Introduction

Modern model theory began with Morley's [**Mor65a**] categoricity theorem: A first order theory is categorical in one uncountable cardinal κ (has a unique model of that cardinality) if and only if it is categorical in all uncountable cardinals. This result triggered the change in emphasis from the study of logics to the study of theories. Shelah's taxonomy of first order theories by the stability classification established the background for most model theoretic researches in the last 35 years. This book lays out of some of the developments in extending this analysis to classes that are defined in non-first order ways. Inspired by [**Sac72, Kei71**], we proceed via short chapters that can be covered in a lecture or two.

There were three streams of model-theoretic research in the 1970's. For simplicity in the discussion below I focus on vocabularies (languages) which contain only countably many relation and function symbols. In one direction workers in algebraic model theory melded sophisticated algebraic studies with techniques around quantifier elimination and developed connections between model theory and algebra. A second school developed fundamental model theoretic properties of a wide range of logics. Many of these logics were obtained by expanding first order logic by allowing longer conjunctions or longer strings of first order quantifiers; others added quantifiers for 'there exist infinitely many', 'there exist uncountably many', 'equicardinality', and many other concepts. This work was summarized in the Barwise-Feferman volume [**BF85**]. The use of powerful combinatorial tools such as the Erdös-Rado theorem on the one hand and the discovery that Chang's conjecture on two cardinal models for arbitrary first theories is independent of ZFC and that various two cardinal theorems are connected to the existence of large cardinals [**CK73**] caused a sense that pure model theory was deeply entwined both with heavy set-theoretic combinatorics and with (major) extensions of ZFC. In the third direction, Shelah made the fear of independence illusory for the most central questions by developing the stability hierarchy. He split all first order theories into 5 classes. Many interesting algebraic structures fall into the three classes (ω-stable, superstable, strictly stable) whose models admit a deep structural analysis. This classification is (set theoretically) absolute as are various fundamental properties of such theories. Thus, for stable theories, Chang's conjecture is proved in ZFC [**Lac72, She78**]. Shelah focused his efforts on the test question: compute the function $I(T, \kappa)$ which gives the number of models of cardinality κ. He achieved the striking main gap theorem. Every theory T falls into one of two classes. T may be intractable, that is $I(T, \kappa) = 2^\kappa$, the maximum, for every sufficiently large κ. Or, every model of T is decomposed as a tree of countable models and the number of models in κ is bounded well below 2^κ. The description of this tree and the proof of the theorem required the development of a far reaching generalization of the Van der Waerden axiomatization of independence in vector spaces and fields. This is not the place for even a cursory survey of the development of stability theory over the last 35 years. However, the powerful tools of the Shelah's calculus of independence and orthogonality are fundamental to the applications of model theory in the 1990's to Diophantine geometry and number theory [**Bou99**].

Since the 1970's Shelah has been developing the intersection of the second and third streams described above: the model theory of the class of models of a sentence in one of a number of 'non-elementary' logics. He builds on Keisler's work [**Kei71**] for the study of $L_{\omega_1,\omega}$ but to extend to other logics he needs a more general framework and the Abstract Elementary Classes (AEC) we discuss below provide one. In the last ten years, the need for such a study has become more widely appreciated as a result of work on both such concrete problems as complex exponentiation and Banach spaces and programmatic needs to understand 'type-definable' groups and to understand an analogue to 'stationary types' in simple theories.

Our goal here is to provide a systematic and intelligible account of some central aspects of Shelah's work and related developments. We study some very specific logics (e.g. $L_{\omega_1,\omega}$) and the very general case of abstract elementary classes. The survey articles by Grossberg [**Gro02**] and myself [**Bal04**] provide further background and motivation for the study of AEC that is less technical than the development here.

An abstract elementary class (AEC) $\boldsymbol{K}$ is a collection of models and a notion of 'strong submodel' $\prec$ which satisfies general conditions similar to those satisfied by the class of models of a first order theory with $\prec$ as elementary submodel. In particular, the class is closed under unions of $\prec$-chains. A Löwenheim-Skolem number is assigned to each AEC: a cardinal κ such that each $M \in \boldsymbol{K}$ has a strong submodel of cardinality κ. Examples include the class of models of a $\forall\exists$ first order theory with $\prec$ as substructure, a complete sentence of $L_{\omega_1,\omega}$ with $\prec$ as elementary submodel in an appropriate fragment of $L_{\omega_1,\omega}$ and the class of submodels of a homogeneous model with $\prec$ as elementary submodel. The models of a sentence of $L_{\omega_1,\omega}(Q)$ (Q is the quantifier 'there exists uncountably many') fit into this context modulo two important restrictions. An artificial notion of 'strong submodel' must be introduced to guarantee the satisfaction of the axioms concerning unions of chains. More important from a methodological viewpoint, without still further and unsatisfactory contortions, the Löwenheim number of the class will be $\aleph_1$.

In general the analysis is not nearly as advanced as in the first order case. We have only approximations to Morley's theorem and only a rudimentary development of stability theory. (There have been significant advances under more specialized assumptions such as homogeneity or excellence [**GH89, HLS05**] and other works of e.g. Grossberg, Hyttinen, Lessmann, and Shelah.) The most dispositive result is Shelah's proof that assuming $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$, if a sentence of $L_{\omega_1,\omega}$ is categorical up to $\aleph_\omega$ then is categorical in all cardinals. Categoricity up to $\aleph_\omega$ is essential [**HS90, BK**].

The situation for AEC is even less clear. One would like at least to show that an AEC could not alternate indefinitely between categoricity and non-categoricity. The strongest result we show here is implicit in [**She99**]. Theorem 15.13 asserts: There is a Hanf number μ (not computed but depending only on the Löwenheim number of $\boldsymbol{K}$) such that if an AEC $\boldsymbol{K}$ satisfying the general conditions of Part 3 is categorial in a *successor* cardinal larger than μ, it is categorical in all larger cardinals. This state of affairs in a major reason that this monograph is titled categoricity. Although a general stability theory for abstract elementary classes is the ultimate goal, the results here depend heavily on assuming categoricity in at least one cardinal.

There are several crucial aspects of first order model theory. By Lindström's theorem [**Lin69**] we know they can be summarized as: first order logic is the only logic (of Lindström type) with Löwenheim number $\aleph_0$ that satisfies the compactness theorem. One corollary of compactness in the first order case plays a distinctive role here, the amalgamation property: two elementary extensions of a fixed model M have a common extension over M. In particular, the first order amalgamation property allows the identification (in a suitable monster model) of a syntactic type (the description of a point by the formulas it satisfies) with an orbit under the automorphism group (we say Galois type).

Some of the results here and many associated results were originally developed using considerable extensions to ZFC. However, later developments and the focus on AEC rather than $L_{\kappa,\omega}$ (for specific large cardinals κ) have reduced such reliance. With one exception, the results in this book are proved in ZFC or in ZFC + $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for finite n; we call this proposition the *very weak generalized continuum hypothesis* VWGCH . The exception is Chapter 17 which relies on the hypothesis that $2^{\mu} < 2^{\mu^+}$ for any cardinal μ; we call this hypothesis the *weak generalized continuum hypothesis*, WGCH. Without this assumption, some crucial results have not been proved in ZFC; the remarkable fact is that such a benign assumption as VWGCH is all that is required. Some of the uses of stronger set theory to analyze categoricity of $L_{\omega_1,\omega}$-sentences can be avoided by the assumption that the class of models considered contains arbitrarily large models.

We now survey the material with an attempt to convey the spirit and not the letter of various important concepts; precise versions are in the text. With a few exceptions that are mentioned at the time all the work expounded here was first discovered by Shelah in a long series of papers.

Part I (Chapters 2-4) contains a discussion of Zilber's quasiminimal excellent classes [**Zil05**]. This is a natural generalization of the study of first order strongly minimal theories to the logic $L_{\omega_1,\omega}$ (and some fragments of $L_{\omega_1,\omega}(Q)$. It clearly exposes the connections between categoricity and homogeneous combinatorial geometries; there are natural algebraic applications to the study of various expansions of the complex numbers. We expound a very concrete notion of 'excellence' for a combinatorial geometry. Excellence describes the closure of an independent n-cube of models. This is a fundamental structural property of countable structures in a class $\boldsymbol{K}$ which implies that $\boldsymbol{K}$ has arbitrarily large models (and more). Zilber's contribution is to understand the connections of these ideas to concrete mathematics, to recognize the relevance of infinitary logic to these concrete problems, and to prove that his particular examples are excellent. These applications require both great insight in finding the appropriate formal context and substantial mathematical work in verifying the conditions laid down. Moreover, his work has led to fascinating speculations in complex analysis and number theory. As pure model theory of $L_{\omega_1,\omega}$, these results and concepts were all established in greater generality by Shelah [**She83a**] more than twenty years earlier. But Zilber's work extends Shelah's analysis in one direction by applying to some extensions of $L_{\omega_1,\omega}$. We explore the connections between these two approaches at the end of Chapter 25. Before turning to that work, we discuss an extremely general framework.

The basic properties of abstract elementary classes are developed in Part II (chapters 5-8). In particular, we give Shelah's presentation theorem which represents every AEC as a pseudoelementary class (class of reducts to a vocabulary L of a first order theory in an expanded language L') that omit a set of types. Many of the key results (especially in Part IV) depend on having Löwenheim number $\aleph_0$. Various successes and perils of translating $L_{\omega_1,\omega}(Q)$ to an AEC (with countable Löwenheim number) are detailed in Chapters 6-8 along with the translation of classes defined by sentences of $L_{\omega_1,\omega}$ to the class of atomic models of a first order theory in an expanded vocabulary. Chapter 8 contains Shelah's beautiful ZFC proof that a sentence of $L_{\omega_1,\omega}(Q)$ that is $\aleph_1$-categorical has a model of power $\aleph_2$.

In Part III (Chapters 9-17) we first study the conjecture that for 'reasonably well-behaved classes', categoricity should be either eventually true or eventually false. We formalize 'reasonably well-behaved' via two crucial hypotheses: amalgamation and the existence of arbitrarily large models. Under these assumptions, the notion of *Galois type over a model* is well-behaved and we recover such fundamental notions as the identification of 'saturated models' with those which are 'model homogeneous'. Equally important, we are able to use the omitting types technology originally developed by Morley to find Ehrenfeucht-Mostowski models for AEC. This leads to the proof that categoricity implies stability in smaller cardinalities and eventually, via a more subtle use of Ehrenfeucht-Mostowski models, to a notion of superstability. The first goal of these chapters is to expound Shelah's proof of a downward categoricity theorem for an AEC (satisfying the above hypothesis) and categorical in a successor cardinal. A key aspect of that argument is the proof that if $\boldsymbol{K}$ is categorical above the Hanf number for AEC's, then two distinct Galois types differ on a 'small' submodel. Grossberg and VanDieren [**GV06c**] christened this notion: tame.

We refine the notion of tame in Chapter 11 and discuss three properties of Galois types: tameness, locality, and compactness. Careful discussion of these notions requires the introduction of cardinal parameters to calibrate the notion of 'small'. We analyze this situation and sketch examples related to the Whitehead conjecture showing how non-tame classes can arise. Grossberg and VanDieren [**GV06b, GV06a**]develop the theory for AEC satisfying very strong tameness hypotheses. Under these conditions they showed categoricity could be transferred upward from categoricity in two successive cardinals. Key to obtaining categoricity transfer from one cardinal λ^+ is the proof that the union of a 'short' chain of saturated models of cardinality λ is saturated. This is a kind of superstability consideration; it requires a further and still more subtle use of the Ehrenfeucht-Mostowski technology and a more detailed analysis of splitting; this is carried out in Chapter 15.

In Chapters 16 and 17 we conclude Part III and explore AEC without assuming the amalgamation property. We show, under mild set-theoretic hypotheses (weak diamond), that an AEC which is categorical in κ and fails the amalgamation property for models of cardinality κ has many models of cardinality κ^+.

In Part IV (Chapters 18-26) we return to the more concrete situation of atomic classes, which, of course, encompasses $L_{\omega_1,\omega}$. Using $2^{\aleph_0} < 2^{\aleph_1}$, one deduces from a theorem of Keisler [**Kei71**] that an $\aleph_1$-categorical sentence ψ in $L_{\omega_1,\omega}$ is ω-stable.

Note however that ω-stability is proved straightforwardly (Chapter 7) if one assumes ψ has arbitrarily large models. In Chapters 18-23, we introduce an independence notion and develop excellence for atomic classes. Assuming cardinal exponentiation is increasing below $\aleph_\omega$, we prove a sentence of $L_{\omega_1,\omega}$ that is categorical up to $\aleph_\omega$ is excellent. In Chapters 24-25 we report Lessmann's [**Les03**] account of proving Baldwin-Lachlan style characterizations of categoricity for $L_{\omega_1,\omega}$ and Shelah's analog of Morley's theorem for excellent atomic classes. We conclude Chapter 25, by showing how to deduce the categoricity transfer theorem for arbitrary $L_{\omega_1,\omega}$-sentences from a (stronger) result for complete sentences. Finally, in the last chapter we explicate the Hart-Shelah example of an $L_{\omega_1,\omega}$-sentence that is categorical up to $\aleph_n$ but not beyond and use it to illustrate the notion of tameness.

The work here has used essentially in many cases that we deal with classes with Löwenheim number $\aleph_0$. Thus, in particular, we have proved few substantive general results concerning $L_{\omega_1,\omega}(Q)$ (the existence of a model in $\aleph_2$ is a notable exception). Shelah has substantial not yet published work attacking the categoricity transfer problem in the context of 'frames'; this work does apply to $L_{\omega_1,\omega}(Q)$ and does not depend on Löwenheim number $\aleph_0$. We do not address this work [**She0x, She00d, She00c**] nor related work which makes essential use of large cardinals ([**MS90, KS96**].

A solid graduate course in model theory is an essential prerequisite for this book. Nevertheless, the only quoted material is very elementary model theory (say a small part of Marker's book [**Mar02**]), and two or three theorems from the Keisler book [**Kei71**] including the Lopez-Escobar theorem characterizing well-orderings. We include in Appendix A a full account of the Hanf number for omitting types. In Appendix B we give the Keisler technology for omitting types in uncountable models. The actual combinatorial principle that extends ZFC and is required for the results here is the Devlin-Shelah weak diamond. A proof of the weak diamond from weak GCH below $\aleph_\omega$ appears in Appendix C. In Appendix D we discuss a number of open problems. Other natural background reference books are [**Mar02, Hod87, She78, CK73**].

The foundation of all this work is Morley's theorem [**Mor65a**]; the basis for transferring this result to infinitary logic is [**Kei71**]. Most of the theory is due to Shelah. In addition to the fundamental papers of Shelah, this exposition depends heavily on various works by Grossberg, Lessmann, Makowski, VanDieren, and Zilber and on conversations with Adler, Coppola, Dolich, Drueck, Goodrick, Hart, Hyttinen, Kesala, Kirby, Kolesnikov, Kueker, Laskowski, Marker, Medvedev, Shelah, and Shkop as well as these authors. The book would never have happened if not for the enthusiasm and support of Rami Grossberg, Monica VanDieren and Andres Villaveces. They brought the subject alive for me and four conferences in Bogota and the 2006 AIM meeting on Abstract Elementary Classes were essential to my understanding of the area. Grossberg, in particular, was a unending aid in finding my way. I thank the logic seminar at the University of Barcelona and especially Enriques Casanovas for the opportunity to present Part IV in the Fall of 2006 and for their comments. I also must thank the University of Illinois at Chicago and the National Science Foundation for partial support during the preparation of this manuscript.