Preface

Classification problems for different types of mathematical structures have been in the center of interests in descriptive set theory during the last 15–20 years. Assume that X is a class of mathematical structures identified modulo an equivalence relation E. This can be, for example, countable groups modulo the isomorphism relation, unitary operators over a fixed space $\mathbb{C}^n$ modulo conjugacy, probability measures over a fixed Polish space modulo identification of measures having the same null sets, or, for instance, reals modulo Turing reducibility.[1] Suppose that Y is another class of mathematical structures identified modulo an equivalence relation F. The classification problem is then to find out whether there is a *definable*, or *effective* injection $\Theta : X/\mathsf{E} \to Y/\mathsf{F}$. Such a map Θ is naturally considered as a classification of objects in X in terms of objects in Y, in a way that respects the quotient structure over E and F, respectively. The existence of such a map can be a result of high importance, especially when objects in Y are of simpler mathematical nature than those in X.

In many cases, it turns out that the classes of structures X and Y can be considered as Borel sets in Polish (that is, separable, completely metrizable) spaces, so that E, F become Borel (as sets of pairs) or, more generally, analytic relations, while reduction maps are usually required to be Borel.[2] Then the problem can be studied by methods of descriptive set theory, where it takes the following form: if E, F are Borel (or more complicated) equivalence relations on Polish sets, resp., X, Y, does there exist a Borel *reduction* of E to F (that is, a Borel map $\vartheta : X \to Y$) satisfying

$$x \mathrel{\mathsf{E}} x' \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(x') : \quad \text{for all } x, x' \in X\,?$$

If such a map ϑ exists, then E is said to be *Borel reducible* to F, symbolically

$$\mathsf{E} \le_{\mathrm{B}} \mathsf{F}\,.$$

Then an injection $\Theta : X/\mathsf{E} \to Y/\mathsf{F}$ can be defined by simply putting $\Theta([x]_{\mathsf{E}}) = [\vartheta(x)]_{\mathsf{F}}$, where $[x]_{\mathsf{E}}$ is the E-equivalence class of x. The *Borel equivalence* or *bi-reducibility* $\sim_{\mathrm{B}}$ and *strict reducibility* $<_{\mathrm{B}}$ are naturally introduced so that

$$\begin{aligned} &\mathsf{E} \sim_{\mathrm{B}} \mathsf{F} \quad \text{iff} \quad \text{both } \mathsf{E} \le_{\mathrm{B}} \mathsf{F} \text{ and } \mathsf{F} \le_{\mathrm{B}} \mathsf{E}, \quad \text{and} \\ &\mathsf{E} <_{\mathrm{B}} \mathsf{F} \quad \text{iff} \quad \mathsf{E} \le_{\mathrm{B}} \mathsf{F} \text{ but } \neg\, \mathsf{F} \le_{\mathrm{B}} \mathsf{E}. \end{aligned}$$

The study of Borel and other effective equivalence relations under Borel reducibility by methods of descriptive set theory revealed a remarkable structure of

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[1] The examples are taken from Hjorth's book [**Hjo00b**] and Kechris' survey paper [**Kec99**], where many more examples of this type are given.

[2] That is, maps with Borel graphs. Baire measurable maps and reductions satisfying certain algebraic requirements are also applied [**Far00**] as well as $\mathbf{\Delta}^1_2$ and more complicated reductions [**Hjo00b, Kan98**], however they are not systematically considered in this book.

mutual $\leq_B$-reducibility and $\leq_B$-irreducibility of Borel equivalence relations of different types. This book presents a selection of basic Borel reducibility/irreducibility results in this area.

Originally, these were informal notes for a short course on Borel reducibility and dichotomy theorems given at Universität Bonn in Winter 2000/2001 for graduate and undergraduate students specializing in set theory. The first purpose of the notes was to give a self-contained treatment of several dichotomy theorems in the theory of reducibility of Borel equivalence relations, mainly those obtained in 1990s, in a form as unified as generally would be possible. This original rather short version was deposited at **arXiv** under the title *Varia, ideals and equivalence relations*. But pursuing the goal of self-containedness, the text has been gradually increased in size about three times with respect to the very first version. The last addition is a brief technical introduction into classical and effective descriptive theory.

Still the book does *not* contain much on such topics as Polish and Borel groups and their actions, ergodic theory, model theory, the structure of countable and hyperfinite equivalence relations, in relation to which one may be advised to study Becker and Kechris [**BK96**] or the recent monographs of Hjorth [**Hjo00b**] and Kechris and Miller [**KM04**].

The prospective reader should have some degree of experience with modern descriptive set theory, including Borel sets and methods of *effective* descriptive set theory. Some knowledge of *forcing* is expected as well, because the Cohen forcing for Polish spaces and the Gandy–Harrington forcing is the *sine qua non* for several of the most important arguments in this book.

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