

# Introduction

For the convenience of the reader, we give an informal resumé of the content, without going into technical details.

**Chapters 1 and 2** contain a brief introduction into descriptive set theory. Here we do not aim to give a really systematic, broad introduction, but rather a technical introduction with basic definitions, some comments, and proofs of several principal theorems of both classical and effective descriptive set theory—exactly those theorems used in the remainder of the book.

We continue in **Chapters 3 and 4** with introductory material on ideals and equivalence relations. We discuss several types of ideals on  $\mathbb{N}$ , and we sketch SOLECKI’s proof that P-ideals, Borel polishable ideals, and the tailmeasure ideals of LSC submeasures on  $\mathbb{N}$  are one and the same. Each Borel ideal  $\mathcal{I}$  on a set  $A$  generates a Borel equivalence relation  $E_{\mathcal{I}}$  on  $\mathcal{P}(A)$  such that  $x E_{\mathcal{I}} y$  iff the symmetric difference  $x \Delta y$  belongs to  $\mathcal{I}$ , for all  $x, y \subseteq A$ . Some other Borel and analytic equivalence relations and important families of them are defined, for instance:

- the equalities  $\Delta_X$  on Borel sets  $X$ ,
- the equivalence relation  $E_0$  of equality of infinite binary sequences (elements of  $2^{\mathbb{N}}$ ) modulo a finite number of terms,
- the equivalence relation  $E_1$  of equality of infinite sequences in  $(2^{\mathbb{N}})^{\mathbb{N}}$  modulo a finite number of terms,
- the equivalence relation  $E_3$  defined on the set  $(2^{\mathbb{N}})^{\mathbb{N}}$  so that  $x E_3 y$  iff  $x(k) E_0 y(k)$  for all  $k$ ,
- the “summable” equivalence relation  $E_2$  defined on the set  $2^{\mathbb{N}}$  so that  $a E_2 b$  iff  $\sum_{k \in a \Delta b} \frac{1}{k} < +\infty$ , where  $a \Delta b$  is the set of all  $k \geq 1$  such that  $a(k) \neq b(k)$ ,
- the density-0 equivalence relation  $Z_0$  defined on  $2^{\mathbb{N}}$  so that  $a Z_0 b$  iff  $\lim_{n \rightarrow \infty} \frac{\#([0, n] \cap (a \Delta b))}{n} = 0$ ,
- the equivalence relations  $\ell^p$  induced on  $\mathbb{R}^{\mathbb{N}}$  by the natural (component-wise) action of additive groups of corresponding Banach spaces,
- the equivalence relation  $T_2$  defined on the set  $\mathbb{R}^{\mathbb{N}}$  so that  $x T_2 y$  iff
 
$$\{x(k) : k \in \mathbb{N}\} = \{y(k) : k \in \mathbb{N}\}$$
 and hence called *the equality of the countable sets of reals*,
- *countable* Borel equivalence relations (those in which each equivalence class is at most countable), and in particular  $E_{\infty}$ , the  $\leq_B$ -largest one in this family,

and some others. Some of these equivalence relations are especially interesting because of their connection with certain large classes of Borel equivalence relations. For instance  $E_0$  is  $\leq_B$ -largest in the family of *hyperfinite* equivalence relations,  $E_\infty$  is  $\leq_B$ -largest in the family of Borel countable equivalence relations,  $T_2$  is connected with various isomorphism relations of countable structures, and so on.

**Chapter 5** introduces the notion of Borel reducibility and presents a diagram of Borel reducibility of some key equivalence relations (in particular those defined above). The diagram, Figure 1 on page 68, begins with equalities on finite, countable, and continuum size Borel sets. This linearly ordered part ends with  $E_0$ , above which the linearity breaks. There exist at least two  $\leq_B$ -incomparable distinguished Borel equivalence relations,  $\leq_B$ -minimal above  $E_0$ , namely  $E_1$  and  $E_3$ , and perhaps also  $E_2$ . Less studied higher levels are most likely even more complicated. The main content of this book consists of the proofs of different reducibility/irreducibility results related to the diagram. These main results are formulated and briefly commented upon in Section 5.6.

Another group of theorems consists of *dichotomy theorems* presented in Section 5.7. One of them, the 1st dichotomy theorem of SILVER (Theorem 5.7.1), asserts that any Borel equivalence relation  $E$  satisfies one of two (obviously incompatible) requirements  $E \leq_B \Delta_{\mathbb{N}}$  or  $\Delta_{2^{\mathbb{N}}} \leq_B E$ . The 2nd dichotomy (Theorem 5.7.2) of HARRINGTON, KECHRIS, and LOUVEAU asserts that any Borel equivalence relation  $E$  satisfies one of two (incompatible) requirements  $E \leq_B \Delta_{2^{\mathbb{N}}}$  or  $E_0 \leq_B E$ . Three more dichotomy theorems clarify the structure of  $\leq_B$ -intervals between  $E_0$  and one of the relations  $E_1, E_2, E_3$ .

**Chapter 6** contains proofs of several assorted reducibility/irreducibility theorems whose only common property is the rather elementary character of their proofs in the sense that only quite standard methods of real analysis and topology are involved. But some of the results are really tricky, in particular, the  $\leq_B$ -incomparability of  $E_2$  and the density-0 equivalence relation  $Z_0$ , or the HJORTH–DOUGHERTY theorem that shows that  $\ell^p$  is Borel reducible to  $\ell^q$  iff  $p \leq q$ .

The following **Chapter 7** is devoted to the class of countable Borel equivalence relations. Studies of recent years demonstrated that this is an extremely rich family of equivalence relations. Among others, it includes hyperfinite equivalence relations, a comparably elementary type among countable Borel ones. It turns out that all countable (Borel) equivalence relations are induced by Borel actions of countable groups (Theorem 7.4.1) Another theorem (Theorem 7.5.1) shows that not all countable Borel equivalence relations are hyperfinite, in particular,  $E_\infty$  is not hyperfinite. We also prove a useful result (Theorem 7.3.1) on  $\sigma$ -additivity of the notions of smoothness and hyperfiniteness as functions of Borel domains.

In the next **Chapter 8** we consider the class of *hyperfinite* equivalence relations. It admits several different but equivalent characterizations, for instance, being induced by a Borel action of the additive group of the integers  $\mathbb{Z}$  or being induced by a Borel action of the group  $\langle \mathcal{P}_{\text{fin}}(\mathbb{N}); \Delta \rangle$  of finite subsets of  $\mathbb{N}$  with the symmetric difference as the group operation. Theorem 8.1.1 in **Chapter 8** proves the equivalence of several known characterizations. Some other theorems on hyperfinite equivalence relations (like the hyperfiniteness of the tail equivalence relations or the classification modulo Borel isomorphism between the domains) are discussed at the end of the chapter.

We come back to non-hyperfinite Borel countable equivalence relations in **Chapter 9**, where some modern results in this area are presented, mainly without proofs, in particular those related to amenable and treeable equivalence relations, as well as those induced by free actions of certain groups. It must be said that all known proofs of modern results on countable equivalence relations are based on advanced techniques of ergodic theory to the extent that makes it inappropriate to present any such proofs in this book.

**Chapter 10** contains proofs of the 1st and 2nd dichotomy theorems. The key ingredient of the proofs is the *Gandy–Harrington forcing*, a technique based on the topology generated by non-empty  $\Sigma_1^1$  sets. Here methods of effective descriptive set theory play an essential role. We also consider a forcing associated to the equivalence relation  $E_0$ : it consists of all  $E_0$ -large Borel sets  $X \subseteq 2^{\mathbb{N}}$ , that is, such that  $E_0 \upharpoonright X$  is not smooth.

**Chapter 11** is devoted to the equivalence relation  $E_1$  and the corresponding ideal  $\mathcal{I}_1$  of all sets  $x \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$  such that  $x \subseteq \{0, 1, \dots, n\} \times \mathbb{N}$  for some  $n$ . The most important property of  $E_1$  is that it is not “polishable”; that is, it does not belong to the family of equivalence relations induced by Borel actions of Polish groups (Theorem 11.8.1). It has been conjectured and verified in some important particular cases that  $E_1$  is a  $\leq_B$ -least equivalence relation among non-“polishable” equivalence relations. For instance, the conjecture is true for equivalence relations of the form  $E_{\mathcal{I}}$ . This result is based on SOLECKI’s theorem on characterization of polishable ideals (Theorem 3.5.1).

Here we prove the 3rd dichotomy (Theorem 5.7.3, in the form of Theorem 11.3.1) of KECHRIS and LOUVEAU: it asserts that any Borel equivalence relation  $E$  such that  $E_0 \leq_B E \leq_B E_1$  satisfies either  $E_0 \sim_B E$  or  $E \sim_B E_1$ , and hence the strict  $<_B$ -interval between  $E_0$  and  $E_1$  is empty. Borel ideals admit an even stronger result (Theorem 11.1.1 of KECHRIS): if  $E_{\mathcal{I}} \leq_B E_1$  then the ideal  $\mathcal{I}$  is Borel isomorphic to exactly one of the ideals  $\mathcal{I}_1$ ,  $\text{Fin}$  (finite subsets of  $\mathbb{N}$ ), or the product of  $\text{Fin}$  and the (trivial) ideal  $\mathcal{P}(\mathbb{N})$  of all subsets of  $\mathbb{N}$ .

**Chapter 12** considers equivalence relations induced by Borel actions of the group  $S_{\infty}$  of all permutations of  $\mathbb{N}$ . This group includes, for instance, various isomorphism relations of countable structures. In particular we prove (Theorem 12.5.2) that any Borel equivalence relation  $E$ , Borel reducible to a Polish action of  $S_{\infty}$ , satisfies  $E \leq_B T_{\xi}$  for some countable ordinal  $\xi$ , where  $\{T_{\xi}\}_{\xi < \omega_1}$  is H. FRIEDMAN’s  $<_B$ -increasing transfinite sequence of Borel equivalence relations. The next **Chapter 13** on *turbulence* makes use of this result.

Turbulent, or, more exactly, *generically* turbulent group actions are characterized by the property that almost all, in the sense of the Baire category, orbits, and even local orbits of the action are somewhere dense. This property separates a class of equivalence relations very different from those induced by actions of  $S_{\infty}$ . Extending, in a certain direction, HJORTH’s results on turbulence, we prove (Theorem 13.5.3) that generically turbulent Borel equivalence relations are not Borel reducible to equivalence relations in a large family  $\mathcal{F}_0$  of all equivalence relations that can be obtained from  $\Delta_{\mathbb{N}}$  (the equality on  $\mathbb{N}$ ) by countable transfinite iterations of the operations of countable power, Fubini product, and some other. Note that not all equivalence relations in  $\mathcal{F}_0$  are “polishable”; for instance,  $E_1$ , a non-“polishable” one, belongs to this family.

**Chapter 14** contains one principal result: the 6th dichotomy (Theorem 5.7.6, in the form of Theorem 14.2.1) of HJORTH and KECHRIS, saying that the strict  $<_{\mathbb{B}}$ -interval between  $\mathbf{E}_0$  and  $\mathbf{E}_3$  is empty, similar to the interval between  $\mathbf{E}_0$  and  $\mathbf{E}_1$  by the 3rd dichotomy theorem. The proof still involves the Gandy–Harrington forcing, but the splitting construction is different and slightly more complicated than the one applied in the proof of the 3rd dichotomy.

The  $<_{\mathbb{B}}$ -interval between  $\mathbf{E}_0$  and the “summable” equivalence relation  $\mathbf{E}_2$  is not known yet to be empty, although it is expected to be such. However, the 4th dichotomy (Theorem 5.7.4, in the form of Theorem 15.2.1) of HJORTH in **Chapter 15** significantly restricts the domain  $<_{\mathbb{B}}$ -below  $\mathbf{E}_2$  to countable Borel equivalence relations. The proof makes use of another splitting construction based on the Gandy–Harrington forcing.

The next **Chapter 16** presents  $\mathbf{c}_0$ -equalities—a family of Borel equivalence relations similar to the density-0 equivalence relation  $\mathbf{Z}_0$ . This family was extensively studied by FARAH, LOUVEAU, and VELICKOVIC. In particular, it was found that it contains a continuum size subfamily of pairwise  $\leq_{\mathbb{B}}$ -incompatible equivalence relations (Theorem 16.6.3).

The problem considered in **Chapter 17** has the opposite character with respect to the results in Chapter 13. We introduce a family of *pinned* (Borel or analytic) equivalence relations  $\mathbf{E}$ —those satisfying the property that in any generic extension of the universe every stable  $\mathbf{E}$ -class contains an element of the ground universe. This family contains, for instance, all orbit equivalence relations of Polish actions of complete left-invariant groups, all Borel equivalence relations with  $\Sigma_3^0$  equivalence classes, some turbulent equivalence relations, and many more. The most notable example of a non-pinned equivalence relation is  $\mathbf{T}_2$ , the equality of countable sets of reals, for which the non-pinned stable class consists of all  $x \in (2^{\mathbb{N}})^{\mathbb{N}}$  in the extension such that the set  $\{x(n) : n \in \mathbb{N}\}$  is equal to the set of all  $a \in 2^{\mathbb{N}}$  in the ground model. Theorem 17.1.3 proves that non-pinned Borel equivalence relations, in particular,  $\mathbf{T}_2$ , are not Borel reducible to pinned ones.

The final **Chapter 18** presents a recent theorem due to ROSENDAL on the cofinality of Borel ideals in the  $\leq_{\mathbb{B}}$ -structure of Borel equivalence relations of general form. In other words, for any Borel equivalence relation  $\mathbf{E}$  there exists a Borel ideal  $\mathcal{I}$  such that  $\mathbf{E} \leq_{\mathbb{B}} \mathbf{E}_{\mathcal{I}}$ . A  $\leq_{\mathbb{B}}$ -cofinal  $\omega_1$ -sequence of Borel equivalence relations of the form  $\mathbf{E}_{\mathcal{I}}$  is defined.

For the convenience of the reader, an appendix (**Appendix A**) is added on some issues related to forcing. It explains the setup and basic terminology of forcing in this book, and it discusses important details related to Cohen and Gandy–Harrington forcing.

**General set-theoretic notation used in this book.**

- $\mathbb{N} = \{0, 1, 2, \dots\}$  : natural numbers;  $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ .
- $X \subseteq Y$  iff  $\forall x (x \in X \implies x \in Y)$  : *the inclusion*, and this holds also in the case when  $X = Y$ .
- $X \subsetneq Y$  or sometimes  $X \subset Y$  means that  $X \subseteq Y$  but  $Y \not\subseteq X$  : *strict inclusion*.
- $X \subseteq^* Y$  means that the difference  $X \setminus Y$  is finite.
- $\{x : \Phi(x)\}$  is the set (or class) of all sets  $x$  such that  $\Phi(x)$ .
- If  $X \subseteq A \times B$  and  $a \in A$ , then  $(X)_a = \{b : \langle a, b \rangle \in X\}$ , a *cross-section*.
- $\text{card } X$  is the cardinality of a set  $X$ , equal to the number of elements of  $X$  whenever  $X$  is finite.
- $\text{dom } P = \{x : \exists y (\langle x, y \rangle \in P)\}$  and  $\text{ran } P = \{y : \exists x (\langle x, y \rangle \in P)\}$  are the *domain* and *range* of any set  $P$  that consists of pairs.
- In particular, if  $P = f$  is a function, then  $\text{dom } f$  and  $\text{ran } f$  are the domain and the range of  $f$ .
- Functions are routinely identified with their graphs; that is, if  $P = f$  is a function, then  $f = \{\langle x, f(x) \rangle : x \in \text{dom } f\}$ , so that  $y = f(x)$  is equivalent to  $\langle x, y \rangle \in f$ .
- Geometrically, if  $P \subseteq \mathbb{X} \times \mathbb{Y}$ , where  $\mathbb{X}$  and  $\mathbb{Y}$  are Polish spaces, then  $\text{pr } P = \{x : \exists y (\langle x, y \rangle \in P)\}$ , *the projection*, will sometimes be used instead of  $\text{dom } P$ , but in fact  $\text{dom } P = \text{pr } P$ .
- $f[X] = \{f(x) : x \in X \cap \text{dom } f\}$ , the *f-image* of  $X$ .
- $f^{-1}[Y] = \{x \in \text{dom } f : f(x) \in Y\}$ , the *f-preimage* of  $Y$ .
- $\Delta$  is the symmetric difference.
- $\exists^\infty x \dots$  means “there exist infinitely many  $x$  such that ...”,  
 $\forall^\infty x \dots$  means “for all but finitely many  $x \dots$  holds”.
- $\{x_a\}_{a \in A}$  is the map  $f$  defined on  $A$  by  $f(a) = x_a, \forall a$ .
- $\mathcal{P}(X) = \{x : x \subseteq X\}$  and  $\mathcal{P}_{\text{fin}}(X) = \{x : x \subseteq X \text{ is finite}\}$ .
- $\emptyset$  is the empty set,  $\Lambda$  is the empty sequence; basically,  $\emptyset = \Lambda$ .
- $X^{<\omega}$  is the set of all finite sequences of elements of a given set  $X$ .
- In particular  $2^{<\omega} \subsetneq \mathbb{N}^{<\omega}$  denote, respectively, the set of all finite sequences of numbers 0, 1 and the set of all finite sequences of natural numbers.
- $\text{lh } s$  is the *length* of a finite sequence  $s$ .
- If  $x$  is any set, then  $s \hat{\ } x$  is the sequence obtained by adjoining  $x$  as the right-most term to a given finite sequence  $s$ .
- $s \subset t$  means that the sequence  $t$  is a proper extension of  $s$ .