CHAPTER 10

Deformations of tight closure and the localization problem

In this last chapter we want to discuss arithmetic and geometric deformations of tight closure, of (strong) semistability, and of the Hilbert-Kunz multiplicity. In arithmetic deformations we look at the dependence of tight closure etc. on varying prime numbers, so the base space of the deformation is $\operatorname{Spec}\mathbb{Z}$.[1] The generic case (over the 0-ideal) in such a situation is in characteristic zero and the special cases (over a maximal ideal) are in varying positive characteristics. In geometric deformations, the characteristic is fixed and positive, and the data depend on certain parameters, the easiest case being $\mathbb{A}^1_K = \operatorname{Spec} K[U]$. Generically we have transcendental elements, while the special instances are algebraic over K. We will give below (Example 10.8) an example of a geometric deformation over $\mathbb{A}^1_K$ which gives a negative answer to the localization problem in tight closure theory (Problem 1.8).

We start with the situation of arithmetic deformations, which is the setting where tight closure in characteristic zero is defined (see also the end of Chapter 1). So let a finitely generated $\mathbb{Z}$-domain $\mathbb{Z} \subseteq S$ be given and consider the generic fiber $S_{\mathbb{Q}} = S \otimes_{\mathbb{Z}} \mathbb{Q} = R$ in characteristic zero and the special fibers $R_p = S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ in positive characteristic p. Recall that for an ideal I in S and an element $f \in S$ one says that $f \in I^*$ holds in R if and only if the corresponding statement holds for almost all prime reductions, that is, if $f \in I_p^*$ holds in R_p for almost all primes p.

How does the containment $f \in I^*$ depend on the prime characteristic? A natural question is the following (see [**54**, Appendix 1], [**47**, Question 13], and [**60**, §4]): suppose that $f \in I^*$ holds for infinitely many prime reductions. Does it then hold for almost all primes?

A related question is how does the Hilbert-Kunz multiplicity of an $\mathfrak{m}$-primary ideal I vary with the prime number? Here we have already mentioned the result of Trivedi (Theorem 9.12) that in graded dimension two the limit $e_{HK}(I_p)$ exists for $p \to \infty$. Is this limit achieved, and if so, for infinitely many prime numbers or even for almost all prime numbers? In terms of semistability, the corresponding question (for projective curves) was first raised by Miyaoka [**76**, Problem 5.4]. Suppose that $C \to \operatorname{Spec}\mathbb{Z}$ is a relative generically smooth projective curve, and let a vector bundle $\mathscr{S}$ on C be given. Then $\mathscr{S}$ induces a vector bundle on every fiber of the curve. Suppose that the restriction of $\mathscr{S}$ to the generic fiber (of characteristic zero) is semistable. It is known that $\mathscr{S}_p$ on C_p is semistable for almost all p; see [**76**, §5]. However, what can be said about the behavior of strong semistability? Miyaoka asked whether there exist infinitely many prime numbers such that the reduction is strongly semistable. Shepherd-Barron [**92**] asked the stronger question whether it is strongly semistable for almost all prime reductions.

We first give a result where everything behaves nicely.

THEOREM 10.1. *Let $\mathbb{Z} \subseteq S$ be a standard-graded normal three-dimensional domain such that* $\operatorname{Proj} S \to \operatorname{Spec}\mathbb{Z}$ *is a relative elliptic curve. Let I be a homogeneous S_+-primary*

[1] One can also consider more general arithmetic base schemes, like the spectrum of the ring of integers in an algebraic number field.

ideal, and $f \in S$ homogeneous. Then the Hilbert-Kunz multiplicity $e_{HK}(I_p)$ is almost constant (constant except for finitely many prime numbers). Moreover, for a homogeneous element f we have either $f \in I_p^*$ or $f \notin I_p^*$ for almost all prime numbers.

PROOF. Almost all fibers are elliptic curves. On an elliptic curve a semistable bundle is strongly semistable (Remark 5.13). Hence a bundle that is semistable in characteristic zero is strongly semistable for almost all prime numbers. This implies that the Harder-Narasimham filtration of the syzygy bundle in characteristic zero induces the strong Harder-Narasimhan filtration for almost all prime numbers. Hence the result follows from the formula for the Hilbert-Kunz multiplicity (Theorem 9.6) and from the numerical characterization of tight closure (Theorem 7.6 and Theorem 7.7). □

However, this result is restricted to elliptic curves; for curves of higher genus the arithmetical behavior is more complicated. With regard to the Hilbert-Kunz multiplicity, Monsky and Han ([**34**], [**33**], [**80**], [**100**]) gave the following example where the Hilbert-Kunz multiplicity in an arithmetic family is not almost constant.

EXAMPLE 10.2. Let $R_p = \mathbb{Z}/(p)[X,Y,Z]/(X^4+Y^4+Z^4)$. Then the Hilbert-Kunz multiplicity of R_p is

$$e_{HK}(R_p) = \begin{cases} 3 \text{ for } p = \pm 1 \bmod 8 \\ 3+1/p^2 \text{ for } p = \pm 3 \bmod 8 . \end{cases}$$

Because of Corollary 9.7 this means that the syzygy bundle $\mathrm{Syz}(x,y,z)$ on the Fermat quartic is strongly semistable for $p = \pm 1 \bmod 8$ (and semistable in characteristic zero), but not strongly semistable for $p = \pm 3 \bmod 8$. Since there are infinitely many primes in every arithmetic progression (Theorem of Dirichlet, see [**90**, Chapitre VI, Section 4, Théorème and Corollaire]), this means that for infinitely many prime reductions the restricted cotangent bundle is not strongly semistable (though it is semistable in characteristic zero). Hence the question of Shepherd-Barron has a negative answer.

The original question of Miyaoka, whether there exist infinitely many prime numbers with strongly semistable reduction, is still open. The following class of examples shows that, under the hypothesis that there exist infinitely many Sophie Germain prime numbers (widely believed to be true), the density of prime numbers with strongly semistable reduction can be arbitrarily small. Recall that a prime number h is called a *Sophie Germain prime* if $2h+1$ is also prime.

EXAMPLE 10.3. Let $h > 5$ be a Sophie Germain prime and set $d = 2h+1$. Consider the syzygy bundle $\mathrm{Syz}(x^2,y^2,z^2)$ on the Fermat curve given by $x^d+y^d+z^d = 0$. It can be shown (see [**11**, Proposition 2]) that, for prime numbers $p \neq \pm 1 \bmod d$, this bundle is not strongly semistable. Hence the density of primes with strongly semistable reduction is $\leq 1/h$, which is arbitrarily small for h large enough.

We now come to the arithmetical behavior of tight closure. The following example given in [**18**] shows that the arithmetical behavior of tight closure is more complicated than first hoped for: an element might belong to the tight closure of an ideal for infinitely many prime numbers, but not for almost all prime numbers. This result also challenges the very definition of tight closure in characteristic zero.

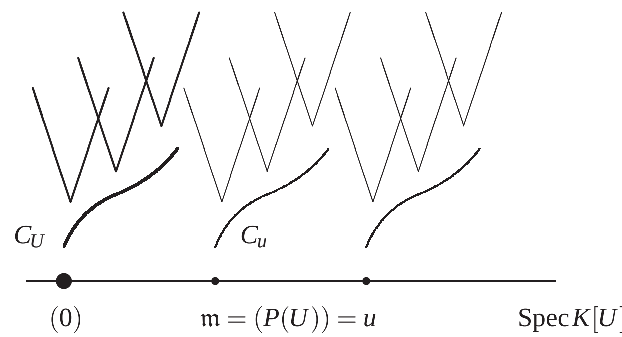
EXAMPLE 10.4. Let $R = K[X,Y,Z]/(X^7+Y^7-Z^7)$, $I = (x^4,y^4,z^4)$, and $f = x^3y^3$. It was proved in [**18**] that $f \in I^*$ holds for $p = 3 \bmod 7$ and does not hold for $p = 2 \bmod 7$.

REMARK 10.5. This example has some interesting consequences. The ideal $\mathfrak{a} = (x,y,z)$ in the forcing algebra

$$A = K[X,Y,Z,U,V,W]/(X^7+Y^7-Z^7, UX^4+VY^4+WZ^4+X^3Y^3)$$

is such that the open subset $D(\mathfrak{a}) \subset \operatorname{Spec} A$ is affine for infinitely many but not for almost all prime reductions. This means that its cohomological dimension fluctuates arithmetically between 0 and 1. In characteristic zero the open subset $D(\mathfrak{a})$ is not affine, so f does belong to the solid closure of I, but it does not belong to the tight closure of I.

We now come to geometric deformations of tight closure. This means basically that we replace $\operatorname{Spec}\mathbb{Z}$, which was our base space for the arithmetic deformations, by an equicharacteristic scheme (of positive characteristic). In fact we will only deal with $\operatorname{Spec} K[U]$ as deformation space. So we have a faithfully flat extension $K[U] \subseteq S$ and are interested in the fibers of the morphism $\operatorname{Spec} S \to \operatorname{Spec} K[U] = \mathbb{A}^1_K$. The (generic or transcendental) fiber over (0) is just $\operatorname{Spec}(S \otimes_{K[U]} K(U))$, which is a ring over the field of rational functions $K(U)$. The (special or algebraic) fibers over a maximal ideal $\mathfrak{m} = (P(U))$ in $K[U]$ are $\operatorname{Spec}(S \otimes_{K[U]} \kappa(\mathfrak{m}))$. We can also identify $\mathfrak{m}$ with a certain value $u \in \bar{K}$ in an algebraic closure. We are again mainly interested in the case where S has relative dimension two over the basis (hence dimension three) and is standard-graded. The corresponding relative projective curve $C = \operatorname{Proj} S$ is then a family over $\mathbb{A}^1$.



In this situation we ask again how the Hilbert-Kunz multiplicity, strong semistability, and tight closure behave in such a geometric family. The behavior of tight closure under such a geometric deformation is related to the localization problem (Problem 1.8) by the following proposition.

PROPOSITION 10.6. *Let K be a field of positive characteristic, and let $K[U] \subset S$ be a three-dimensional standard-graded domain such that almost all fibers are a normal standard-graded domain of dimension two. Let $M \subseteq K[U] \subset S$ be the multiplicative system of all non-zero polynomials in U. Let $I = (f_1,\dots,f_n)$ be a homogeneous S_+-primary ideal, generated by homogeneous elements of degree $d_i = \deg(f_i)$ with syzygy bundle* $\operatorname{Syz}(f_1,\dots,f_n)$. *Let $f \in S$ be a homogeneous element of degree $\geq (\sum_{i=1}^n d_i)/(n-1)$. If the syzygy bundle is generically strongly semistable, then $f \in I^*$ in S_M. If tight closure localizes, then $f \in I^*$ also holds for almost all algebraic specializations.*

PROOF. Note that $S_M = S \otimes_{K[U]} (K[U]_M) = S \otimes_{K[U]} K(U)$. So the ring localized at the given multiplicative system is the generic fiber ring in the family. The statement about the containment in the tight closure follows from the degree bound (Theorem 6.4). So suppose that $f \in I^*$ in S_M. If tight closure localizes, then there exists an element $h \in M$ such that $hf \in I^*$ holds in S. By the persistence of tight closure (Theorem 1.12) we have for every algebraic specialization $U \mapsto u \in \overline{K}$ that $hf \in I^*$ in S_u. Since $h(u) = 0$ for only finitely many values $u \in \overline{K}$ it follows that $f \in I^*$ in S_u for almost all algebraic values. □

This means that we can disprove localization if we find a geometric deformation where $f \in I^*$ holds generically, but $f \notin I^*$ holds for infinitely many algebraic values $u \in \overline{K}$. Such a behavior can only happen if the syzygy bundle is strongly semistable in the transcendental case, but not strongly semistable for infinitely many algebraic values (so this cannot happen for a family of elliptic curves). Monsky has given in [**78**] the following example, where the Hilbert-Kunz multiplicity varies with an algebraic parameter. This example can also be interpreted in terms of strong semistability (see also [**99**, Section 5]).

EXAMPLE 10.7. Let $K = \mathbb{F}_2$ and $S = K[U][X,Y,Z]/(g)$ for

$$g = Z^4 + Z^2XY + Z(X^3 + Y^3) + (U + U^2)X^2Y^2 .$$

This example was studied by Monsky in the context of Hilbert-Kunz theory. He showed that the Hilbert-Kunz multiplicity of $S_{\kappa(\mathfrak{p})} = S \otimes_{K[U]} \kappa(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec} K[U]$, is

$$e_{HK}(S_{\kappa(\mathfrak{p})}) = \begin{cases} 3 \ \text{ for } \ \kappa(\mathfrak{p}) = K(U) \\ 3 + 1/4^m \ \text{ for } \ \kappa(\mathfrak{p}) = K(u) \subset \overline{K}, \ \text{ where } \ m = \deg(K(u)|K) . \end{cases}$$

By Corollary 9.7 this means that the syzygy bundle $\operatorname{Syz}(x,y,z)$ on $C_\mathfrak{p} = \operatorname{Proj} S_{\kappa(\mathfrak{p})}$ is strongly semistable in the transcendental situation and not strongly semistable in all algebraic situations. Monsky shows in fact that the syzygy bundle $\operatorname{Syz}(x^{\tilde{q}}, y^{\tilde{q}}, z^{\tilde{q}})(\frac{3\tilde{q}}{2} - 1)$ has a global non-zero section on $C_\mathfrak{p}$, where $\kappa(\mathfrak{p})$ is algebraic of degree m and $\tilde{q} = 2^{m+1}$ [**78**, Theorem 3.1]. Thus this pull-back is not semistable anymore. Hence we have a destabilizing short exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow \operatorname{Syz}(x^{\tilde{q}}, y^{\tilde{q}}, z^{\tilde{q}}) \longrightarrow \mathscr{M} \longrightarrow 0 .$$

Suppose that $\mathscr{L}$ has degree $6\tilde{q} + a$ and $\mathscr{M}$ has degree $6\tilde{q} - a$ (the sum must be $12\tilde{q}$). Then we have $\nu_1 = -\frac{3}{2} + \frac{a}{4\tilde{q}}$ and $\nu_2 = -\frac{3}{2} - \frac{a}{4\tilde{q}}$. Hence by Theorem 9.6 we must have

$$3 + (\frac{1}{4})^m = 2((-\frac{3}{2} + \frac{a}{4\tilde{q}})^2 + (-\frac{3}{2} - \frac{a}{4\tilde{q}})^2 - 3) = 2(2\frac{9}{4} - 3 + 2\frac{a^2}{16\tilde{q}^2}) = 3 + \frac{a^2}{4 \cdot 4^{m+1}} ;$$

therefore $a = 4$. So $\mathscr{L} \otimes \mathscr{O}(\frac{3\tilde{q}}{2} - 1)$ has degree 0 and it has a section, thus $\mathscr{L} = \mathscr{O}(1 - \frac{3\tilde{q}}{2})$. It follows that

$$0 \longrightarrow \mathscr{O}(1) \longrightarrow \operatorname{Syz}(x^{\tilde{q}}, y^{\tilde{q}}, z^{\tilde{q}})(\frac{3\tilde{q}}{2}) \longrightarrow \mathscr{O}(-1) \longrightarrow 0 \quad (*)$$

must be the destabilizing sequence.

Building on Monsky's example we have the following example which gives a negative answer to the localization problem. We describe here how the disproving of this problem reduces it to an elementary ideal membership problem, which was finally settled in [**19**].

EXAMPLE 10.8. Let $K = \mathbb{F}_2$ and $S = K[U][X,Y,Z]/(g)$ for

$$g = Z^4 + Z^2XY + Z(X^3 + Y^3) + (U + U^2)X^2Y^2 .$$

Consider the ideal

$$I = (x^4, y^4, z^4) \ \text{ and } \ f = y^3z^3 \ \text{ in } \ S_{\kappa(\mathfrak{p})}, \ \mathfrak{p} \in \operatorname{Spec} K[U] .$$

Since $\mathrm{Syz}(x,y,z)$ is strongly semistable for U transcendental, the same is true for $\mathrm{Syz}(x^4,y^4,z^4) = F^{2*}(\mathrm{Syz}(x,y,z))$. Hence by the degree bound Theorem 6.4 we know that $f \in I^*$ holds in the transcendental case (in $S_{K(U)}$). There is however strong numerical evidence, which we present in the following, that $f \notin I^*$ in $S_{\kappa(\mathfrak{p})}$ for all algebraic instances $\mathfrak{p} = (u)$, $u \in \overline{K}$.

In order to show that $f \notin I^*$ in $S_{\kappa(\mathfrak{p})}$ we have to show for a test element t and a certain power $q = 2^e$ (depending on $u \in \overline{K}$) that $t f^q \neq I^{[q]}$. We first need a lemma on test elements.

LEMMA 10.9. *For every point $\mathfrak{p} \in \operatorname{Spec} K[U]$ the test ideal of $S_{\kappa(\mathfrak{p})}$ is $(S_{\kappa(\mathfrak{p})})_{\geq 2}$.*

PROOF. The test ideal $\tau = \tau_{S_{\kappa(\mathfrak{p})}}$ is the annihilator of the tight closure 0^* inside $H^2_{\mathfrak{m}}(S_{\kappa(\mathfrak{p})}) \cong H^1(D(\mathfrak{m}), \mathscr{O})$ (for the test ideal see [**55**], in particular Proposition 4.1), where $\mathfrak{m}$ is the graded irrelevant ideal. We claim that $0^* = H^2_{\mathfrak{m}}(S_{\kappa(\mathfrak{p})})_{\geq 0}$ (note that the local cohomology module is graded). It is clear that 0^* contains everything of nonnegative degree. An element of negative degree belongs to the tight closure only if it is annihilated by some Frobenius power (Corollary 4.14). We consider a non-zero class of negative degree, $c \in H^1(C, \mathscr{O}(-n))$. This gives a non-trivial extension $0 \to \mathscr{O}_C(-n) \to \mathscr{S} \to \mathscr{O}_C \to 0$ and dually $0 \to \mathscr{O}_C \to \mathscr{F} \to \mathscr{O}_C(n) \to 0$. In such a situation every quotient bundle of $\mathscr{F}$ has positive degree (Proposition 4.12(i)), and we have $\deg(\mathscr{F}) = n \deg(C) \geq 4 > 2 \geq \frac{2}{p}(g(C)-1)$. Hence by [**8**, Theorem 2.7] or directly [**41**, Corollary 7.7] the rank-two bundle $\mathscr{F}$ is ample and every Frobenius pull-back of it stays ample. Hence the Frobenius cannot trivialize the extension.

We have $(H^2_{\mathfrak{m}}(S_{\kappa(\mathfrak{p})}))_m = 0$ for $m \geq 2$, hence $(S_{\kappa(\mathfrak{p})})_{\geq 2}$ annihilates $0^* = (H^2_{\mathfrak{m}}(S_{\kappa(\mathfrak{p})}))_{\geq 0}$. Therefore $(S_{\kappa(\mathfrak{p})})_{\geq 2} \subseteq \tau$ (we only need this inclusion). We also have $(H^2_{\mathfrak{m}}(S_{\kappa(\mathfrak{p})}))_0 = K\langle \frac{z^3}{x^2y}, \frac{z^3}{xy^2}, \frac{z^2}{xy} \rangle$ and $(H^2_{\mathfrak{m}}(S_{\kappa(\mathfrak{p})}))_1 = K\langle \frac{z^3}{xy} \rangle$, and the annihilators of the three cohomology classes of degree zero are (y,z), (x,z), and (x,y). Hence no linear form annihilates $(H^2_{\mathfrak{m}}(S_{\kappa(\mathfrak{p})}))_0$, and we have equality.

We also show that the Frobenius is injective on the degree zero part of the cohomology module. The class z^2/xy is mapped to itself, since

$$\frac{z^4}{x^2y^2} = \frac{z^2xy + z(x^3+y^3) + (u+u^2)x^2y^2}{x^2y^2} = \frac{z^2}{xy}.$$

The class z^3/x^2y is mapped to

$$\frac{z^6}{x^4y^2} = \frac{z^2(z^2xy + z(x^3+y^3) + (u+u^2)x^2y^2)}{x^4y^2}$$
$$= \frac{(z^2xy + z(x^3+y^3) + (u+u^2)x^2y^2)xy + z^3x^3}{x^4y^2} = \frac{z^3}{xy^2},$$

so the other two basis classes swap. In particular this p-linear map is a bijection over a finite field. □

Lemma 10.9 shows particularly that y^2 is a test element. The next lemma helps to simplify the computations.

LEMMA 10.10. *Let $u \in \overline{K}$, $m = \deg(K(u)|K)$ and set $q = 2^m$. Then $y^2 f^q \notin I^{[q]}$ if and only if $y f^{\frac{q}{2}} \notin I^{[\frac{q}{2}]}$.*

PROOF. One direction is clear. For the other direction we tensor the sequence $(*)$ in Example 10.7 with $\mathscr{O}(1)$ and get ($\tilde{q} = 2q$)

$$0 \longrightarrow \mathscr{O}(2) \longrightarrow \mathrm{Syz}(x^{2q}, y^{2q}, z^{2q})(3q+1) \longrightarrow \mathscr{O} \longrightarrow 0\,.$$

The element $yf^{\frac{q}{2}} = yy^{3\frac{q}{2}}z^{3\frac{q}{2}}$ determines the cohomology class $\delta(yf^{q/2})$ in $H^1(C, \mathrm{Syz}(x^{2q}, y^{2q}, z^{2q})(3q+1))$, which is 0 if and only if the element belongs to the ideal $(x^{2q}, y^{2q}, z^{2q}) = I^{[q/2]}$. So by assumption this class is not zero. Since $H^1(C, \mathscr{O}(2)) = 0$ we have an isomorphism

$$H^1(C, \mathrm{Syz}(x^{2q}, y^{2q}, z^{2q})(3q+1)) \cong H^1(C, \mathscr{O})\,.$$

The Frobenius maps the class $\delta(yf^{\frac{q}{2}})$ to the class $\delta(y^2 f^q)$ in $H^1(C, \mathrm{Syz}(x^{4q}, y^{4q}, z^{4q})(6q+2))$. The Frobenius acts, however, injectively on the zero degree of the cohomology, as shown in the proof of Lemma 10.9. Hence $y^2 f^q \notin (x^{4q}, y^{4q}, z^{4q}) = I^{[q]}$. $\square$

The following numerical observation (Lemma 10.10 makes the computation easier by halving the Frobenius exponent of I) gave important numerical evidence that in Example 10.8 the conclusion of Proposition 10.6 does not hold and hence localization cannot hold.

OBSERVATION 10.11. Let $u \in \overline{K}$ be an algebraic element of degree m, and let $\mathfrak{p} \in \mathrm{Spec}\,\mathbb{F}_2[U]$ be the corresponding closed point. Then the following hold in $S_{\kappa(\mathfrak{p})}$: For $q' = 2^e$, $e < m$, we have

$$y(yz)^{\frac{3q'}{2}} \in (x^{2q'}, y^{2q'}, z^{2q'})$$

and (more importantly) for $q = 2^m$ we have

$$y(yz)^{\frac{3q}{2}} \notin (x^{2q}, y^{2q}, z^{2q}), \quad \text{hence } y^2 f^q \notin I^{[q]} \text{ and } f \notin I^*\,.$$

This has been computed for all algebraic elements of degree ≤ 8, that is, for all elements in $\mathbb{F}_{256} \subset \overline{K}$ with the help of Almar Kaid and CoCoA. For this just take an irreducible polynomial $h(U)$ of degree m and check the ideal membership in $\mathbb{F}_2[U, X, Y, Z]/(g, h)$. Moreover, for each degree $9, 10, 11, 12$ and for one chosen irreducible polynomial of that degree, the same behavior was observed in computations done by Martin Kreuzer and Daniel Heldt.

After these notes were written, P. Monsky succeeded in proving that this behavior holds for all algebraic values $u \in \overline{K}$, thus completing the proof that tight closure does not commutate with localization; see [**19**].