forall g, h \in G. \tag{1.73}$$

On the left-hand-side of (1.73) we are using ordinary matrix multiplication, and on the right-hand-side, to define gh, multiplication in G. The number of rows of a given $M(g)$ is called the dimension of the representation.

An *action* of G on a set S is a map from $G \times S \to S$, denoted by $g(s)$ for $g \in G$, $s \in S$, which satisfies

$$g(h(s)) = (gh)(s) \quad \forall\, g, h \in G,\ s \in S. \tag{1.74}$$

Let V be a finite dimensional $\mathbb{C}$ vector space, with basis $w_1, w_2, \ldots w_n$. Any action of G on V makes V into a $\mathbb{C}G$ module. A module is called *irreducible* if it has no submodules other than $\{0\}$ and itself. Maschke's theorem [**JL01**] says that every V can be expressed as a direct sum of irreducible submodules.

If we form a matrix $M(g)$ whose ith row consists of the coefficients of the w_j when expanding $g(w_i)$ in the w basis, then $\{M(g), g \in G\}$ is a representation of G. In general $\{M(g), g \in G\}$ will depend on the choice of basis, but the trace of the matrices will not. The trace of the matrix $M(g)$ is called the character of the module (under the given action), which we denote $\operatorname{char}(V)$. If $V = \bigoplus_{j=1}^d V_j$, where each V_j is irreducible, then the basis can be ordered so that the matrix M will be in block form, where the sizes of the blocks are the dimensions of the V_j. Clearly $\operatorname{char}(V) = \sum_{j=1}^d \operatorname{char}(V_j)$. It turns out that there are only a certain number of possible functions which occur as characters of irreducible modules, namely one for each conjugacy class of G. These are called the irreducible characters of G.

In the case $G = S_n$, the conjugacy classes are in one-to-one correspondence with partitions $\lambda \in \operatorname{Par}(n)$, and the irreducible characters are denoted $\chi^\lambda(\sigma)$. The dimension of a given V_j with character χ^λ is known to always be f^λ. The value of a given $\chi^\lambda(\sigma)$ depends only on the conjugacy class of σ. For the symmetric group the conjugacy class of an element is determined by rearranging the lengths of the disjoint cycles of σ into nonincreasing order to form a partition called the cycle-type of σ. Thus we can talk about $\chi^\lambda(\beta)$, which means the value of χ^λ at any permutation of cycle type β. For example, $\chi^{(n)}(\beta) = 1$ for all $\beta \vdash n$, so $\chi^{(n)}$ is called the trivial character. Also, $\chi^{1^n}(\beta) = (-1)^{n-\ell(\beta)}$ for all $\beta \vdash n$, so χ^{1^n} is called the sign character, since $(-1)^{n-\ell(\beta)}$ is the sign of any permutation of cycle type β.

One reason Schur functions are important in representation theory is the following.

THEOREM 1.29. *When expanding the s_λ into the p_λ basis, the coefficients are the χ^λ. To be exact*

$$p_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu) s_\lambda$$
$$s_\lambda = \sum_{\mu \vdash n} z_\mu^{-1} \chi^\lambda(\mu) p_\mu.$$

Let $\mathbb{C}[X_n] = \mathbb{C}[x_1, \ldots, x_n]$. Given $f(x_1, \ldots, x_n) \in \mathbb{C}[X_n]$ and $\sigma \in S_n$, then

$$\sigma f = f(x_{\sigma_1}, \ldots, x_{\sigma_n}) \tag{1.75}$$

defines an action of S_n on $\mathbb{C}[X_n]$. We should mention that here we are viewing the permutation σ in a different fashion than when discussing permutation statistics, since here σ is sending the variable in the σ_1st coordinate to the first coordinate, i.e.

$$\sigma = \begin{pmatrix} \ldots & \sigma_1 & \ldots \\ \ldots & 1 & \ldots \end{pmatrix}, \tag{1.76}$$

which is σ^{-1} in our previous notation. To verify that (1.75) defines an action, note that in this context $\beta(\sigma(f)) = f(x_{\sigma_{\beta_1}}, \ldots)$ while

$$\beta\sigma = \begin{pmatrix} \ldots & \beta_1 & \ldots \\ \ldots & 1 & \ldots \end{pmatrix} \begin{pmatrix} \ldots & \sigma_{\beta_1} & \ldots \\ \ldots & \beta_1 & \ldots \end{pmatrix} = \begin{pmatrix} \ldots & \sigma_{\beta_1} & \ldots \\ \ldots & 1 & \ldots \end{pmatrix}. \tag{1.77}$$

Let V be a subspace of $\mathbb{C}[X_n]$. Then

$$V = \sum_{i=0}^{\infty} V^{(i)}, \tag{1.78}$$

where $V^{(i)}$ is the subspace consisting of all elements of V of homogeneous degree i in the x_j. Each $V^{(i)}$ is finite dimensional, and this gives a "grading" of the space V. We define the *Hilbert series* $\mathcal{H}(V; q)$ of V to be the sum

$$\mathcal{H}(V; q) = \sum_{i=0}^{\infty} q^i \dim(V^{(i)}), \tag{1.79}$$

where dim indicates the dimension as a $\mathbb{C}$ vector space.

Assume that V is a subspace of $\mathbb{C}[X_n]$ fixed by the S_n action. We define the *Frobenius series* $\mathcal{F}(V; q)$ of V to be the symmetric function

$$\sum_{i=0}^{\infty} q^i \sum_{\lambda \in \mathrm{Par}(i)} \mathrm{Mult}(\chi^\lambda, V^{(i)}) s_\lambda, \tag{1.80}$$

where $\mathrm{Mult}(\chi^\lambda, V^{(i)})$ is the multiplicity of the irreducible character χ^λ in the character of $V^{(i)}$ under the action. In other words, if we decompose $V^{(i)}$ into irreducible S_n-submodules, $\mathrm{Mult}(\chi^\lambda, V^{(i)})$ is the number of these submodules whose trace equals χ^λ.

A polynomial in $\mathbb{C}[X_n]$ is *alternating*, or an alternate, if

$$\sigma f = (-1)^{\mathrm{inv}(\sigma)} f \quad \forall\, \sigma \in S_n. \tag{1.81}$$

The set of alternates in V forms a subspace called the subspace of alternates, or anti-symmetric elements, denoted V^ϵ. This is also an S_n-submodule of V.

PROPOSITION 1.29.1. *The Hilbert series of* V^ϵ *equals the coefficient of* s_{1^n} *in the Frobenius series of* V*, i.e.*

$$\mathcal{H}(V^\epsilon; q) = \langle \mathcal{F}(V; q), s_{1^n} \rangle . \tag{1.82}$$

PROOF. Let B be a basis for $V^{(i)}$ with the property that the matrices $M(\sigma)$ are in block form. Then $b \in B$ is also in $V^{\epsilon(i)}$ if and only if the row of $M(\sigma)$ corresponding to b has entries $(-1)^{\mathrm{inv}(\sigma)}$ on the diagonal and 0's elsewhere, i.e. is a block corresponding to χ^{1^n}. Thus

$$\langle \mathcal{F}(V; q), s_{1^n} \rangle = \sum_{i=0}^{\infty} q^i \dim(V^{\epsilon(i)}) = \mathcal{H}(V^\epsilon; q). \tag{1.83}$$

□

REMARK 1.30. Since the dimension of the representation corresponding to χ^λ equals f^λ, which by (1.46) equals $< s_\lambda, h_{1^n} >$, we have

$$\langle \mathcal{F}(V; q), h_{1^n} \rangle = \mathcal{H}(V; q). \tag{1.84}$$

EXAMPLE 1.31. Since a basis for $\mathbb{C}[X_n]$ can be obtained by taking all possible monomials in the x_i,

$$\mathcal{H}(\mathbb{C}[X_n]; q) = (1 - q)^{-n}. \tag{1.85}$$

Taking into account the S_n-action, it is known that

$$\begin{aligned} \mathcal{F}(\mathbb{C}[X_n]; q) &= \sum_{\lambda \in \mathrm{Par}(n)} s_\lambda \frac{\sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)}}{(q)_n} \\ &= \sum_{\lambda \in \mathrm{Par}(n)} s_\lambda s_\lambda(1, q, q^2, \ldots) = \prod_{i,j} \frac{1}{(1 - q^i x_j z)}|_{z^n}. \end{aligned} \tag{1.86}$$

The Ring of Coinvariants. The set of symmetric polynomials in the x_i, denoted $\mathbb{C}[X_n]^{S_n}$, which is generated by $1, e_1, \ldots e_n$, is called the *ring of invariants*. The quotient ring $R_n = \mathbb{C}[x_1, \ldots, x_n]/ < e_1, e_2, \ldots, e_n >$, or equivalently $\mathbb{C}[x_1, \ldots, x_n]/ < p_1, p_2, \ldots, p_n >$, obtained by moding out by the ideal generated by all symmetric polynomials of positive degree is known as the *ring of coinvariants*. It is known that R_n is finite dimensional as a $\mathbb{C}$-vector space, with $\dim(R_n) = n!$, and more generally that

$$\mathcal{H}(R_n; q) = [n]!. \tag{1.87}$$

E. Artin [**Art42**] derived a specific basis for R_n, namely

$$\{ \prod_{1 \le i \le n} x_i^{\alpha_i}, \, 0 \le \alpha_i \le i - 1 \}. \tag{1.88}$$

Also,

$$\mathcal{F}(R_n; q) = \sum_{\lambda \in \mathrm{Par}(n)} s_\lambda \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)}, \tag{1.89}$$

a result that Stanley [**Sta79**], [**Sta03**] attributes to unpublished work of Lusztig. This shows the Frobenius series of R_n is $(q)_n$ times the Frobenius series of $\mathbb{C}[X_n]$.

Let

$$\Delta = \det \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ & & \vdots & \\ 1 & x_n & \dots & x_n^{n-1} \end{bmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

be the Vandermonde determinant. The *space of harmonics* H_n can be defined as the $\mathbb{C}$ vector space spanned by V_n and its partial derivatives of all orders. Haiman [**Hai94**] provides a detailed proof that H_n is isomorphic to R_n as an S_n module, and notes that an explicit isomorphism α is obtained by letting $\alpha(h), h \in H_n$, be the element of $\mathbb{C}[X_n]$ represented modulo $< e_1, \dots, e_n >$ by h. Thus $\dim(H_n) = n!$ and moreover the character of H_n under the S_n-action is given by (1.89). He also argues that (1.89) follows immediately from (1.86) and the fact that H_n generates $\mathbb{C}[X_n]$ as a free module over $\mathbb{C}[X_n]^{S_n}$.