Glossary of Notation

Unless otherwise stated all rings are commutative with 1. A multiplicatively closed set of a ring R always contains 1_R. A ring homomorphism $R \to S$ maps 1_R to 1_S. This is also called an algebra S/R. We try to use the standard notation and language of algebraic geometry.

$\mathbb{N}$	$\{0, 1, 2, \dots\}$	
$\mathbb{N}_+$	$\{1, 2, 3, \dots\}$	
$\mathfrak{U}(X)$	set of open subsets of X	1
$\mathrm{Ab}(X)$	category of abelian sheaves on X	1
s_x	germ of a section s at x	1
$\mathrm{Supp}(s)$	support of a section s	1
$\mathscr{F}(U)$	group of sections on an open set U	1
$\mathscr{F}_x$	stalk of $\mathscr{F}$ at x	1
$\Gamma_Y(X, \mathscr{F})$	group of sections with support in Y	1
$H^i_Y(X, \mathscr{F})$	i-th cohomology with support in Y	1
$H^i_x(X, \mathscr{F})$	i-th local cohomology at a point x	1
$\widetilde{M}$	sheaf associated to a module M	1
$\widetilde{\mathscr{F}}$	flasque sheaf associated to $\mathscr{F}$	2
$j_!\mathscr{F}$	extension of a sheaf by zero	2
$\mathscr{F}\|_U$	restriction of a sheaf to an open set U	2
$\mathrm{Mod}(X)$	category of sheaves of modules on a ringed space X	4
$H^i(X, \mathscr{F})$	i-th global cohomology	4
$j_*\mathscr{F}$	direct image sheaf	5
${}_RM$	an S-module considered as an R-module via $R \to S$	6
$\mathrm{Spec}\, R$	spectrum of a ring R	6
$D(f)$	non-vanishing set of f on an affine scheme	7
M_f	localization of a module at f	7
Ann	annihilator of an element or ideal or module	7
$V(\mathfrak{a})$	vanishing set of an ideal $\mathfrak{a}$ on an affine scheme	8
$V(f)$	vanishing set of a set f of polynomials	8
$\Gamma_{\mathfrak{a}}(M)$	sections of a module with support in $V(\mathfrak{a})$	8

$H^i_{\mathfrak{a}}(M)$	i-th cohomology with support in $V(\mathfrak{a})$	8
$H^i_{\mathfrak{m}}(M)$	i-th local cohomology at a maximal ideal $\mathfrak{m}$	8
$\mathscr{J}_Y$	ideal sheaf of Y	9
$\mathscr{O}^r_X$	direct sum of r copies of $\mathscr{O}_X$	9
$k(x)$	residue field of $\mathscr{O}_{X,x}$	9
$\mathbb{P}^d_A$	d-dimensional projective space over a ring A	11
$C^\bullet(\mathfrak{U}, \mathscr{F})$	alternating Čech complex with respect to a covering $\mathfrak{U}$	11
$\check{H}^\bullet(\mathfrak{U}, \mathscr{F})$	Čech cohomology	11
$\widetilde{C}^\bullet(\mathfrak{U}, \mathscr{F})$	normalized Čech complex	11
$\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$	sheafified Čech complex	12
dim	(Krull) dimension of a ring or module or scheme	16
$C^\bullet[1]$	complex with degree shift by 1	16
$\bigwedge R^d$	exterior algebra of R^d	22
$K_\bullet(t, M)$	homological Koszul complex	22
$K^\bullet(t, M)$	cohomological Koszul complex	23
$H_\bullet(t, M)$	homology of $K_\bullet(t, M)$	23
$H^\bullet(t, M)$	homology of $K^\bullet(t, M)$	23
M_S	localization with respect to S	23
Δ	transition determinant	26
μ_x	multiplication by x	27
$\mathfrak{S}(R)$	socle of a zero-dimensional local or graded ring R	29
$\varinjlim$	direct (injective) limit	29
Φ_t	canonical map $M/(t)M \to H^d_{\mathfrak{m}}(M)$	32
$\begin{bmatrix} m \\ t \end{bmatrix}$	generalized fraction of m over denominator set t	32
$\Omega^\bullet_{R/k}$	algebra of differential forms of R/k	35
Res_R	residue for a complete reduced local ring R	39, 93
ρ_t	canonical map induced by the residue	40, 89
$f(0)$	constant term of a power series f	41
$J,\ J_f$	Jacobian determinant (of a set f of polynomials)	43
$\varprojlim$	inverse (projective) limit	45
S_+	homogeneous maximal ideal of a graded ring S	47
$\mathrm{Proj}\, S$	projective scheme of a graded ring S	47
$D_+(f)$	non-vanishing set of a homogeneous element $f \in S$ on $\mathrm{Proj}\, S$	47
$M_{(f)}$	homogeneous localization of a graded module	47
$M(n)$	graded module shifted by n	47
M^*	sheaf associated to a module	47

$\dim_k$	dimension of a k-vector space	49
$\Gamma_*\mathscr{F}$	graded module associated to a sheaf	49
$\Gamma(X,\mathscr{F})$	module of global sections	50
$\chi(\mathscr{F})$	Euler characteristic of $\mathscr{F}$	50
$\mathrm{Supp}(\mathscr{F})$	support of a sheaf $\mathscr{F}$	50
$P_{\mathscr{F}}$	Hilbert polynomial of $\mathscr{F}$	50
$p_a(X)$	arithmetic genus of X	51
$\Omega^p_{X/A}$	sheaf of p-forms on a scheme X/A	52
$R(X)$	field of rational functions on an integral scheme X/A	52
$\int_X$	integral for a projective variety X	55, 101
$\delta_{\mathscr{F}}$	duality isomorphism	55, 94, 102
$\widehat{R},\ \widehat{M}$	completion of a local ring R or an R-module M	56
Res_x	residue at a closed point x	56, 95
$X(k)$	set of k-rational points of X	57
$V_+(I)$	zero-set of a homogeneous ideal I in $\mathrm{Proj}\, S$	58
$\mathbb{A}^d_k$	affine space over k	58
$G(f) = Gf$	degree form of a polynomial f	58
$T_P(V)$	tangent space of a variety V at P	61
$\mathrm{grad}\, f$	gradient of a polynomial f	63
$\langle\ ,\ \rangle$	standard scalar product	63
H_∞	hyperplane at infinity	63
$f_{X_i} = \frac{\partial f}{\partial X_i}$	partial derivative of f with respect to X_i	63
$\mathrm{tr}(\varphi)$	trace of an endomorphism	65
$\mathrm{tr}_{S/R}$	trace map of an algebra	65
$\omega_{S/R}$	canonical (dualizing) module	65, 84
$Q(R)$	full ring of quotients of a ring R	66
$\mathfrak{C}_{S/R}$	Dedekind complementary module	66
$\mathfrak{d}_D(S/R)$	Dedekind different	66
$\mathfrak{f}_{T/S}$	conductor of an algebra T/S	67
$\mathfrak{k}(S_{\mathfrak{P}})$	residue field of $S_{\mathfrak{P}}$	68
$\mu(M)$	minimal number of generators of M	68
$\mu_{\mathfrak{P}}(M)$	minimal number of generators of $M_{\mathfrak{P}}$	68
$r_{\mathfrak{P}}$	Cohen-Macaulay type at $\mathfrak{P}$	68
$\mathrm{Max}\, S$	set of maximal ideals of S	69
δ	canonical multiplication map $S \otimes_R S \to S$	70
$\mathfrak{d}_N(S/R)$	Noether different	70
$\mathfrak{d}_K(S/R)$	Kähler different	70
Δ^t_x	Bézoutian	72

τ_t^x	linear form associated to the Bézoutian	72
d_x^t	generalized Jacobian	74
$\omega_{V/k}$	sheaf of regular differential forms (also called canonical or dualizing sheaf)	84
$\operatorname{Reg} V$	set of regular points of V	84
$c_{V/k}$	fundamental class of a variety	85
$\mathscr{F}^*,\ \mathscr{F}^{**}$	dual (double dual) of a sheaf $\mathscr{F}$	85
$\operatorname{Pic} V$	Picard group of a variety V	86
$\mathfrak{C}_{V/W}$	complementary module of a finite morphism $V \to W$	87
$\mu_x(X)$	multiplicity of X at x	96
$\Delta(X)$	centroid of a zero-dimensional affine scheme	96
$p_g(V)$	geometric genus of a projective variety	103
$\operatorname{Coh}(V)$	category of coherent sheaves on V	110
$\delta^i_{\mathscr{F}}$	i-duality isomorphism	111
$\operatorname{Sing}(X)$	singular locus of X	112
$\delta(X)$	singularity degree of X	113
$c(X)$	conductor degree of X	113
$\|\mu\|$	$\mu_0 + \cdots + \mu_d$, $\mu = (\mu_0, \ldots, \mu_d) \in \mathbb{N}^{d+1}$	115
$\mu!$	$\mu_0! \cdots \mu_d!$, $\mu = (\mu_0, \ldots, \mu_d) \in \mathbb{N}^{d+1}$	115
∂^μ	$(\frac{\partial}{\partial X_0})^{\mu_0} \cdots (\frac{\partial}{\partial X_d})^{\mu_d}$, $\mu = (\mu_0, \ldots, \mu_d) \in \mathbb{N}^{d+1}$	115
X^μ	$X_0^{\mu_0} \cdots X_d^{\mu_d}$, $\mu = (\mu_0, \ldots, \mu_d) \in \mathbb{N}^{d+1}$	116
$P_F(\partial)$	differential operator associated to $F = \{F_0, \ldots, F_d\}$	115
ev_0	evaluation at zero	116
$\operatorname{polysol}(I(\partial))$	polynomial solutions of the differential equations associated to an ideal I	116
$\mathbb{P}(V)$	projective space associated to a vector space V	117
$X(\Delta)$	toric variety associated to a lattice polytope Δ	119
$\Delta_f(X, Z)$	representative of Bézoutian of $f = \{f_1, \ldots, f_d\}$	120
$\deg_X f$	degree of f in a set of variables X	123
$\sqrt{(f)}$	radical of the ideal (f)	127
Ω_0	homogeneous d-form used for residues	128, 136
T	the torus of $\mathbb{P}^d_k$ or a toric variety	129, 131
$\mathbb{G}_m$	multiplicative group	129
$\left[\dfrac{G\,\Omega_0}{F_0 \cdots F_d}\right]$	homogeneous generalized fraction	129, 138
N, M	dual lattices	130
$\sigma^\vee$	dual of a convex cone σ	130
$\sigma(1)$	set of 1-dimensional faces of σ	130

A_σ	semigroup algebra of σ	130
χ^m	character associated to $m \in M$	130, 131
V_σ	affine toric variety corresponding to σ	131
ω	the d-form $\frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d}$ or the $(d+1)$-form $\frac{dt_0}{t_0} \cdots \frac{dt_d}{t_d}$	131, 132
$\operatorname{Int} \sigma^\vee$	interior of $\sigma^\vee$	131
$C(\Delta)$	cone associated to a lattice polytope Δ	131
S_Δ	semigroup algebra of a lattice polytope Δ	132
L_Δ	quotient field of S_Δ	132
I_Δ	dualizing ideal corresponding to Δ	132
Res_f	toric residue (first and second versions)	133
u_ρ	minimal generator of $\rho \cap N$	133
D_ρ	divisor associated to minimal generator ρ	133
S	total coordinate ring of the toric variety	133
$\operatorname{div}(\chi^m)$	divisor of the character χ^m	134
$\operatorname{Cl}(X)$	group of torus-invariant Weil divisors modulo linear equivalence	134
Δ_d	standard d-simplex	134
β_0	sum of the degrees of the variables	135
δ	the critical degreee $\sum_{i=0}^d \deg F_i - \beta_0$	138
Res_F	toric residue (homogeneous versions)	138, 140
Σ_Δ	normal fan of Δ	138
Res_f	global residue	144
$\Delta(f)$	Newton polytope of a Laurent series f	144
Δ_F^σ	polynomial of toric residue ± 1	148

1. Local Cohomology Functors

In this section some properties of local cohomology with values in sheaves are presented. Later in § 5 and § 10 residues will be defined for local cohomology classes of sheaves of differential forms. The reader is assumed to be familiar with basic notions and facts of homological algebra and sheaf theory as explained in [**39**, Chap. II and III].

Let X be a topological space, $Y \subset X$ a closed subset, and let $\mathfrak{U}(X)$ be the set of open subsets of X. Let $\mathrm{Ab}(X)$ denote the category of sheaves of abelian groups on X, called *abelian sheaves*. Let $\mathscr{F} \in \mathrm{Ab}(X)$ be given. For a section s of $\mathscr{F}$ in a neighborhood U of a point $x \in X$, the germ of s at x is denoted by s_x, and the support $\mathrm{Supp}(s)$ of s is the set of all $y \in U$ with $s_y \neq 0$.

Definition 1.1. $\Gamma_Y(X, \mathscr{F}) := \{s \in \mathscr{F}(X) \mid \mathrm{Supp}(s) \subset Y\}$ is called *the group of sections of $\mathscr{F}$ with support in Y.*

Lemma 1.2. *$\Gamma_Y(X, -)$ is a left-exact functor.*

Proof. Clearly $\Gamma_Y(X, -)$ is a functor. Let an exact sequence of abelian sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

be given. Then $0 \to \mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x \to 0$ is an exact sequence of abelian groups for all $x \in X$. Likewise the sequence $0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X)$ is exact. Since obviously $\Gamma_Y(X, \mathscr{F}) = \ker(\Gamma_Y(X, \mathscr{G}) \to \Gamma_Y(X, \mathscr{H}))$, the exactness of

$$0 \longrightarrow \Gamma_Y(X, \mathscr{F}) \longrightarrow \Gamma_Y(X, \mathscr{G}) \longrightarrow \Gamma_Y(X, \mathscr{H})$$

follows immediately. □

Since $\mathrm{Ab}(X)$ has sufficiently many injective objects the right-derived functors of $\Gamma_Y(X, -)$ are defined.

Definition 1.3. Let $H^i_Y(X, -)$ be the i-th right-derived functor of $\Gamma_Y(X, -)$. For an abelian sheaf $\mathscr{F}$ on X we call $H^i_Y(X, \mathscr{F})$ *the i-th cohomology on X with values in $\mathscr{F}$ and support in Y.*

Important special cases are:

a) $Y = X$: Then we simply write $H^i(X, \mathscr{F})$ and call this *the i-th cohomology of X with values in $\mathscr{F}$.*
b) $Y = \{x\}$ with a closed point $x \in X$: Then we write $H^i_x(X, \mathscr{F})$ and call this *the local cohomology of X at x with values in $\mathscr{F}$.*

If X carries additional structure, for example is a ringed space or a scheme, and if $\mathscr{F}$ is an $\mathscr{O}_X$-module, the cohomology is always understood in the above sense as a derived functor of abelian sheaves.

Definition 1.4. An abelian sheaf $\mathscr{F}$ is called *flasque* if for all open sets $V \subset U$ the restriction morphism $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective. A module M over a ring R is called *flasque* if its associated sheaf $\widetilde{M}$ on $\mathrm{Spec}\, R$ is flasque.

Examples 1.5.

a) A constant sheaf is flasque on an irreducible space.

b) If $\mathscr{F}$ is a presheaf, we define for each $U \in \mathfrak{U}(X)$ the abelian group

$$\widetilde{\mathscr{F}}(U) = \prod_{x \in U} \mathscr{F}_x,$$

with restriction morphisms given by the canonical projections. In this way we obtain a flasque sheaf on X, *the flasque sheaf associated to* $\mathscr{F}$. If $\mathscr{F}$ is already a sheaf, then the canonical morphism $\mathscr{F} \to \widetilde{\mathscr{F}}$ is a monomorphism. One constructs an exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \longrightarrow \mathscr{F}^1 \longrightarrow \cdots$$

with $\mathscr{F}^0 := \widetilde{\mathscr{F}}$ and $\mathscr{F}^i := \widetilde{\mathscr{C}^i}$, where $\mathscr{C}^i := \operatorname{coker}(\mathscr{F}^{i-1} \to \mathscr{F}^i)$. Here $\mathscr{F}^{-1} = \mathscr{F}$. This exact sequence is called the *canonical flasque resolution of* $\mathscr{F}$.

Lemma 1.6. *Let* $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ *be an exact sequence in* $\operatorname{Ab}(X)$ *and let* $\mathscr{F}$ *be flasque. Then:*

a) $0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{G}) \to \Gamma_Y(X, \mathscr{H}) \to 0$ *is exact.*

b) $\mathscr{G}$ *is flasque if and only if* $\mathscr{H}$ *is.*

Proof. a) We have only to show that $\Gamma_Y(X, \mathscr{G}) \to \Gamma_Y(X, \mathscr{H})$ is surjective. Let a section $s \in \Gamma_Y(X, \mathscr{H})$ be given. Consider the set M of all pairs (V, t) with $V \in \mathfrak{U}(X), t \in \mathscr{G}(U)$ such that $\beta(t) = s|_V$. We order M by

$$(V, t) \leq (V', t') \iff V \subset V',\ t'|_V = t$$

Then M is inductively ordered and by Zorn's lemma has a maximal element (V, t). If $V \neq X$, then we can choose $x \in X \setminus V$ and a neighborhood U of x with a section $t' \in \mathscr{G}(U)$ such that $\beta(t') = s|_U$. We have $t|_{V \cap U} - t'|_{V \cap U} \in \mathscr{F}(V \cap U)$, and since $\mathscr{F}$ is flasque there exists $t_0 \in \mathscr{F}(U)$ with $t_0|_{V \cap U} = t|_{V \cap U} - t'|_{V \cap U}$. Then $t' + t_0 \in \mathscr{G}(U)$ and $t \in \mathscr{G}(V)$ define a section $t^* \in \mathscr{G}(V \cup U)$. It follows that $\beta(t^*) = s|_{V \cup U}$, which contradicts the maximality of (V, t). Therefore $V = X$ and there exists $t \in \mathscr{G}(X)$ with $\beta(t) = s$.

Now set $U := X \setminus Y$. Since $\operatorname{Supp}(s) \subset Y$ we have $t|_U \in \mathscr{F}(U)$, which is regarded here as a subgroup of $\mathscr{G}(U)$. Choose $t_0 \in \mathscr{F}(X)$ with $t_0|_U = t|_U$ and set $t^* := t - t_0$. Then $t^* \in \Gamma_Y(X, \mathscr{G})$ and $\beta(t^*) = s$, which is what we had to show.

b) Applying the above in the case $Y = X$, it follows that for open sets $V \subset U$ the rows of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F}(U) & \to & \mathscr{G}(U) & \to & \mathscr{H}(U) & \to & 0 \\ & & \downarrow \rho & & \downarrow \rho' & & \downarrow \rho'' & & \\ 0 & \to & \mathscr{F}(V) & \to & \mathscr{G}(V) & \to & \mathscr{H}(V) & \to & 0 \end{array}$$

are exact, where ρ is surjective since $\mathscr{F}$ is flasque. It follows that ρ' is surjective if and only ρ'' is. □

Lemma 1.7. *Let* $(X, \mathscr{O}_X)$ *be a ringed space and* $\mathscr{I}$ *an injective* $\mathscr{O}_X$*-module. Then* $\mathscr{I}$ *is a flasque sheaf.*

Proof. For $U \in \mathfrak{U}(X)$ let $j : U \to X$ be the inclusion and $\mathscr{O}_U := j_!(\mathscr{O}_X|_U)$ the extension of $\mathscr{O}_X|_U$ by zero, that is the sheaf associated to the presheaf

$$V \longmapsto \begin{cases} \mathscr{O}_X(V) & \text{for } V \subset U \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\mathscr{O}_{U,x} = \begin{cases} \mathscr{O}_x & \text{for } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the sequence $0 \to \mathscr{O}_{U'} \to \mathscr{O}_U$ is exact for $U' \subset U, U' \in \mathfrak{U}(X)$. Since $\mathscr{I}$ is injective, the sequence

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_U, \mathscr{I}) \longrightarrow \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_{U'}, \mathscr{I}) \longrightarrow 0$$

is also exact. However $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_U, \mathscr{I}) = \Gamma(U, \mathscr{I}) = \mathscr{I}(U)$ canonically, and since

$$\mathscr{I}(U) \longrightarrow \mathscr{I}(U') \longrightarrow 0$$

is exact, the sheaf $\mathscr{I}$ is flasque. □

In particular each injective abelian sheaf on a topological space X is flasque.

PROPOSITION 1.8. *Let X be a topological space and $Y \subset X$ a closed subset. Then for each flasque sheaf $\mathscr{F}$ on X we have*

$$H^i_Y(X, \mathscr{F}) = 0 \textit{ for } i > 0.$$

PROOF. There is an exact sequence $0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0$ with an injective sheaf $\mathscr{I}$. By 1.7 $\mathscr{I}$ is flasque, hence so is $\mathscr{G}$ by 1.6 b). Furthermore, the sequence

$$0 \longrightarrow \Gamma_Y(X, \mathscr{F}) \longrightarrow \Gamma_Y(X, \mathscr{I}) \longrightarrow \Gamma_Y(X, \mathscr{G}) \longrightarrow 0$$

is exact by 1.6 a) and $H^i_Y(X, \mathscr{I}) = 0$ for $i > 0$ by the construction of derived functors. The long exact cohomology sequence

$$\begin{array}{ccccccccccc} 0 \to H^1_Y(X,\mathscr{F}) & \to & H^1_Y(X,\mathscr{I}) & \to & H^1_Y(X,\mathscr{G}) & \to & H^2_Y(X,\mathscr{F}) & \to & H^2_Y(X,\mathscr{I}) & \to & \cdots \\ & & \| & & & & & & \| & & \\ & & 0 & & & & & & 0 & & \end{array}$$

shows that $H^1_Y(X, \mathscr{F}) = 0$ and $H^{i+1}_Y(X, \mathscr{F}) \cong H^i_Y(X, \mathscr{G})$ for $i \geq 1$. Since $\mathscr{G}$ is flasque it follows by induction that $H^i_Y(X, \mathscr{F}) = 0$ for all $i \geq 1$. □

If $T = \{T^i\}_{i \geq 0}$ is a δ-functor on an abelian category $\mathfrak{A}$ with values in another abelian category, then an object $\mathscr{F}$ of $\mathfrak{A}$ is called *T-acyclic* if $T^i(\mathscr{F}) = 0$ for $i > 0$. Flasque sheaves are acyclic for the cohomology on X with support in Y. Assume

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}^0 \longrightarrow \mathscr{G}^1 \longrightarrow \cdots \tag{1.1}$$

is a T-acyclic resolution of $\mathscr{F} \in \mathfrak{A}$, that is an exact sequence for which all $\mathscr{G}^i$ $(i \geq 0)$ are T-acyclic. Applying T^0 we obtain a complex

$$T^0(\mathscr{G}^\bullet):\ 0 \longrightarrow T^0(\mathscr{G}^0) \longrightarrow T^0(\mathscr{G}^1) \longrightarrow T^0(\mathscr{G}^2) \longrightarrow \cdots.$$

LEMMA 1.9. *$T^i(\mathscr{F}) \cong H^i(T^0(\mathscr{G}^\bullet))$.*

PROOF. Set $\mathscr{F} =: \mathscr{F}^0$ and decompose (1.1) into short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathscr{F}^0 \longrightarrow \mathscr{G}^0 \longrightarrow \mathscr{F}^1 \longrightarrow 0 \\ 0 \longrightarrow \mathscr{F}^1 \longrightarrow \mathscr{G}^1 \longrightarrow \mathscr{F}^2 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow \mathscr{F}^i \longrightarrow \mathscr{G}^i \longrightarrow \mathscr{F}^{i+1} \longrightarrow 0 \\ \vdots \end{gathered}$$

where $\mathscr{F}^i := \ker(\mathscr{G}^i \to \mathscr{G}^{i+1})$ $(i \geq 0)$. Applying T^0 and the long exact cohomology sequence we obtain exact sequences

$$0 \longrightarrow T^0(\mathscr{F}^i) \longrightarrow T^0(\mathscr{G}^i) \longrightarrow T^0(\mathscr{F}^{i+1}) \longrightarrow T^1(\mathscr{F}^i) \longrightarrow 0,$$

isomorphisms

$$T^p(\mathscr{F}^{p+1}) \cong T^p(\mathscr{F}^p) \text{ for } p \geq 1,$$

and commutative triangles

$$\begin{array}{ccc} T^0(\mathscr{G}^i) & \longrightarrow & T^0(\mathscr{G}^{i+1}) \\ & \searrow \quad \nearrow & \\ & T^0(\mathscr{F}^{i+1}), & \end{array}$$

from which one deduces that

$$\begin{aligned} T^0(\mathscr{F}^i) &= H^0(T^0(\mathscr{G}^\bullet)) \\ T^1(\mathscr{F}^i) &= H^{i+1}(T^0(\mathscr{G}^\bullet)) \text{ for } i > 0 \end{aligned}$$

and

$$T^p(\mathscr{F}^0) = T^{p-1}(\mathscr{F}^1) = \cdots = T^1(\mathscr{F}^{p-1}) = H^p(T^0(\mathscr{G}^\bullet)),$$

which is what we had to show. □

The lemma shows in particular: If an abelian sheaf $\mathscr{F}$ has a flasque resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}^0 \longrightarrow \mathscr{G}^1 \longrightarrow \cdots,$$

then

$$H^i_Y(X, \mathscr{F}) = H^i(\Gamma_Y(X, \mathscr{G}^\bullet)),$$

i.e. local cohomology can be computed by means of flasque resolutions.

PROPOSITION 1.10. *Let $(X, \mathscr{O}_X)$ be a ringed space. The local cohomology functors $H^i_Y(X, \mathscr{F})$ for $\mathscr{F} \in \mathrm{Mod}(X)$ are the derived functors of $\Gamma_Y(X, \mathscr{F})$ in the category $\mathrm{Mod}(X)$ of $\mathscr{O}_X$-modules, that is they can be computed with injective resolutions in $\mathrm{Mod}(X)$. In particular all $H^i_Y(X, \mathscr{F})$ are modules over $\Gamma_Y(X, \mathscr{O}_X)$ and all homomorphisms $H^i_Y(X, \mathscr{F}) \to H^i_Y(X, \mathscr{G})$ associated with morphisms $\mathscr{F} \to \mathscr{G}$ in $\mathrm{Mod}(X)$ are $\Gamma_Y(X, \mathscr{O}_X)$-linear.*

This is clear by 1.9 since injective $\mathscr{O}_X$-modules are flasque by 1.7.

By the universal property of derived functors, the inclusion

$$\Gamma_Y(X, \mathscr{F}) \subset \Gamma(X, \mathscr{F})$$

corresponds to a homomorphism of δ-functors

$$H^i_Y(X, \mathscr{F}) \to H^i(X, \mathscr{F}) \ (i \geq 0)$$

called *the canonical homomorphism from local into global cohomology.* Later this will play a basic role in the formulation of the residue theorem.

Let $U := X \setminus Y$. The restriction $\Gamma(X, \mathscr{F}) \to \Gamma(U, \mathscr{F}|_U)$ likewise corresponds to a canonical homomorphism of δ-functors

$$H^i(X, \mathscr{F}) \to H^i(U, \mathscr{F}|_U) \ (i \geq 0).$$

PROPOSITION 1.11. *There exists a functorial long exact sequence*

$$\begin{aligned} 0 &\longrightarrow \Gamma_Y(X,\mathscr{F}) \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow \Gamma(U,\mathscr{F}|_U) \longrightarrow \\ &H^1_Y(X,\mathscr{F}) \longrightarrow H^1(X,\mathscr{F}) \longrightarrow H^1(U,\mathscr{F}|_U) \longrightarrow \\ &H^2_Y(X,\mathscr{F}) \longrightarrow \cdots \end{aligned}$$

PROOF. For each flasque sheaf $\mathscr{G}$ on X the canonical sequence

$$0 \longrightarrow \Gamma_Y(X,\mathscr{G}) \longrightarrow \Gamma(X,\mathscr{G}) \longrightarrow \Gamma(U,\mathscr{G}|_U) \longrightarrow 0$$

is exact, as is easily seen. Assume now that a flasque resolution (1.1) of $\mathscr{F}$ is given. Then there is an exact sequence of complexes

$$0 \longrightarrow \Gamma_Y(X,\mathscr{G}^\bullet) \longrightarrow \Gamma(X,\mathscr{G}^\bullet) \longrightarrow \Gamma(U,(\mathscr{G}|_U)^\bullet) \longrightarrow 0$$

and by 1.9 the long exact cohomology sequence associated with it gives the sequence we are looking for. □

Now let $V \subset X$ be an open set such that $Y \subset V$.

LEMMA 1.12. *There is a canonical isomorphism of δ-functors*

$$H^i_Y(X,\mathscr{F}) \cong H^i_Y(V,\mathscr{F}|_V).$$

In other words: The cohomology with support in Y depends only on the neighborhoods of Y.

PROOF. The canonical map

$$\Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(V,\mathscr{F}|_V)\ (s \mapsto s|_V)$$

defines a functorial isomorphism. It is clear that the map is injective. In order to show surjectivity let a section $s \in \Gamma_Y(V,\mathscr{F}|_V)$ be given, and set $U := X \setminus Y$. Then s and the zero section on U define a section $t \in \Gamma(X,\mathscr{F})$ with $\mathrm{Supp}(t) \subset Y$ and $t|_V = s$.

Passing to derived functors leads to the desired isomorphism since $\mathscr{F} \to \mathscr{F}|_V$ is an exact functor. □

LEMMA 1.13. *Let $X' \subset X$ be a closed subset with $Y \subset X'$, and let $j : X' \to X$ denote the inclusion map. Then for abelian sheaves $\mathscr{F}$ on X' there is an isomorphism of δ-functors*

$$H^i_Y(X',\mathscr{F}) \cong H^i_Y(X,j_*\mathscr{F})\ (i \geq 0).$$

PROOF. The canonical isomorphism $\Gamma(X',\mathscr{F}) \cong \Gamma(X,j_*\mathscr{F})$ induces an isomorphism

$$\Gamma_Y(X',\mathscr{F}) \cong \Gamma_Y(X,j_*\mathscr{F}).$$

From a flasque resolution (1.1) of $\mathscr{F}$ we obtain a flasque resolution

$$0 \to j_*\mathscr{F} \to j_*\mathscr{G}^0 \to j_*\mathscr{G}^1 \to \cdots$$

of $j_*\mathscr{F}$ since the functor j_* is exact in our situation. Thus we have

$$H^i_Y(X',\mathscr{F}) \cong H^i(\Gamma_Y(X',\mathscr{G}^\bullet)) \cong H^i(\Gamma_Y(X,j_*\mathscr{G}^\bullet)) \cong H^i_Y(X,j_*\mathscr{F}).$$ □

7. Duality and Residue Theorems for Projective Space

The theorems proved here about differential forms on projective space are the basis for later generalizations to arbitrary projective varieties. Though only a special case of much more general results, the residue theorem on $\mathbb{P}^d$ leads to uniform proofs and generalizations of various classical results about intersections of hypersurfaces. The section concludes with some applications.

Let X be a scheme over a ring A. The module of differentials $\Omega^1_{X/A}$ of X over A is the $\mathscr{O}_X$-module with the property: If $U = \operatorname{Spec} B$ is an open affine subset of X, then $\Omega^1_{X/A}|_U = \widetilde{\Omega^1_{B/A}}$, where $\Omega^1_{B/A}$ is the module of Kähler differentials of B/A. If $\mathfrak{p} \in \operatorname{Spec} B$ is the prime ideal corresponding to the point $x \in X$, then

$$(\Omega^1_{X/A})_x = (\Omega^1_{B/A})_{\mathfrak{p}} = \Omega^1_{B_{\mathfrak{p}}/A} = \Omega^1_{\mathscr{O}_x/A}.$$

Similarly, for each $p \in \mathbb{N}$, the sheaf $\Omega^p_{X/A}$ of *p-forms* (*differential forms of degree* p) is defined by

$$\Omega^p_{X/A}|_U := \widetilde{\Omega^p_{B/A}} = \bigwedge^p \widetilde{\Omega^1_{B/A}}$$

for $U = \operatorname{Spec} B$ as above. Moreover, for all $p \in \mathbb{N}$, there is *exterior differentiation* $d : \Omega^p_{X/A} \to \Omega^{p+1}_{X/A}$, defined as the morphism of sheaves of A-modules which for each U is induced by the differentiation

$$d_{B/A} : \Omega^p_{B/A} \longrightarrow \Omega^{p+1}_{B/A} \qquad (b_0\, db_1 \cdots db_p \longmapsto db_0\, db_1 \cdots db_p).$$

In particular, $d : \mathscr{O}_X \to \Omega^1_{X/A}$ is given by the universal derivation

$$d_{B/A} : B \longrightarrow \Omega^1_{B/A} \qquad (b \longmapsto d_{B/A} b).$$

We call d the *universal derivation* of $\mathscr{O}_{X/A}$.

If X is an integral scheme with generic point x, then

$$(\Omega^p_{X/A})_x = \Omega^p_{R(X)/A} = \bigwedge^p \Omega^1_{R(X)/A},$$

where $R(X) = \mathscr{O}_x$ is the field of rational functions on X. We call $\Omega^p_{R(X)/A}$ the $R(X)$-*vector space of rational p-forms* on X and consider it as a constant sheaf on X. Then we have a canonical morphism of sheaves

$$\Omega^p_{X/A} \longrightarrow \Omega^p_{R(X)/A}$$

which in general is not injective.

In this section, differential forms of degree d on $X := \mathbb{P}^d_A = \operatorname{Proj} S$ play the main role, where $S = A[X_0, \dots, X_d]$ is the polynomial ring in the variables $X_0, \dots, X_d$ over A. We give an alternate description of the sheaf $\Omega^d_{X/A}$ in this case.

The module of differentials

$$\Omega^1_{S/A} = \bigoplus_{i=0}^{d} S\, dX_i$$

is a graded S-module where $\deg dX_i = 1$ for $i = 0 \dots, d$. The universal derivation

$$d : S \longrightarrow \Omega^1_{S/A} \qquad \Big(f \longmapsto \sum_{j=0}^{d} \frac{\partial f}{\partial X_j}\, dX_j\Big)$$

is homogeneous of degree 0. Let $\widetilde{\Omega^1_{S/A}}$ be the sheaf on $X = \operatorname{Proj} S$ associated to the graded S-module $\Omega^1_{S/A}$. Then clearly

$$\widetilde{\Omega^1_{S/A}} \cong \bigoplus_{i=0}^{d} \mathscr{O}_X(-1).$$

Furthermore, the universal derivation d induces a morphism

$$\widetilde{d} : \widetilde{S} \longrightarrow \widetilde{\Omega^1_{S/A}}$$

of sheaves of A-modules, which for homogeneous polynomials in S of the same degree is described over $D_+(g)$ by the formula

$$\widetilde{d}\Big(\frac{f}{g}\Big) = \frac{g df - f dg}{g^2}.$$

For each open set $U \subset X$ the map $\widetilde{d}_U$ is an A-derivation of $\widetilde{S}(U) = \mathscr{O}_X(U)$ to $\widetilde{\Omega^1_{S/A}}(U)$.

The S-linear map

$$e : \bigoplus_{i=0}^{d} S\, dX_i \longrightarrow S \qquad (dX_i \longmapsto X_i)$$

is homogeneous of degree 0 and a surjection onto S_+. It induces an $\mathscr{O}_X$-module homomorphism

$$\varepsilon : \widetilde{\Omega^1_{S/A}} \longrightarrow \widetilde{S} = \mathscr{O}_X$$

which is an epimorphism (though e is not). The connection between $\widetilde{\Omega^1_{S/A}}$ and $\Omega^1_{X/A}$ is given by

PROPOSITION 7.1. *We have* $\ker \varepsilon \cong \Omega^1_{X/A}$ *and* $\operatorname{im} \widetilde{d} \subset \ker \varepsilon$. *The diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow & \Omega^1_{X/A} & \longrightarrow & \widetilde{\Omega^1_{S/A}} & \xrightarrow{\ \varepsilon\ } & \mathscr{O}_X & \longrightarrow 0 \\ & {\scriptstyle d}\nwarrow & & \nearrow{\scriptstyle \widetilde{d}} & & & \\ & & \mathscr{O}_X & & & & \end{array}$$

is commutative and the top row is exact. This row is called the Euler sequence.

PROOF. It suffices to consider the situation over the open affine set

$$D_+(X_0) = \operatorname{Spec} A\Big[\frac{X_1}{X_0}, \dots, \frac{X_d}{X_0}\Big].$$

Here, $\widetilde{\Omega^1_{S/A}}(D_+(X_0))$ is the free $A\big[\frac{X_1}{X_0}, \dots, \frac{X_d}{X_0}\big]$-module with basis $\big\{\frac{dX_0}{X_0}, \dots, \frac{dX_d}{X_0}\big\}$. Since $\varepsilon\big(\frac{dX_i}{X_0}\big) = \frac{X_i}{X_0}$ $(i = 1, \dots, d)$ and $\varepsilon\big(\frac{dX_0}{X_0}\big) = 1$, the differentials

$$\frac{X_j}{X_0} \cdot \frac{dX_0}{X_0} - \frac{dX_j}{X_0} = \frac{X_j\, dX_0 - X_0\, dX_j}{X_0^2} = \widetilde{d}\Big(\frac{X_j}{X_0}\Big) \qquad (j = 1, \dots, d)$$

form a basis of $\ker \varepsilon$. Further if $f \in S$ is homogeneous of degree ν, then

$$\varepsilon\Big(\widetilde{d}\Big(\frac{f}{X_0^\nu}\Big)\Big) = \varepsilon\Big(\frac{X_0^\nu\, df - \nu f X_0^{\nu-1}\, dX_0}{X_0^{2\nu}}\Big) = \frac{X_0^\nu \cdot \nu \cdot f - \nu \cdot f \cdot X_0^\nu}{X_0^{2\nu}} = 0$$

by Euler's formula, which states that $df = \sum_{j=0}^{d} \frac{\partial f}{\partial X_j}\, dX_j$ is sent to $\nu \cdot f$ by the map $dX_j \mapsto X_j$ $(j = 0, \dots, d)$. It follows that $\operatorname{im} \widetilde{d}|_{D_+(X_0)} \subset \ker \varepsilon|_{D_+(X_0)}$ and that $\widetilde{d}$ induces the universal derivation $d : A[\frac{X_1}{X_0}, \dots, \frac{X_d}{X_0}] \to \ker \varepsilon$. This proves the proposition. □

By taking exterior powers, the Euler sequence gives a canonical isomorphism

$$\widetilde{\Omega^{d+1}_{S/A}} \cong \Omega^d_{X/A} \otimes_{\mathscr{O}_X} \mathscr{O}_X \cong \Omega^d_{X/A}. \tag{7.1}$$

Set $t_i := \frac{X_i}{X_0}$ $(i = 1, \dots, d)$. The isomorphism (7.1) identifies the differential form

$$f\, dt_1 \cdots dt_d \in \Omega^d_{X/A}(D_+(X_0)) \qquad (f \in k[t_1, \dots, t_d]) \tag{7.2}$$

with the element

$$\frac{F\, dX_0 \cdots dX_d}{X_0^{\deg f + d + 1}} \in \widetilde{\Omega^{d+1}_{S/A}}(D_+(X_0)), \tag{7.3}$$

where $F := X_0^{\deg f} \cdot f\big(\frac{X_1}{X_0}, \dots, \frac{X_d}{X_0}\big)$ is the homogenization of f. The isomorphism (7.1) is the desired description of $\Omega^d_{X/A}$ as the sheaf $\widetilde{M}$ of a graded S-module M.

Suppose that A is a domain, $R(X)$ the field of rational functions of X/A, and $L := Q(S)$ the quotient field of S. Then $R(X) \subset L$ in a canonical way, and (7.1) induces a canonical $R(X)$-linear injection

$$\Omega^d_{R(X)/A} \longrightarrow \Omega^{d+1}_{L/A}, \tag{7.4}$$

where for $i = 0, \dots, d$ the form $(-1)^i d\big(\frac{X_0}{X_i}\big) \cdots \widehat{d\big(\frac{X_i}{X_i}\big)} \cdots d\big(\frac{X_d}{X_i}\big)$ is mapped onto

$$\frac{1}{X_i^{d+1}}\, dX_0 \cdots dX_d.$$

As $\widetilde{\Omega^1_{S/A}} \cong \bigoplus_{i=0}^d \mathscr{O}_X(-1)$, we obtain from (7.1) that $\Omega^d_{X/A} \cong \mathscr{O}_X(-d-1)$, and more generally

$$\Omega^d_{X/A}(n) \cong \mathscr{O}_X(n - d - 1) \quad \text{for } n \in \mathbb{Z}. \tag{7.5}$$

From now on let $A = k$ be a field. By (7.5) and 6.4 we have

PROPOSITION 7.2.

a) $H^p(X, \Omega^d_{X/k}(n)) = 0$ *for* $p \neq 0, d$ *and all* $n \in \mathbb{Z}$.
b) $H^d(X, \Omega^d_{X/k}(n)) = 0$ *for* $n \geq 1$.
c) $\dim_k H^d(X, \Omega^d_{X/k}(-n)) = \binom{d+n}{d}$ *for* $n \geq 0$.

Furthermore, $\Gamma(X, \Omega^d_{X/k}(n)) \cong \Gamma(X, \mathscr{O}_X(n - d - 1))$ *for* $n \leq d$*, in particular* $\Gamma(X, \Omega^d_{X/k}) = 0$.

Proposition 6.3 is a more precise assertion than 7.2 c). Let $\mathfrak{m} := (X_1, \dots, X_d)$ and $\mathfrak{M} := \mathfrak{m} S_{\mathfrak{m}}$. Then $H^d(X, \Omega^d_{X/k}(n))$ can be identified in canonical manner with the k-vector space spanned by the fractions

$$\begin{bmatrix} dX_0 \cdots dX_d \\ X_0^{a_0}, \dots, X_d^{a_d} \end{bmatrix} \in H^{d+1}_{\mathfrak{M}}(\Omega^{d+1}_{S_{\mathfrak{m}}/k}), \text{ where } a_i \in \mathbb{N}_+ \text{ and } \sum_{i=0}^d a_i = d + 1 - n.$$

In particular, for $n = 0$ we have

$$H^d(X, \Omega^d_{X/k}) = k \cdot \begin{bmatrix} dX_0 \cdots dX_d \\ X_0, \dots, X_d \end{bmatrix}.$$

This basis element is independent of the choice of coordinates. Let $Y_i = \sum_{j=0}^d c_{ij} X_j$ $(i = 0, \dots, d)$, where $c_{ij} \in k$ and $\Delta := \det(c_{ij}) \neq 0$. Then by 4.18

$$\begin{bmatrix} dY_0 \cdots dY_d \\ Y_0, \dots, Y_d \end{bmatrix} = \begin{bmatrix} \Delta\, dX_0 \cdots dX_d \\ Y_0, \dots, Y_d \end{bmatrix} = \begin{bmatrix} dX_0 \cdots dX_d \\ X_0 \dots, X_d \end{bmatrix}.$$

DEFINITION 7.3. The *integral over* X is the canonical map

$$\int_X : H^d(X, \Omega^d_{X/k}) \longrightarrow k$$

which sends $\alpha \cdot \begin{bmatrix} dX_0 \cdots dX_d \\ X_0, \dots, X_d \end{bmatrix}$ to α for all $\alpha \in k$.

THEOREM 7.4 (Duality for $\mathbb{P}^d_k$). *For $X = \mathbb{P}^d_k$, the pair $(\Omega^d_{X/k}, \int_X)$ represents the functor*

$$\mathrm{Hom}_k(H^d(X, -), k)$$

for coherent $\mathscr{O}_X$-modules. In other words: For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical k-linear map

$$\delta_{\mathscr{F}} : \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \Omega^d_{X/k}) \longrightarrow \mathrm{Hom}_k(H^d(X, \mathscr{F}), k) \quad \Big(\ell \longmapsto \int_X \circ H^d(X, \ell)\Big)$$

is bijective.

PROOF. a) We consider at first the case $\mathscr{F} = \mathscr{O}_X(-n)$ with $n \in \mathbb{Z}$. Then

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X(-n), \Omega^d_{X/k}) \cong \Gamma(X, \Omega^d_{X/k}(n)) \cong \Gamma(X, \mathscr{O}_X(n-d-1)).$$

This vector space vanishes for $n \leq d$, and so does $H^d(X, \mathscr{O}_X(-n))$ by 6.4.

Assume therefore that $n \geq d+1$. As a k-vector space, $\Gamma(X, \Omega^d_{X/k}(n))$ has the basis

$$\Big\{ X_0^{a_0} \cdots X_d^{a_d}\, dX_0 \cdots dX_d \mid a_i \in \mathbb{N}\ (i = 0, \dots, d),\ \sum_{i=0}^d a_i = n-d-1 \Big\}$$

and $H^d(X, \mathscr{O}_X(-n))$ has the basis

$$\Big\{ \begin{bmatrix} 1 \\ X_0^{b_0}, \dots, X_d^{b_d} \end{bmatrix} \mid b_i \in \mathbb{N}_+\ (i = 0, \dots, d),\ \sum_{i=0}^d b_i = n \Big\}$$

The map $\delta_{\mathscr{O}_X(-n)}$ is induced by the pairing

$$\Big(X_0^{a_0} \cdots X_d^{a_d}\, dX_0 \cdots dX_d,\ \begin{bmatrix} 1 \\ X_0^{b_0}, \dots, X_d^{b_d} \end{bmatrix} \Big) \longmapsto \int_X \begin{bmatrix} X_0^{a_0} \cdots X_d^{a_d}\, dX_0 \cdots dX_d \\ X_0^{b_0}, \dots, X_d^{b_d} \end{bmatrix}.$$

The last expression is 1 if $b_i = a_i + 1$ for $i = 0, \dots, d$ and vanishes otherwise. Therefore the above bases are dual with respect to $\int_X$, and $\delta_{\mathscr{O}_X(-n)}$ is bijective.

b) We can proceed now as in the proof of the local duality theorem 5.13. Since the theorem is true for $\mathscr{F} = \mathscr{O}_X(n)$ with $n \in \mathbb{Z}$, it is true for finite direct sums of such sheaves. For each coherent sheaf there exists an exact sequence of the following form

$$\bigoplus_{i=1}^{s} \mathscr{O}_X(m_i) \longrightarrow \bigoplus_{j=1}^{r} \mathscr{O}_X(n_j) \longrightarrow \mathscr{F} \longrightarrow 0.$$

As both functors $\mathrm{Hom}_{\mathscr{O}_X}(-, \Omega^d_{X/k})$ and $\mathrm{Hom}_k(H^d(X,-),k)$ are left exact, it follows immediately that $\delta_{\mathscr{F}}$ is also an isomorphism. □

COROLLARY 7.5. *Let* $\mathscr{F}^* := \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$ *be the dual of a locally free sheaf* $\mathscr{F}$ *of finite rank. Then*

$$\mathrm{Hom}_k(H^d(X,\mathscr{F}),k) \cong \Gamma(X, \mathscr{F}^*(-d-1)).$$

PROOF. This follows from the isomorphisms

$$\begin{aligned}\mathrm{Hom}_k(H^d(X,\mathscr{F}),k) &\cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \Omega^d_{X/k}) \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}^* \otimes_{\mathscr{O}_X} \Omega^d_{X/k}) \\ &\cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}^*(-d-1)) \cong \Gamma(X, \mathscr{F}^*(-d-1)).\end{aligned}$$ □

For the construction of residues at closed points of the projective space we need

LEMMA 7.6. *Let* $(R, \mathfrak{m})$ *be a noetherian local ring,* $(\widehat{R}, \widehat{\mathfrak{m}})$ *its completion,* M *a finite* R*-module, and* $\widehat{M} := \widehat{R} \otimes_R M$ *the completion of* M*. Then there exist for all* $p \in \mathbb{N}$ *canonical isomorphisms*

$$H^p_{\mathfrak{m}}(M) \cong H^p_{\widehat{\mathfrak{m}}}(\widehat{M}).$$

PROOF. Each system $t = \{t_1, \dots, t_d\}$ of parameters of R is also one of $\widehat{R}$. By § 4, there is a canonical isomorphism $H^p_{\mathfrak{m}}(M) \cong \varinjlim H^p(t^\rho, M)$. Moreover $H^p(t^\rho, \widehat{M}) \cong \widehat{R} \otimes_R H^p(t^\rho, M)$. By 4.8, the support of $H^p(t^\rho, M)$ consists only of the maximal ideal $\mathfrak{m}$. Therefore it is complete and $\widehat{R} \otimes_R H^p(t^\rho, M) \cong H^p(t^\rho, M)$. The lemma now follows from $H^p_{\widehat{\mathfrak{m}}}(\widehat{M}) \cong \varinjlim H^p(t^\rho, \widehat{M})$. □

For $p = d$ we see that any element of $H^d_{\widehat{\mathfrak{m}}}(\widehat{M})$ can be written as a generalized fraction $\begin{bmatrix} m \\ t \end{bmatrix}$ with $m \in M$ and a system t of parameters of R.

For the remainder of this section let k be an algebraically closed field. If $x \in \mathbb{P}^d_k$ is a closed point with local ring $\mathscr{O}_x$, then $\widehat{\mathscr{O}_x} = k[[t_1, \dots, t_d]]$ is a ring of formal power series in variables $t_1, \dots, t_d$ over k and $\widehat{\mathscr{O}_x} \otimes_{\mathscr{O}_x} (\Omega^d_{X/k})_x = \Omega^d_{k[[t_1,\dots,t_d]]/k}$, the dth exterior power of the universally finite module of differentials of $\widehat{\mathscr{O}_x}/k$. Set $R := k[[t_1 \dots, t_d]]$ and $\mathfrak{m} := (t_1, \dots, t_d)$.

DEFINITION 7.7. The *residue* at x, denoted by Res_x, is the composition of the canonical isomorphism $H^d_x(\Omega^d_{X/k}) \xrightarrow{\sim} H^d_{\mathfrak{m}}(\Omega^d_{R/k})$ from 7.6 with the residue map $\mathrm{Res}_R : H^d_{\mathfrak{m}}(\Omega^d_{R/k}) \to k$ from § 5.

REMARK 7.8. Keeping Proposition 6.3 in mind, we see that the integral $\int_X$ is the map induced by Res_R, where $R := \widehat{S}_{\mathfrak{m}}$, $\mathfrak{m} = (X_0, \dots, X_d)$. In other words, $\int_X$ comes from the residue at the origin of $\mathbb{A}^{d+1}_k$, the vertex of the cone of $\mathbb{P}^d_k$.

In the following we denote by $X(k)$ the set of closed points of $X := \mathbb{P}^d_k$, that is the points $x \in X$ with $\dim \mathscr{O}_x = d$. For $x \in X(k)$, let $i_x : H^d_x(\Omega^d_{X/k}) \to H^d(X, \Omega^d_{X/k})$ be the canonical map from local to global cohomology and

$$i : \bigoplus_{x \in X(k)} H^d_x(\Omega^d_{X/k}) \longrightarrow H^d(X, \Omega^d_{X/k})$$

the map induced by $\{i_x\}_{x \in X(k)}$. Further let

$$\Sigma_{\mathrm{Res}} : \bigoplus_{x \in X(k)} H^d_x(\Omega^d_{X/k}) \longrightarrow k$$

be the "residue sum":

$$\Sigma_{\mathrm{Res}}(\{\omega_x\}_{x \in X(k)}) = \sum_{x \in X(k)} \mathrm{Res}_x(\omega_x)$$

for each family $\{\omega_x\}_{x \in X(k)} \in \bigoplus_{x \in X(k)} H^d_x(\Omega^d_{X/k})$.

THEOREM 7.9 (Residue Theorem for $\mathbb{P}^d_k$). *For $X = \mathbb{P}^d_k$, the following diagram commutes:*

$$\begin{array}{ccc} \bigoplus_{x \in X(k)} H^d_x(\Omega^d_{X/k}) & \xrightarrow{\ i\ } & H^d(X, \Omega^d_{X/k}) \\ & {}_{\Sigma_{\mathrm{Res}}}\searrow \quad \swarrow{}_{\int_X} & \\ & k & \end{array}$$

In particular we have $\sum_{x \in X(k)} \mathrm{Res}_x(\omega_x) = 0$ for $\{\omega_x\} \in \bigoplus_{x \in X(k)} H^d_x(\Omega^d_{X/k})$ if and only if $\{\omega_x\}$ is an element of $\ker i$.

PROOF. It suffices to show that for any $x \in X(k)$ the following diagram commutes

$$\begin{array}{ccc} H^d_x(\Omega^d_{X/k}) & \xrightarrow{\ i\ } & H^d(X, \Omega^d_{X/k}) \\ & {}_{\mathrm{Res}_x}\searrow \quad \swarrow{}_{\int_X} & \\ & k. & \end{array} \tag{7.6}$$

As all maps in the diagram are independent of the coordinates, we may assume that $x = \langle 1, 0, \dots, 0 \rangle$. Then $\widehat{\mathscr{O}_x} = k[[t_1, \dots, t_d]]$ with the affine coordinate functions $t_i = \frac{X_i}{X_0}$ $(i = 1, \dots, d)$, and it suffices to show

$$i_x \begin{bmatrix} dt_1 \cdots dt_d \\ t_1^{a_1}, \dots, t_d^{a_d} \end{bmatrix} = \begin{cases} \begin{bmatrix} dX_0 \cdots dX_d \\ X_0^{d+1-\sum a_i},\ X_1^{a_1}, \dots, X_d^{a_d} \end{bmatrix} & \text{for } a_1 = \cdots = a_d = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{7.7}$$

where we write elements of local or global cohomology as generalized fractions as indicated above. The formula results from the description 3.10 of the canonical map from local to global cohomology by Čech cocycles.

Let $\mathfrak{U} = \{U_i\}_{i=0\dots,d}$ be the standard open covering of $\mathbb{P}^d_k$ with $U_i := D_+(X_i)$ $(i = 0, \dots, d)$. Further let $\mathfrak{U}' := \{U_0 \cap U_i\}_{i=1,\dots,d}$ be the corresponding covering of $U_0 \setminus \{x\}$. Then $\mathfrak{U}'$ is just the covering of $\mathbb{A}^d_k \setminus \{O\} = \operatorname{Spec} k[t_1, \dots, t_d] \setminus \{(t_1, \dots, t_d)\}$ given by the open sets $D(t_i)$ $(i = 1, \dots, d)$.

By the definition of generalized fractions, the Čech cocycle

$$\xi := \frac{dt_1 \cdots dt_d}{t_1^{a_1} \cdots t_d^{a_d}} \in \Gamma\big((U_0 \cap U_1) \cap \cdots \cap (U_0 \cap U_d)), \Omega^d_{X/k}|_{U_0 \setminus \{x\}}\big)$$

from $C^{d-1}(\mathfrak{U}', \Omega^d_{X/k}|_{U_0\setminus\{x\}})$ represents the element

$$\begin{bmatrix} dt_1 \cdots dt_d \\ t_1^{a_1}, \dots, t_d^{a_d} \end{bmatrix} \in H^d_x(\Omega^d_{X/k}).$$

Similarly, the Čech cocycle

$$\eta := \frac{dX_0 \cdots dX_d}{X_0^{d+1-\sum a_i} X_1^{a_1} \cdots X_d^{a_d}} \in \Gamma(U_0 \cap \cdots \cap U_d, \Omega^d_{X/k}) = \Gamma(U_0 \cap \cdots \cap U_d, \widetilde{\Omega^{d+1}_{S/k}})$$

from $C^d(\mathfrak{U}, \Omega^d_{X/k})$ represents for $a_1 = \cdots = a_d = 1$ the element

$$\begin{bmatrix} dX_0 \cdots dX_d \\ X_0, X_1^{a_1}, \dots, X_d^{a_d} \end{bmatrix} \in H^d(X, \Omega^d_{X/k}),$$

and otherwise is a coboundary, namely the image by the differential d of the element

$$\alpha := \Big\{ \frac{m_{0\dots\hat{k}\dots d}}{X_0^{\alpha_0} \cdots \widehat{X_k^{\alpha_k}} \cdots X_d^{\alpha_d}} \Big\}_{k=0,\dots,d} \in C^d(\mathfrak{U}, \widetilde{\Omega^{d+1}_{S/k}}),$$

with $\alpha_0 := \sum_{i=1}^d \alpha_i - d > 0, m_{1,\dots,d} := X_0^{\alpha_0 - 1} dX_0 \cdots dX_d$, and $m_{0\dots\hat{k}\dots d} := 0$ for $k = 1, \dots, d$, as in the description of d following formula (4.2). Here we use (7.1) to identify the differential form $dt_1 \cdots dt_d$ with $\frac{1}{X_0^{d+1}} dX_0 \cdots dX_d$ and hence ξ with η. It follows from 3.10 that formula (7.7) is true, which proves the residue theorem. □

We now derive a variant of the residue theorem for zero-dimensional complete intersection subschemes of $\mathbb{P}^d_k$ that will yield many applications. Let $F_1, \dots, F_d \in S$ be homogeneous polynomials which form a regular sequence. Then the zero-set $V_+(F_1, \dots, F_d)$ is a zero-dimensional subscheme of $X = \mathbb{P}^d_k$. Its support is a finite set $Y = \{y_1, \dots, y_s\}$ of closed points of X. We may choose coordinates so that $Y \cap V_+(X_0) = \emptyset$. Then $X = D_+(X_0) \cup \bigcup_{i=1}^d D_+(F_i)$. Let $f_1, \dots, f_d \in k[t_1, \dots, t_d]$ be the dehomogenizations of the F_i, i.e. $f_i(t_1, \dots, t_d) := F_i(1, t_1, \dots, t_d)$ $(i = 1, \dots, d)$. Then Y is the support of $V(f_1, \dots, f_d) \subset \mathbb{A}^d_k =: X'$ and

$$X' \setminus Y = \textstyle\bigcup_{i=1}^d D(f_i) = \bigcup_{i=1}^d (D_+(X_0) \cap D_+(F_i)).$$

The set $f := \{f_1, \dots, f_d\}$ is a system of parameters of $\mathcal{O}_{X,y_j}$ for all $j = 1, \dots, s$. For an $h \in k[t_1 \dots, t_d]$ let ω_j denote the generalized fraction

$$\begin{bmatrix} h\, dt_1 \cdots dt_d \\ f_1, \dots, f_d \end{bmatrix} \in H^d_{y_j}(\Omega^d_{X'/k}) \ (j = 1, \dots, s).$$

Let $G(f_i) := F_i(0, t_1, \dots, t_d)$ and $G(h)$ be the forms of maximal degree of f_i and h, which we call their *degree forms*. Since the hypersurfaces $F_i = 0$ have no common point on the hyperplane $X_0 = 0$ at infinity, $\{G(f_1) \dots, G(f_d)\}$ is a regular sequence in $k[t_1, \dots, t_d]$, and the origin $O \in \mathbb{A}^d_k$ is the only common zero of the $G(f_i)$. Hence the generalized fraction $\begin{bmatrix} G(h)\, dt_1 \cdots dt_d \\ G(f_1), \dots, G(f_d) \end{bmatrix} \in H^d_O(\Omega^d_{X'/k})$ is defined.

THEOREM 7.10 (Affine Residue Theorem). *Assume* $\deg h \le \sum_{i=1}^d \deg f_i - d$. *Then*

$$\sum_{j=1}^s \operatorname{Res}_{y_j} \omega_j = \operatorname{Res}_O \begin{bmatrix} G(h)\, dt_1 \cdots dt_d \\ G(f_1), \dots, G(f_d) \end{bmatrix}.$$

The sum vanishes if $\deg h < \sum_{i=1}^d \deg f_i - d$.

PROOF. Let $\mathfrak{U} := \{D_+(X_0), D_+(F_1), \dots, D_+(F_d)\}$ and

$$\mathfrak{U}' := \{D(f_1), \dots, D(f_d)\} = \{D_+(X_0) \cap D_+(F_1), \dots, D_+(X_0) \cap D_+(F_d)\}$$

be the coverings of X and $X' \setminus Y$. We have

$$H^d_Y(\Omega^d_{X/k}) = \bigoplus_{j=1}^{s} H^d_{y_j}(\Omega^d_{X/k}),$$

and the canonical map $H^d_Y(\Omega^d_{X/k}) \to H^d(X, \Omega^d_{X/k})$ is induced by the map i of the residue theorem. The cohomology class $\{\omega_j\}_{j=1,\dots,s} \in H^d_Y(\Omega^d_{X'/k})$ can be represented by the cocycle

$$\frac{h\,dt_1 \cdots dt_d}{f_1 \cdots f_d} \in C^{d-1}(\mathfrak{U}', \Omega^d_{X'/k}),$$

and its image in $H^d(X, \Omega^d_{X/k})$ is zero if $\deg h < \sum_{i=1}^d \deg f_i - d$, otherwise it is the cohomology class represented by the cocycle

$$\frac{H\,dX_0 \cdots dX_d}{X_0 F_1 \cdots F_d} \in C^{d-1}(\mathfrak{U}, \Omega^d_{X/k}),$$

where H denotes the homogenization of h. In the first case, thanks to the residue theorem, $\sum_{j=1}^s \operatorname{Res}_{y_j}(\omega_j) = 0$, and in the second case

$$\sum_{j=1}^{s} \operatorname{Res}_{y_j}(\omega_j) = \int_X \begin{bmatrix} H\,dX_0 \cdots dX_d \\ X_0, F_1, \dots, F_d \end{bmatrix} = \operatorname{Res}_{\mathfrak{M}} \begin{bmatrix} H\,dX_0 \cdots dX_d \\ X_0, F_1, \dots, F_d \end{bmatrix}.$$

Here $\mathfrak{M}$ denotes the maximal ideal of $S_{\mathfrak{m}}$ with $\mathfrak{m} = (X_0, \dots, X_d)$, and the generalized fraction is to be considered as an element of $H^{d+1}_{\mathfrak{M}}(\Omega^{d+1}_{S_{\mathfrak{m}}/k})$.

By the same argument as above we have that

$$\operatorname{Res}_O \begin{bmatrix} G(h)\,dt_1 \cdots dt_d \\ G(f_1), \dots, G(f_d) \end{bmatrix} = 0$$

if $\deg h < \sum_{i=1}^d \deg f_i - d$, and in case of equality

$$\operatorname{Res}_O \begin{bmatrix} G(h)\,dt_1 \cdots dt_d \\ G(f_1), \dots, G(f_d) \end{bmatrix} = \operatorname{Res}_{\mathfrak{M}} \begin{bmatrix} G(h)\,dX_0 \cdots dX_d \\ X_0, G(f_1), \dots, G(f_d) \end{bmatrix},$$

where on the right hand side the degree forms $G(h)$ and $G(f_i)$ have to be considered as polynomials in $X_1, \dots, X_d$. It remains to show that when $\deg h = \sum_{i=1}^d \deg f_i - d$, we have

$$\operatorname{Res}_{\mathfrak{M}} \begin{bmatrix} H\,dX_0 \cdots dX_d \\ X_0, F_1, \dots, F_d \end{bmatrix} = \operatorname{Res}_{\mathfrak{M}} \begin{bmatrix} G(h)\,dX_0 \cdots dX_d \\ X_0, G(f_1), \dots, G(f_d) \end{bmatrix}.$$

Choose $\{a_1, \dots, a_d\} \in \mathbb{N}^d_+$ such that $X_i^{a_i} \in (X_0, F_1, \dots, F_d)$ and write

$$X_i^{a_i} = c_{i0} X_0 + \sum_{j=1}^{d} c_{ij} F_j \qquad (c_{ij} \in S,\ j = 1, \dots, d).$$

Set $\overline{c}_{ij} := c_{ij}(0, X_1, \dots, X_d)$ $(i, j = 1, \dots, d)$ and

$$\Delta := \det(c_{ij})_{i,j=1,\dots,d} \quad \text{and} \quad \overline{\Delta} := \det(\overline{c}_{ij})_{i,j=1,\dots,d}.$$

Then by the transformation law 5.5 b),

$$\operatorname{Res}_{\mathfrak{M}} \begin{bmatrix} H\,dX_0 \cdots dX_d \\ X_0, F_1, \dots, F_d \end{bmatrix} = \operatorname{Res}_{\mathfrak{M}} \begin{bmatrix} \Delta H\,dX_0 \cdots dX_d \\ X_0, X_1^{a_1}, \dots, X_d^{a_d} \end{bmatrix},$$

and

$$\operatorname{Res}_{\mathfrak{M}}\begin{bmatrix} G(h)\,dX_0\cdots dX_d \\ X_0, G(f_1),\dots,G(f_d)\end{bmatrix} = \operatorname{Res}_{\mathfrak{M}}\begin{bmatrix} \overline{\Delta}G(h)\,dX_0\cdots dX_d \\ X_0, X_1^{a_1},\dots,X_d^{a_d}\end{bmatrix}.$$

Since

$$\Delta H \equiv \overline{\Delta}G(h) \bmod X_0\cdot S,$$

it follows from 5.5 c) that the two residues agree. The last statement of the theorem follows from 5.10 applied to Res_O. □

In the situation of 7.10 the socle $\mathfrak{S}$ of the graded k-algebra

$$A := k[X_1,\dots,X_d]/(G(f_1),\dots,G(f_d))$$

is the k-vector space generated by the image of a homogeneous transition determinant Δ from $G(f) := \{G(f_1),\dots,G(f_d)\}$ to $X := \{X_1,\dots,X_d\}$ (see 4.12). This determinant has degree $\delta := \sum_{i=1}^d \deg f_i - d$, hence $\mathfrak{S}$ is the homogeneous component of A of degree δ and also the highest non-vanishing component. For each $h\in k[X_1,\dots,X_d]$ there exists a "remainder"

$$r\in k[X_1,\dots,X_d] \text{ with } \deg r\le\delta$$

such that

$$h\equiv r \bmod (f_1,\dots,f_d).$$

In fact, if $\deg h\le\delta$ there is nothing to show. If $\deg h := \delta' > \delta$, then we have that $G(h)\in(G(f_1),\dots,G(f_d))$. Write

$$G(h)=\sum_{i=1}^d b_iG(f_i)$$

with homogeneous $b_i\in k[X_1,\dots,X_d]$ of degree $\delta'-\deg f_i$ $(i=1,\dots,d)$. Then the polynomial $h' := h-\sum_{i=1}^d b_if_i$ has degree $<\delta'$. Repeating this process a finite number of times we find an r as desired.

If $\deg r=\delta$, then

$$G(r)\equiv\kappa\cdot\Delta \bmod (G(f_1),\dots,G(f_d))$$

with a unique $\kappa\in k\setminus\{0\}$.

REMARK 7.11. Theorem 7.10 holds true without degree assumption on h if $G(h)$ is replaced by $G(r)$ with a remainder r as above. If $\deg r<\delta$, then the residue sum vanishes. If $\deg r=\delta$, then

$$\sum_{j=1}^s \operatorname{Res}_{y_j}\omega_j=\kappa.$$

PROOF. The first statement follows from 5.5 c). Given 7.10, the second follows from 5.10 and the third from the transformation law 5.5 b) since

$$\sum_{j=1}^s \operatorname{Res}_{y_j}\omega_j=\operatorname{Res}_O\begin{bmatrix} G(r)\,dX_1\cdots dX_d\\ G(f)\end{bmatrix}=\operatorname{Res}_O\begin{bmatrix}\kappa\,dX_1\cdots dX_d\\ X\end{bmatrix}=\kappa. \qquad \square$$

Observe that if we write

$$G(r)=\kappa\cdot\Delta+\sum_{i=1}^d c_iG(f_i)$$

with homogeneous $c_i \in k[X_1, \dots, X_d]$ and $\kappa \in k$ (possibly $= 0$), then

$$h \equiv \kappa \cdot \Delta + r' \bmod (f_1, \dots, f_d) \tag{7.8}$$

with $r' := r - \sum_{i=1}^{d} c_i f_i$ of degree $< \delta$.

Theorem 7.10 is also true in the "weighted case", i.e. if the variables X_i are given weights $\alpha_i \in \mathbb{N}_+$. For a non-cohomological proof, based on the work of Scheja and Storch, see [**63**, 4.8 a)]. Essentially the same proof is used also in [**67**] to prove 7.10 in the two-dimensional case. In the weighted case, a proof (with applications) that uses $k = \mathbb{C}$ and complex analytic arguments can be found in [**13**].

We turn to applications of theorem 7.10. Given polynomials $f_i \in k[X_1, \dots, X_d]$, the residues of generalized fractions with denominator set $\{f_1, \dots, f_d\}$ are intersection invariants of the hypersurfaces $H_i := V(f_i) \subset \mathbb{A}_k^d$ $(i = 1, \dots, d)$. If the numerator ω is chosen properly the residue has a geometric interpretation. The residue theorem then yields global relations among the invariants. In the following some examples will be given which can be derived from the residue theorem on $\mathbb{P}_k^d$.

Let $Y := V_+(F_1, \dots, F_d)$ be a reduced zero-dimensional complete intersection subscheme of $\mathbb{P}_k^d$. Here the F_i $(i = 1, \dots, d)$ are homogeneous polynomials in $k[X_0, \dots, X_d]$ which form a regular sequence. Let $n := \prod_{i=1}^{d} \deg F_i$ be the degree of Y.

We choose coordinates so that $Y \cap V_+(X_0) = \emptyset$. Then in $\mathbb{A}_k^d := \mathbb{P}_k^d \setminus V_+(X_0)$ we have

$$Y = V(f_1, \dots, f_d)$$

where the $f_i := F(1, X_1, \dots, X_d)$ form a regular sequence in $k[X_1, \dots, X_d]$. Moreover $\{G(f_1), \dots, G(f_d)\}$ with the degree forms $G(f_i) := F_i(0, X_1, \dots, X_d)$ is also a regular sequence in $k[X_1, \dots, X_d]$, and the origin $O \in \mathbb{A}_k^d$ is the only common zero of these polynomials.

Reducedness of Y is equivalent to each of the following conditions:

a) Y consists of n distinct closed points of $\mathbb{A}_k^d$.
b) $\mathscr{O}_{Y,P} = k$ for each $P \in Y$.
c) With $J := \frac{\partial(f_1, \dots, f_d)}{\partial(X_1, \dots, X_d)}$ we have $J(P) \neq 0$ for each $P \in Y$.
d) The hypersurfaces $H_i := V(f_i)$ $(i = 1, \dots, d)$ intersect transversally at every $P \in Y = H_1 \cap \dots \cap H_d$, i.e. the points $P \in Y$ are regular on each H_i, and the tangent hyperplanes $T_P(H_i)$ $(i = 1, \dots, d)$ have only P in common.

We may also consider Y as the intersection of the curve $C := V(f_1, \dots, f_{d-1})$ with the hypersurface H_d. Then reducedness of Y means that all $P \in Y$ are regular on C and on H_d, and that the tangent $T_P(C)$ is not contained in $T_P(H_d)$.

THEOREM 7.12. *Under the above assumptions let $f \in k[X_1, \dots, X_d]$ be given such that*

$$\deg f \leq \sum_{i=1}^{d} \deg f_i + \rho \deg f_d - d$$

where $\rho \in \mathbb{N}$ and char k does not divide $\rho!$. Set $R_0 := \frac{f}{J}$ and for $\sigma \in \mathbb{N}$ with $0 < \sigma \leq \rho$

$$R_\sigma := \frac{1}{\sigma} J^{-1} \frac{\partial(f_1, \dots, f_{d-1}, R_{\sigma-1})}{\partial(X_1, \dots, X_d)}.$$

Then

$$\sum_{P\in Y} R_\rho(P) = \mathrm{Res}_O \begin{bmatrix} G(f)\,dX_1\cdots dX_d \\ G(f_1),\dots,G(f_{d-1}),G(f_d)^{\rho+1}\end{bmatrix}.$$

The sum vanishes if $\deg f < \sum_{i=1}^d \deg f_i + \rho\deg f_d - d$.

PROOF. The sequence $\{f_1,\dots,f_d\}$ is a system of parameters of $\mathscr{O}_{\mathbb{P}^d_k,P}$ for every $P\in Y$. By 5.11 we have

$$\mathrm{Res}_P \begin{bmatrix} f\,dX_1\cdots dX_d \\ f_1,\dots,f_{d-1},f_d^{\rho+1}\end{bmatrix} = R_\rho(P)$$

and the claim follows from 7.10. □

In the following only the cases $\rho = 0$ and $\rho = 1$ play a role. For $\rho = 0$ we obtain

COROLLARY 7.13. *If* $\deg f \le \sum_{i=1}^d \deg f_i - d$, *then*

$$\sum_{P\in Y}\frac{f(P)}{J(P)} = \mathrm{Res}_O \begin{bmatrix} G(f)\,dX_1\cdots dX_d \\ G(f_1),\dots,G(f_d)\end{bmatrix}.$$

The sum vanishes if $\deg f < \sum_{i=1}^d \deg f_i - d$.

The last statement is called the Jacobi formula [**53**] or Euler-Jacobi formula. It is the subject of many publications and applications ([**6, 7, 13, 22, 31, 60, 63, 65**]). We have obtained 7.13 as a consequence of the residue theorem 7.10. Conversely, using a deformation argument, it is also possible to deduce 7.10 from 7.13, see [**69**].

For $\rho = 1$ example 5.11 implies

COROLLARY 7.14 (B. Segre [**90**]). *If* $\deg f \le \sum_{i=1}^d \deg f_i + \deg f_d - d$, *then*

$$\sum_{P\in Y} J(P)^{-3}\Big[J(P)\frac{\partial(f_1,\dots,f_{d-1},f)}{\partial(X_1,\dots,X_d)}(P) - f(P)\frac{\partial(f_1,\cdots,f_{d-1},J)}{\partial(X_1,\dots,X_d)}(P)\Big] =$$
$$\mathrm{Res}_O \begin{bmatrix} G(f)\,dX_1\cdots dX_d \\ G(f_1),\dots,G(f_{d-1}),G(f_d)^2\end{bmatrix}.$$

The sum vanishes if $\deg f < \sum_{i=1}^d \deg f_i + \deg f_d - d$.

The Euler-Jacobi formula yields

THEOREM 7.15 (Cayley-Bacharach Theorem). *For Y as above let $H\subset\mathbb{P}^d_k$ be a hypersurface of degree* $< \sum_{i=1}^d \deg F_i - d$. *If H contains $n-1$ points of Y, then $Y\subset H$.*

PROOF. In the Euler-Jacobi formula all terms but one vanish. Then all terms must vanish, which implies the statement. □

EXAMPLE 7.16. If two cubic curves C_1 and C_2 in $\mathbb{P}^2_k$ have exactly 9 points of intersection and another cubic C contains 8 of them, then $C_1\cap C_2\subset C$. This is known as Euler-Cramer paradox. On can derive Pascal's theorem about conics quickly from this statement, and the theorem of Pappus which is a special case of that of Pascal, see for instance [**67**, 5.16].

Let us consider now Y as the intersection of the curve $C := V(f_1, \dots, f_{d-1})$ with the hypersurface $H := V(f_d)$. We define the following elements of $k[X_1, \dots, X_d]^d$:

$$N := \operatorname{grad} f_d = \Big(\frac{\partial f_d}{\partial X_1}, \dots, \frac{\partial f_d}{\partial X_d}\Big)$$
$$J_i := \det(\operatorname{grad} f_1, \dots, \operatorname{grad} f_{d-1}, e_i) \quad (i = 1, \dots, d)$$

where $e_i = (0, \dots, 1, \dots, 0)$ is the i-th standard basis vector of k^d, and

$$T := (J_1, \dots, J_d), \ T_0 := J^{-1} T$$

We then have $J = \langle T, N \rangle$ with the standard scalar product $\langle \ , \ \rangle$. For $P \in Y$ the vector $T(P) = (J_1(P), \dots, J_d(P))$ is tangent to C at P and $N(P) = \operatorname{grad} f_d(P)$ is normal to $T_P(H)$. The tangent vector $T_0(P) := J(P)^{-1} T(P)$ is normed so that $\langle T_0(P), N(P) \rangle = 1$.

The following application of 7.13 is due to R. Hübl (private communication):

Theorem 7.17. *With the above notation and assumptions, we have*

a) *If H is not a hyperplane, then*

$$\sum_{P \in Y} T_0(P) = 0.$$

b) *If H is a hyperplane, then $\sum_{P \in Y} T_0(P)$ depends only on the schemes $\bar{H}_i \cap H_\infty$ $(i = 1, \dots, d)$, where $\bar{H}_i$ denotes the projective closure of H_i and H_∞ is the hyperplane at infinity.*

Proof. Apply 7.13 with $f = J_i$. We have $\operatorname{Res}_P \begin{bmatrix} J_i \, dX_1 \cdots dX_d \\ f_1, \dots, f_d \end{bmatrix} = \frac{J_i(P)}{J(P)}$ and

$$\deg J_i \le \sum_{i=1}^{d-1} \deg f_i - d + 1$$

hence by 7.10

$$\sum_{P \in Y} \frac{J_i(P)}{J(P)} = \operatorname{Res}_O \begin{bmatrix} G(J_i) \, dX_1 \cdots dX_d \\ G(f_1), \dots, G(f_d) \end{bmatrix}.$$

If H is not a hyperplane, then $\deg J_i < \sum_{i=1}^{d} \deg f_i - d$ and the residue vanishes. Otherwise we have equality, and the residue depends only on the degree forms $G(f_1), \dots, G(f_d)$.

In the situation of 7.17 b) the sum of tangent vectors does not change if $f_1, \dots, f_d$ are replaced by a system of polynomials $g_1, \dots, g_d$ defining hypersurfaces $H_i' := V(g_i)$ such that H_i' and H_i have the same schemes at infinity $(i = 1, \dots, d)$ and $Y' := H_1' \cap \dots \cap H_d'$ is reduced, for example, if the hyperplane H is replaced by a parallel hyperplane H' and $Y' := C \cap H'$ is reduced. □

In the following we use the notation f_{X_i} for the partial derivative $\frac{\partial f}{\partial X_i}$ of a polynomial f. With C and H as above let

$$T_i := (f_d)_{X_d} e_i - (f_d)_{X_i} e_d \text{ and } h_i := \langle T_i, N \rangle \ (i = 1, \dots, d).$$

Then $\langle T_i(P), N(P) \rangle = 0$, i.e. the vectors $T_i(P)$ are tangent to the hypersurface H at P.

THEOREM 7.18. *For $i = 1, \ldots, d-1$ we have*

$$\sum_{P \in Y} \langle T_0(P), T_i(P) \rangle = \mathrm{Res}_O \begin{bmatrix} G(h_i)\, dX_1 \cdots dX_d \\ G(f_1), \ldots, G(f_d) \end{bmatrix}.$$

As above the sum depends only on the schemes at infinity of the hypersurfaces H_i ($i = 1, \ldots, d$).

PROOF. Apply 7.13 with $f = h_i$ and note that either $\deg h_i < \sum_{i=1}^{d} \deg f_i - d$ or, in case of equality,

$$G(h_i) = \det(\mathrm{grad}\, G(f_1), \ldots, \mathrm{grad}\, G(f_{d-1}), G(T_i))$$

with $G(T_i) := G(f_d)_{X_d} e_i - G(f_d)_{X_i} e_d$. □

This theorem generalizes a theorem of G. Humbert [**50**] about the angles under which two plane curves intersect. When $d = 2$ set $f := f_1$, $g := f_2$, $C_1 := V(f)$ and $C_2 := V(g)$. Then $T = (f_{X_2}, -f_{X_1})$, $N = (g_{X_1}, g_{X_2})$ and $T_1 = (g_{X_2}, -g_{X_1})$, hence

$$\langle T_0, T_1 \rangle = \frac{f_{X_1} g_{X_1} + f_{X_2} g_{X_2}}{f_{X_1} g_{X_2} - f_{X_2} g_{X_1}}$$

For $P \in Y := C_1 \cap C_2$ and $h := \langle T, T_1 \rangle$ we denote the invariant

$$\mathrm{Res}_P \begin{bmatrix} h\, dX_1 dX_2 \\ f, g \end{bmatrix} = \langle T_0(P), T_1(P) \rangle$$

by $a_P(C_1, C_2)$. Then by 7.18

$$\sum_{P \in Y} a_P(C_1, C_2) = \mathrm{Res}_O \begin{bmatrix} [G(f)_{X_1} G(g)_{X_1} + G(f)_{X_2} G(g)_{X_2}]\, dX_1 dX_2 \\ G(f), G(g) \end{bmatrix}.$$

See the discussion in [**67**, p. 119–122], where G. Humbert's theorem is derived from the last formula. Another higher dimensional version of this theorem can be found in [**47**].

The theory of residues can also be used to prove certain global relations about the curvature of plane algebraic curves such as the Reiss relation and its generalization by B. Segre ([**90, 91**]), which follows from 7.14 (see also [**67**, chap. 12]). Similar relations about the curvature of algebraic curves in higher dimensions and algebraic hypersurfaces can be found in chapter 6 of the thesis of G. Quarg [**85**]. See also [**66**], where some of his results are outlined.