CHAPTER 1

Basic definitions

The aim of this chapter is to fix the notation used in the rest of the book. We discuss the definition of Koszul cohomology in the algebraic context (graded modules over the symmetric algebra of a vector space) and the geometric context (graded modules associated to a line bundle on a projective variety) and briefly discuss the relation of Koszul cohomology with minimal resolutions and its functorial properties.

1.1. The Koszul complex

Let V be a vector space of dimension $r+1$ over a field k of characteristic zero and let

$$\langle - \;,\; - \rangle : V^\vee \times V \to k$$

be the duality pairing. Given a nonzero element $x \in V^\vee$, the corresponding map $\langle x, - \rangle : V \to k$ extends uniquely to an anti–derivation

$$\iota_x : \bigwedge^* V \to \bigwedge^* V$$

of the exterior algebra of degree -1. This derivation is defined inductively by putting $\iota_x|_V = \langle x, - \rangle : V \to k$ and

$$\iota_x(v \wedge v_1 \wedge \ldots \wedge v_{p-1}) = \langle x, v \rangle . v_1 \wedge \ldots \wedge v_{p-1} - v \wedge \iota_x(v_1 \wedge \ldots \wedge v_{p-1}).$$

The resulting maps

$$\iota_x : \bigwedge^p V \to \bigwedge^{p-1} V$$

are called *contraction* (or *inner product*) maps; they are dual to the exterior product maps

$$\lambda_x : \bigwedge^{p-1} V^\vee \xrightarrow{\wedge x} \bigwedge^p V^\vee$$

and satisfy $\iota_x \circ \iota_x = 0$. Hence we obtain a complex

$$K_\bullet(x) : (0 \to \bigwedge^{r+1} V \to \ldots \to \bigwedge^p V \xrightarrow{\iota_x} \bigwedge^{p-1} V \xrightarrow{\iota_x} \bigwedge^{p-2} V \to \ldots \to k \to 0)$$

called the *Koszul complex.*

Note that for any $\alpha \in k^*$, the complexes $K_\bullet(x)$ and $K_\bullet(\alpha x)$ are isomorphic; hence the Koszul complex depends only on the point $[x] \in \mathbb{P}(V^\vee)$.

Lemma 1.1. *Given nonzero elements $\xi \in V$, $x \in V^\vee$, let $\lambda_\xi : \bigwedge^{p-1} V \xrightarrow{\wedge \xi} \bigwedge^p V$ be the map given by wedge product with ξ. We have*

$$\iota_x \circ \lambda_\xi + \lambda_\xi \circ \iota_x = \langle x, \xi \rangle . \operatorname{id} .$$

Proof: It suffices to verify the statement on decomposable elements. To this end, we compute

$$\begin{aligned}(\lambda_\xi \circ \iota_x)(v_1 \wedge \ldots \wedge v_p) &= \sum_i (-1)^i \langle x, v_i\rangle.\xi \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p \\ (\iota_x \circ \lambda_\xi)(v_1 \wedge \ldots \wedge v_p) &= \langle x, \xi\rangle.v_1 \wedge \ldots \wedge v_p \\ &\quad + \sum_i (-1)^{i-1} \langle x, v_i\rangle.\xi \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p\end{aligned}$$

and the statement follows. □

COROLLARY 1.2. *For every nonzero element $x \in V^\vee$, the Koszul complex $K_\bullet(x)$ is an exact complex of k-vector spaces.*

REMARK 1.3. Put $W_x = \ker(\langle x, -\rangle : V \to k)$. Taking exterior powers in the resulting short exact sequence

$$0 \to W_x \to V \to k \to 0$$

we obtain exact sequences

$$0 \to \bigwedge^p W_x \to \bigwedge^p V \to \bigwedge^{p-1} W_x \to 0$$

for all $p \geq 1$. Using these exact sequences, one checks by induction on p that

$$\bigwedge^p W_x = \ker(\iota_x : \bigwedge^p V \to \bigwedge^{p-1} V) = \operatorname{im}(\iota_x : \bigwedge^{p+1} V \to \bigwedge^p V)$$

for all $p \geq 1$. Hence the contraction map $\iota_x : \bigwedge^p V \to \bigwedge^{p-1} V$ factors through $\bigwedge^{p-1} W_x$.

1.2. Definitions in the algebraic context

Let M be a graded module over the symmetric algebra S^*V. Let $\iota : \bigwedge^p V \to \bigwedge^{p-1} V \otimes V$ be the dual of the wedge product map $\lambda : \bigwedge^{p-1} V^\vee \otimes V^\vee \to \bigwedge^p V^\vee$. Note that we have the identification

$$\begin{aligned}\bigwedge^p V &\hookrightarrow \bigwedge^{p-1} V \otimes V \cong \operatorname{Hom}(V^\vee, \bigwedge^{p-1} V) \\ v_1 \wedge \ldots \wedge v_p &\mapsto (x \mapsto \iota_x(v_1 \wedge \ldots \wedge v_p)).\end{aligned}$$

The graded S^*V–module structure of M induces maps $\mu : V \otimes M_q \to M_{q+1}$ for all q. Define a map

$$\delta : \bigwedge^p V \otimes M_q \to \bigwedge^{p-1} V \otimes M_{q+1}$$

by the composition

$$\begin{array}{ccc} \bigwedge^p V \otimes M_q & \xrightarrow{\iota \otimes \mathrm{id}} & \bigwedge^{p-1} V \otimes V \otimes M_q \\ & \searrow^{\delta} & \downarrow{\scriptstyle \mathrm{id} \otimes \mu} \\ & & \bigwedge^{p-1} V \otimes M_{q+1}. \end{array}$$

Concretely, this means that

$$\delta(v_1 \wedge \ldots \wedge v_p \otimes m) = \sum_i (-1)^i v_1 \wedge \ldots \widehat{v_i} \ldots \wedge v_p \otimes v_i m.$$

DEFINITION 1.4. The *Koszul cohomology group* $K_{p,q}(M, V)$ is the cohomology at the middle term of the complex

$$\bigwedge^{p+1} V \otimes M_{q-1} \xrightarrow{\delta} \bigwedge^p V \otimes M_q \xrightarrow{\delta} \bigwedge^{p-1} V \otimes M_{q+1}. \tag{1.1}$$

Remark 1.5. In the above definition we do not require that M is module of finite type over S^*V. (In practice we shall often restrict ourselves to this case.) In any case, if $\dim_k M_q < \infty$ for all q then the Koszul cohomology groups $K_{p,q}(M,V)$ are finite–dimensional k–vector spaces.

An element $x \in V^\vee$ induces a derivation

$$\partial_x : S^*V \to S^*V$$

of degree -1 on the symmetric algebra, which is defined inductively by the rule

$$\partial_x(v.v_1 \dots v_{p-1}) = \partial_x(v).v_1 \dots v_{p-1} + v.\partial_x(v_1 \dots v_{p-1}).$$

If we choose coordinates $X_0, \dots, X_r$ on V, with duals $x_i \in V^\vee$, the resulting map

$$\partial_{x_k} : S^pV \to S^{p-1}V$$

sends a homogeneous polynomial f of degree p to the partial derivative $\frac{\partial f}{\partial X_k}$. Using the natural map

$$\begin{array}{ccc} S^{q+1}V & \to & S^qV \otimes V \cong \mathrm{Hom}(V^\vee, S^qV) \\ f & \mapsto & (x \mapsto \iota_x(f)) \end{array}$$

and the wedge product map $\lambda : \bigwedge^{p-1}V \otimes V \to \bigwedge^p V$ we define the map

$$D : \bigwedge^{p-1}V \otimes S^{q+1}V \to \bigwedge^p V \otimes S^qV$$

as the composition

$$\begin{array}{ccc} \bigwedge^{p-1}V \otimes S^{q+1}V & \xrightarrow{\mathrm{id} \otimes \iota} & \bigwedge^{p-1}V \otimes V \otimes S^qV \\ & \overset{D}{\searrow} & \Big\downarrow \lambda \otimes \mathrm{id} \\ & & \bigwedge^p V \otimes S^qV. \end{array}$$

Proposition 1.6. *We have $K_{0,0}(S^*V, V) \cong k$, and $K_{p,q}(S^*V, V) = 0$ for all $(p,q) \neq (0,0)$.*

Proof: The first part follows from the definition. To prove the second part, choose coordinates $X_0, \dots, X_r$ on V and note that

$$D : \bigwedge^p V \otimes S^{q+1}V \to \bigwedge^{p+1}V \otimes S^qV$$

is given by

$$X_{i_1} \wedge \dots \wedge X_{i_p} \otimes f \mapsto \sum_{k=0}^{r} X_k \wedge X_{i_1} \wedge \dots \wedge X_{i_p} \otimes \frac{\partial f}{\partial X_k}.$$

The Euler formula

$$\sum_{k=0}^{r} X_k \frac{\partial f}{\partial X_k} = (q+1).f$$

implies that

$$D \circ \delta + \delta \circ D = (p+q+1).\mathrm{id},$$

hence the Koszul complex is exact. □

Recall that given a graded module M and an integer n, the graded module $M(n)$ is defined by the rule $M(n)_d = M_{d+n}$; this means that one shifts the grading of M to the left by n steps.

COROLLARY 1.7. *We have an exact complex of graded S^*V–modules*

$$K_\bullet(k) : (0 \to \bigwedge^{r+1} V \otimes S^*V(-r-1) \to \bigwedge^{r} V \otimes S^*V(-r) \to \dots \\ \dots \to \bigwedge^{2} V \otimes S^*V(-2) \to V \otimes S^*V(-1) \to S^*V \to k \to 0). \tag{1.2}$$

Proof: Put together the Koszul complexes for S^*V with various degree shifts. □

1.3. Minimal resolutions

Let $M = \oplus_q M_q$ be a graded S^*V–module of finite type. Let $\mathfrak{m} = \oplus_{d\geq 1} S^d V \subset S^*V$ be the irrelevant ideal of $S^*V = S$.

DEFINITION 1.8. A graded free resolution

$$\dots \to F_{p+1} \xrightarrow{\varphi_{p+1}} F_p \to \dots \to F_0 \to M \to 0 \tag{1.3}$$

is called *minimal* if $\varphi_{p+1}(F_{p+1}) \subset \mathfrak{m}.F_p$ for all p.

REMARK 1.9. Put $\overline{F}_i = F_i \otimes k$, $\overline{\varphi}_i = \varphi \otimes \mathrm{id} : \overline{F}_i \to \overline{F}_{i-1}$. The resolution $F_\bullet$ is minimal if and only if $\overline{\varphi}_i = 0$ for all i.

PROPOSITION 1.10. *Every finitely generated graded S^*V–module admits a minimal free resolution. The minimal free resolution is unique up to isomorphism.*

Proof: The statement is proved in [**Ei95**, §20.1] for the case of local rings. The proof in the graded case is analogous. □

REMARK 1.11.

(i) If one replaces the condition that M is finitely generated by the conditions
  (a) $\dim_k M_q < \infty$ for all q;
  (b) there exists n_0 such that $M_q = 0$ for all $q < n_0$

  then M admits a minimal free resolution of the form

$$\dots \to \bigoplus_{q\geq q_1} S(-q) \otimes M_{1,q} \to \bigoplus_{q \geq q_0} S(-q) \otimes M_{0,q} \to M \to 0;$$

  see [**Gre84a**, (1.b)].

(ii) Bruns and Herzog [**BH93**] note that the statement of Proposition 1.10 extends to so–called *–local rings, a class of rings that includes local rings and symmetric algebras.

DEFINITION 1.12. Let $F_\bullet \to M$ be a minimal graded free resolution. Write $F_i = \oplus_j S(-j)^{\beta_{i,j}} = \oplus_j M_{i,j} \otimes S(-j)$, where $M_{i,j}$ is a k–vector space of dimension $\beta_{i,j}$.

(i) The numbers $\beta_{i,j}$ are called the *graded Betti numbers* of the module M;
(ii) The vector space $M_{0,q}$ is called the space of generators of M of degree q. If $p \geq 1$ then $M_{p,q}$ is called the space of *syzygies* of order p and degree q of the module M.

PROPOSITION 1.13. *We have*

$$K_{p,q}(M,V) \cong M_{p,p+q}.$$

Proof: The symmetry property of the Tor functor implies that we can calculate $\mathrm{Tor}_p^S(M,k)$ via a free resolution of one of the two factors. The Koszul complex $K_\bullet(k)$ provides a free resolution of k by Corollary 1.7. Hence

$$\mathrm{Tor}_p^S(M,k)_{p+q} = H_p(K_\bullet(k)\otimes M)_{p+q} = K_{p,q}(M,V).$$

If we compute $\mathrm{Tor}_p^S(M,k)$ via the free resolution $F_\bullet$ of M, we find

$$\mathrm{Tor}_p^S(M,k)_{p+q} = H_p(F_\bullet \otimes k)_{p+q} = M_{p,p+q}$$

by Remark 1.9, and the statement follows. □

COROLLARY 1.14. *Put* $\kappa_{p,q} = \dim_k K_{p,q}(M,V)$*. We have* $\kappa_{p,q} = \beta_{p,p+q}$*.*

COROLLARY 1.15 (Hilbert syzygy theorem). *Any graded* S^*V *module* M *of finite type has a graded free resolution* $F_\bullet \to M$ *of length at most* $\dim V$*.*

Proof: Put $r = \dim \mathbb{P}(V^\vee)$. Clearly $\kappa_{p,q} = 0$ for $p \geq r+2$, hence $M_{p,q} = 0$ for $p \geq r+2$ by Proposition 1.13. □

DEFINITION 1.16. The *Betti table* associated to a graded S^*V–module M is given by

$$\begin{matrix} \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\ \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \\ \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix} \quad = \quad \begin{matrix} \kappa_{0,0} & \kappa_{1,0} & \kappa_{2,0} & \cdots \\ \kappa_{0,1} & \kappa_{1,1} & \kappa_{2,1} & \cdots \\ \kappa_{0,2} & \kappa_{1,2} & \kappa_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

REMARK 1.17. The number of columns with non-zero elements in the Betti diagram equals the *projective dimension* of the module M. See Remark 2.39 for the meaning of the number of rows in the Betti diagram.

EXAMPLE 1.18. The coordinate ring S_X of the twisted cubic $X \subset \mathbb{P}^3$ admits a resolution

$$0 \to S(-3)^2 \to S(-2)^3 \to S \to S_X \to 0.$$

The associated Betti diagram is (we put a dash '–' if the corresponding entry is zero)

$$\begin{matrix} 1 & - & - \\ - & 3 & 2. \end{matrix}$$

1.4. Definitions in the geometric context

Let X be a projective variety over $\mathbb{C}$, and let L be a holomorphic line bundle on X. Put $V = H^0(X,L)$.

DEFINITION 1.19. The Koszul cohomology group $K_{p,q}(X,L)$ is the Koszul cohomology of the graded S^*V–module

$$R(L) = \bigoplus\nolimits_q H^0(X, L^q).$$

Concretely, this means that $K_{p,q}(X,L)$ is the cohomology at the middle term of the complex

$$\bigwedge^{p+1}V \otimes H^0(X,L^{q-1}) \xrightarrow{\delta} \bigwedge^{p}V \otimes H^0(X,L^{q}) \xrightarrow{\delta} \bigwedge^{p-1}V \otimes H^0(X,L^{q+1}),$$

where the differential

$$\bigwedge^{p+1}V \otimes H^0(X,L^{q-1}) \xrightarrow{\delta} \bigwedge^{p}V \otimes H^0(X,L^{q})$$

is given by

$$\delta(v_1 \wedge \ldots \wedge v_{p+1} \otimes s) = \sum_i (-1)^i v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_{p+1} \otimes (v_i.s).$$

More generally, if $V \subset H^0(X,L)$, and $\mathcal{F}$ is a coherent sheaf on X, then we define

$$\begin{aligned} R(\mathcal{F},L) &= \bigoplus_q H^0(X, \mathcal{F} \otimes L^q) \\ K_{p,q}(X;\mathcal{F},L,V) &= K_{p,q}(R(\mathcal{F},L),V). \end{aligned}$$

If $\mathcal{F} = \mathcal{O}_X$ we write $K_{p,q}(X,L,V)$.

Remark 1.20. The Koszul cohomology groups can be introduced for a projective scheme X over an algebraically closed field, and a line bundle L on X. Many of the general facts presented here work well also for schemes. However, for convenience, in the course of this book we shall mostly work with complex varieties.

Remark 1.21. The above definition admits an obvious generalization to higher cohomology groups. Consider the graded S^*V–module

$$R^i(\mathcal{F},L) = \bigoplus_q H^i(X, \mathcal{F} \otimes L^q)$$

and put $K^i_{p,q}(X;\mathcal{F},L) = K_{p,q}(R^i(\mathcal{F},L),V)$. For technical reasons it is sometimes useful to study these groups; cf. [**Gre84a**].

In general the module $R(\mathcal{F},L)$ is not necessarily finitely generated. The following result shows that a mild positivity condition on L suffices.

Proposition 1.22. *If L is globally generated and $L \ncong \mathcal{O}_X$, then $R(\mathcal{F},L)$ is a finitely generated S^*V–module.*

Proof: Let $\varphi : X \to \mathbb{P}(V^\vee)$ be the morphism associated to L. Consider the coherent sheaf $\mathcal{G} = \varphi_*\mathcal{F}$ on $\mathbb{P}(V^\vee)$. The projection formula shows that

$$R(\mathcal{F},L) \cong \bigoplus_q H^0(\mathbb{P}(V^\vee), \mathcal{G} \otimes \mathcal{O}_P(q)).$$

Standard results on coherent sheaves imply that

(i) there exists $N \in \mathbb{Z}$ such that the module

$$R(\mathcal{F},L)' = \bigoplus_{q \geq N} H^0(\mathbb{P}(V^\vee), \mathcal{G} \otimes \mathcal{O}_P(q))$$

is a submodule of S^*V, hence finitely generated.

(ii) there exists $M \in \mathbb{Z}$ such that $H^0(\mathbb{P}(V^\vee), \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}}(q)) = 0$ for all $q < M$.

One then obtains a finite set of generators of $R(\mathcal{F},L)$ by adding a basis for the finite–dimensional k–vector space

$$\bigoplus_{M \leq q < N} H^0(\mathbb{P}(V^\vee, \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}}(q))$$

to a set of generators of the module $R(\mathcal{F},L)'$. □

REMARK 1.23. Note that we may shift the second index in the Koszul cohomology groups by changing the coherent sheaf:

$$\begin{aligned} K_{p,q}(X;\mathcal{F},L) &= K_{p,q-1}(X;\mathcal{F}\otimes L,L) = \dots \\ &= K_{p,1}(X;\mathcal{F}\otimes L^{q-1},L) = K_{p,0}(X;\mathcal{F}\otimes L^q,L). \end{aligned}$$

Given a vector bundle E on X and a section $\sigma \in H^0(X, E^\vee)$, the construction of Section 1.1 generalizes and gives a Koszul complex of vector bundles

$$\mathcal{K}_\bullet(\sigma) : (\bigwedge^r E \to \dots \to \bigwedge^p E \xrightarrow{\iota_\sigma} \bigwedge^{p-1} E \to \dots \to \mathcal{O}_X).$$

If σ is a *regular section*, i.e. the rank of E coincides with the codimension of the zero locus $Z = V(\sigma)$, this complex provides a resolution of the ideal sheaf $\mathcal{I}_Z$ of Z; cf. [**FL85**]. More generally, given a line bundle L and a homomorphism of vector bundles $\sigma : E \to L$, we obtain a complex of vector bundles

$$\mathcal{K}_\bullet(\sigma) : (\dots \to \bigwedge^p E \otimes L^q \xrightarrow{\delta} \bigwedge^{p-1} E \otimes L^{q+1} \to \dots)$$

where δ is defined as the composition

$$\bigwedge^p E \otimes L^q \xrightarrow{\iota \otimes \mathrm{id}} \bigwedge^{p-1} E \otimes E \otimes L^q \xrightarrow{\mathrm{id} \otimes (\mu \circ \sigma)} \bigwedge^{p-1} E \otimes L^{q+1}.$$

Given a line bundle L, put $V = H^0(X, L)$. Applying the above construction to the evaluation map $\mathrm{ev} : V \otimes \mathcal{O}_X \to L$, we obtain a Koszul complex of vector bundles

$$\mathcal{K}_\bullet(X, L) : (\dots \to \bigwedge^p V \otimes L^q \xrightarrow{\delta} \bigwedge^{p-1} V \otimes L^{q+1} \to \dots).$$

The Koszul complex $K_\bullet(X, L)$ associated to the S^*V–module $R(L)$ is obtained from this complex by taking global sections.

EXAMPLE 1.24. Let V be a k–vector space of dimension $r+1$. Applying the above construction to the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}(V^\vee)$, we obtain an exact complex of locally free sheaves

$$(1.4)\quad 0 \to \bigwedge^{r+1} V \otimes \mathcal{O}_{\mathbb{P}}(-r-1) \to \dots \to \bigwedge^2 V \otimes \mathcal{O}_{\mathbb{P}}(-2) \to V \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}} \to 0.$$

Note that complex (1.4) coincides with the sheafification of the complex of graded S^*V–modules appearing in Corollary 1.7. (And conversely, this complex can be recovered from the complex of sheaves (1.4) by taking global sections.)

1.5. Functorial properties

1.5.1. Algebraic case. Consider the category $\mathcal{C}$ whose objects are pairs (M, V), where V is a finite–dimensional k–vector space and M is a graded S^*V–module. Given a homomorphism $f : M \to M'$ of graded S^*V–modules and a linear map $g : V \to V'$, let

$$S(g) : S^*V \to S^*V'$$

be the induced homomorphism of symmetric algebras. We say that $(f, g) : (M, V) \to (M', V')$ is a morphism in $\mathcal{C}$ if

$$f(\lambda.x) = S(g)(\lambda).f(x) \quad \forall \lambda \in S^*V, \quad \forall x \in M.$$

Given a morphism $\varphi = (f, g)$ as above, we obtain induced maps of Koszul groups

$$\varphi_* : K_{p,q}(M, V) \to K_{p,q}(M', V')$$

for all p, q. Hence Koszul cohomology defines a covariant functor

$$\begin{array}{rcl} K_{\mathrm{alg}}:\mathcal{C} & \to & (BiGr-Vect)_k \\ (M,V) & \mapsto & K_{*,*}(M,V) \end{array}$$

from $\mathcal{C}$ to the category of bigraded k–vector spaces.

LEMMA 1.25. *An exact sequence* $0\to A\to B\to C\to 0$ *of graded* S^*V*–modules induces a long exact sequence of Koszul groups*

$$K_{p+1,q-1}(C,V)\to K_{p,q}(A,V)\to K_{p,q}(B,V)\to K_{p,q}(C,V)\to K_{p-1,q+1}(A,V).$$

Proof: Let

$$K^{\bullet}(M)=(\ldots\to \bigwedge^{p+1}V\otimes M_{q-1}\to \bigwedge^{p}V\otimes M_q\to \bigwedge^{p-1}V\otimes M_{q+1}\to\ldots)$$

be the Koszul complex associated to a graded S^*V–module M. The result follows by taking the long exact homology sequence associated to the short exact sequence

$$0\to K^{\bullet}(A)\to K^{\bullet}(B)\to K^{\bullet}(C)\to 0.$$

□

This leads to the following easy but useful corollary.

COROLLARY 1.26. *Let* $f:A\to B$ *be a homomorphism of graded* S^*V*–modules*

(i) *If* f *is injective then*
  (a) $K_{p,q}(A,V)\to K_{p,q}(B,V)$ *is injective if* $A_{q-1}=B_{q-1}$*;*
  (b) $K_{p,q}(A,V)\to K_{p,q}(B,V)$ *is surjective if* $A_q=B_q$*.*

(ii) *If* f *is surjective then*
  (a) $K_{p,q}(A,V)\to K_{p,q}(B,V)$ *is injective if* $A_q=B_q$*;*
  (b) $K_{p,q}(A,V)\to K_{p,q}(B,V)$ *is surjective if* $A_{q+1}=B_{q+1}$*.*

Proof: To prove (i), put $C=\ker f$, apply Lemma 1.25 and note that $C_i=0$ implies $K_{p,i}(C,V)=0$. For (ii), apply the same reasoning with $C=\mathrm{coker}\ f$.

□

1.5.2. Geometric case. Let $\mathcal{V}$ be the category whose objects are pairs (X,L), with X a projective variety defined over $\mathbb{C}$ and L a holomorphic line bundle on X. A morphism in $\mathcal{V}$ is a pair $(f,f^{\#}):(X,L)\to(Y,M)$ with $f:X\to Y$ a morphism of k–varieties and $f^*M\xrightarrow{f^{\#}}L$ a homomorphism of line bundles. Given a morphism $\varphi=(f,f^{\#})$ as above, we obtain maps

$$H^0(Y,M^q)\xrightarrow{f^*}H^0(X,f^*M^q)\xrightarrow{f^{\#}}H^0(X,L^q).$$

Put $V=H^0(X,L)$, $W=H^0(Y,M)$. Given a morphism $\varphi=(f,f^{\#})$ as above, we obtain an induced morphism $(R(M),W)\to(R(L),V)$ in $\mathcal{C}$. Hence the preceding construction defines contravariant functors

$$\begin{array}{rcl} R:\mathcal{V} & \to & \mathcal{C} \\ (X,L) & \mapsto & (R(L),V). \end{array}$$

and

$$\begin{array}{rcl} K_{\mathrm{geom}}:\mathcal{V} & \to & (BiGr-Vect)_k \\ (X,L) & \mapsto & K_{*;*}(X,L). \end{array}$$

Note that $K_{\mathrm{geom}}=K_{\mathrm{alg}}\circ R$.

Remark 1.27. There exists an obvious extension of the functors R and K_{geom} to the category $\widetilde{\mathcal{V}}$ of quadruples $(X, \mathcal{F}, L, V)$ with $\mathcal{F} \in \text{Coh}(X)$, $L \in \text{Pic}(X)$, $V \subseteq H^0(X, L)$. A morphism in this category is a 3–tuple $(f, f^{\#}, f^{\#\#}) : (X, L, \mathcal{F}, V) \to (Y, M, \mathcal{G}, W)$ with

$$f : X \to Y, \quad f^{\#} : f^*M \to L, \quad f^{\#\#} : f^*\mathcal{G} \to \mathcal{F}$$

such that the induced map $H^0(Y, M) \to H^0(X, L)$ sends W to V.

Definition 1.28. Given a pair $(X, L) \in \mathcal{V}$ there is a natural homomorphism of graded S^*V–modules $\pi : S^*V \to R(L)$. Put

$$S(X) = \text{im}(\pi), \quad I = \ker(\pi).$$

Remark 1.29. If L is globally generated with associated morphism $\varphi_L : X \to \mathbb{P}H^0(X, L)^\vee$, then $S(X)$ is the coordinate ring of $\varphi_L(X)$ and I its ideal.

Proposition 1.30. *Notation as above. The inclusion $S(X) \subset R(L)$ induces a homomorphism*

$$K_{p,q}(S(X), V) \to K_{p,q}(R(L), V) = K_{p,q}(X, L).$$

We have

(i) $K_{p,1}(S(X), W) \cong K_{p,1}(X, L, W)$ *for every linear subspace* $W \subset V$ *if* $\dim X \geq 1$*;*
(ii) $K_{p,q}(S(X), V) \cong K_{p-1,q+1}(I, V)$ *for all* $q \geq 1$.

Proof: The first isomorphism follows by applying Corollary 1.26 to the inclusion $S(X) \subset R(L)$, using the equalities $S_0(X) = R_0(X) = \mathbb{C}$, $S_1(X) = R_1(L) = V$. The second statement follows from the long exact sequence of Koszul groups associated to $0 \to I \to S^*V \to S(X) \to 0$ and the vanishing $K_{p,q}(S^*V, V) = 0$ for all $q \geq 1$ (Proposition 1.6). □

Let $f : X \to S$ be a projective morphism of schemes, and let $L \in \text{Pic}(X/S)$. Recall [**HAG77**, II, Ex. 5.16] that given a torsion-free coherent sheaf $\mathcal{F}$ of $\mathcal{O}_S$–modules, its p–th exterior power $\bigwedge^p \mathcal{F}$ is defined as the sheaf associated to the presheaf

$$U \mapsto \bigwedge^p \mathcal{F}(U).$$

In particular, there exist natural wedge product and contraction maps

$$\lambda : \bigwedge^{p-1} \mathcal{F} \otimes \mathcal{F} \to \bigwedge^p \mathcal{F}, \quad \iota : \bigwedge^p \mathcal{F} \to \bigwedge^{p-1} \mathcal{F} \otimes \mathcal{F}.$$

Proposition 1.31. *Let $f : X \to S$ be a flat, projective morphism such that S is integral. There exist a coherent sheaf $\mathcal{K}_{p,q}(X/S, L)$ on S and a non-empty Zariski open subset $U \subset S$ such that*

$$\mathcal{K}_{p,q}(X/S, L) \otimes k(s) \cong K_{p,q}(X_s, L_s)$$

for all $s \in U$.

Proof: Put $\mathcal{E} = f_*L$; by flatness, $\mathcal{E}$ is torsion-free. Using the natural maps

$$\iota : \bigwedge^p \mathcal{E} \to \bigwedge^{p-1} \mathcal{E} \otimes \mathcal{E}, \quad \mu : f_*L \otimes f_*(L^q) \to f_*(L^{q+1}),$$

we obtain a homomorphism of $\mathcal{O}_S$–modules

$$\delta : \bigwedge^p \mathcal{E} \otimes f_*(L^q) \to \bigwedge^{p-1} \mathcal{E} \otimes f_*(L^{q+1}).$$

The sheaf $\mathcal{K}_{p,q}(X/S, L)$ is defined as the cohomology sheaf at the middle term of the resulting complex of sheaves of $\mathcal{O}_S$–modules

$$\bigwedge^{p+1}\mathcal{E} \otimes f_*(L^{q-1}) \xrightarrow{\delta} \bigwedge^{p}\mathcal{E} \otimes f_*(L^{q}) \xrightarrow{\delta} \bigwedge^{p-1}\mathcal{E} \otimes f_*(L^{q+1}). \tag{1.5}$$

As f is proper, the sheaves $f_*(L^q)$ are coherent for all $q \geq 0$. Hence the cohomology sheaf $\mathcal{K}_{p,q}(X/S, L)$ is a coherent sheaf of $\mathcal{O}_S$–modules.

Since f is flat and L is locally free, L^q is flat over S for all $q \geq 0$. Hence the function

$$F_q : s \mapsto \dim_{k(s)} H^0(X_s, L_s^q)$$

is upper semicontinuous for all $q \geq 0$ [**HAG77**, III, Thm. 12.8]. The function F_q takes its minimal value on a nonempty Zariski open subset $U_q \subset S$. Put

$$U = U_0 \cap U_{q-1} \cap U_q \cap U_{q+1}.$$

Since the functions F_0, F_{q-1}, F_q and F_{q+1} are constant on U, the maps

$$f_*(L^k) \otimes k(s) \to H^0(X_s, L_s^k) \tag{1.6}$$

are isomorphisms for $k \in \{0, q-1, q, q+1\}$ by Grauert's theorem; cf. [**HAG77**, III, Cor. 12.9]. Let $i_s : \operatorname{Spec} k(s) \hookrightarrow S$ be the inclusion of a point. Since the functor $\mathcal{F} \mapsto \mathcal{F} \otimes k(s) = i_s^*\mathcal{F}$ is exact, we have

$$H^p(i_s^*\mathcal{F}^\bullet) \cong i_s^*\mathcal{H}^p(\mathcal{F}^\bullet)$$

for every complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_S$–modules. Put $V = H^0(X_s, L_s)$ and apply the functor i_s^* to the complex (1.5). Using the isomorphisms (1.6) we obtain a complex

$$\bigwedge^{p+1}V \otimes H^0(X_s, L_s^{q-1}) \xrightarrow{\delta_s} \bigwedge^{p}V \otimes H^0(X_s, L_s^{q}) \xrightarrow{\delta_s} \bigwedge^{p-1}V \otimes H^0(X_s, L_s^{q+1})$$

for all $s \in U$. By construction δ_s is the usual Koszul differential. Hence

$$\mathcal{K}_{p,q}(X/S, L) \otimes k(s) \cong K_{p,q}(X_s, L_s)$$

for all $s \in U$. □

Remark 1.32. The construction of the sheaf $\mathcal{K}_{p,q}(X/S, L)$ appears in [**V93**] in the case that $\mathcal{E}$ is locally free. More generally, if $\mathcal{F}$ is a coherent sheaf on X that is flat over S one can define coherent sheaves $\mathcal{K}^i_{p,q}(X/S; \mathcal{F}, L)$ such that

$$\mathcal{K}^i_{p,q}(X/S; \mathcal{F}, L) \otimes k(s) \cong K^i_{p,q}(X_s; \mathcal{F}_s, L_s)$$

for all s belonging to a suitable Zariski open subset of S.

Corollary 1.33 (cf. [**BG85**]). *Notation as above. The function*

$$s \mapsto \kappa_{p,q}(X_s, L_s)$$

is upper semicontinuous on the Zariski open subset $U \subset S$.

Proof: Nakayama's lemma implies that for every coherent sheaf $\mathcal{F}$ on S, the function

$$s \mapsto \dim_{k(s)}(\mathcal{F} \otimes k(s))$$

is upper semicontinuous; cf. [**HAG77**, III, Example 12.7.2]. □

1.6. Notes and comments

The Koszul complex originally came up in the context of differential graded algebras. Koszul [**Ko50**] considered the following situation, see also [**Hae87**] and [**Hal87**]. Let $p : E \to X$ be a principal fiber bundle with fiber G over a differentiable manifold X (a typical example is obtained by taking E a Lie group, G a Lie subgroup, and $X = E/G$ a homogeneous space). The complex $A^*(E)$ of differential forms on E admits the following description. Let $P_G = \langle x_1, \ldots, x_r\rangle \subset H^*(G, \mathbb{R})$ be the subspace of primitive elements. There exist G-invariant differential forms ξ_i on E such that the restriction of ξ_i to the fiber G represents the class x_i and such that $d\xi_i = p^* c_i$ with $c_i \in A^*(X)$. Then $\bigwedge^* P_G \otimes A^*(X)$ is a differential algebra with differential d defined by $d(x_i \otimes 1) = 1 \otimes c_i$, $d(1 \otimes b) = 1 \otimes db$, and it can be shown that the natural map

$$\psi : \bigwedge^* P_G \otimes A^*(X) \to A^*(E)$$

defined by $\psi(x_i \otimes 1) = \xi_i$, $\psi(1 \otimes b) = p^* b$ is a quasi-isomorphism. If moreover $A^*(X)$ is formal [**DMGS75**], one can replace $A^*(X)$ with $H^*(X)$. The de Rham cohomology of E is then computed by the complex $\bigwedge^* P_G \otimes H^*(B)$.

The general algebraic situation is the following. Given a commutative ring A and a positive integer n, one puts a grading on the tensor algebra $A \otimes_{\mathbb{Z}} \bigwedge^* \mathbb{Z}^n$ by putting the elements of the canonical basis $\{e_i\}_i$ of $\mathbb{Z}^n$ in degree one. Given an element $a = (a_1, \ldots, a_n)$ of A^n, one defines a differential d_A on $A \otimes_{\mathbb{Z}} \bigwedge^* \mathbb{Z}^n \cong \bigwedge^*(A^n)$ by mapping A to 0 and any e_i to a_i; by construction this map shifts the degrees by -1. This differential coincides with the contraction i_a, where a is seen as an element of the dual module $(A^n)^\vee$. The resulting graded differential algebra is denoted by $K_\bullet(a)$; one readily sees that it is isomorphic to the tensor product $K_\bullet(a_1) \otimes \ldots \otimes K_\bullet(a_n)$. If M is an A-module, there is an induced differential d_M on the graded modules $K_\bullet(a, M) = M \otimes_{\mathbb{Z}} \bigwedge^* \mathbb{Z}^n$, respectively $K^\bullet(a, M) = \mathrm{Hom}_A(K_\bullet(a), M)$.