Preface

Quasisymmetric maps are homeomorphisms that distort relative distance in a uniform, scale-invariant fashion. Their role in geometric function theory is pervasive. Euclidean quasiconformal maps are locally quasisymmetric. Quasisymmetry provides a metric substitute for quasiconformality suitable for the study of maps defined on lower-dimensional submanifolds or subsets of an ambient Euclidean space. On the other hand, quasiconformal maps of Euclidean spaces of sufficiently high dimension, or of sufficiently nice domains therein, are quasisymmetric. It is by now well understood that quasisymmetry is a remarkably ubiquitous condition which also arises in many settings outside of classical function theory. For instance, the boundary extensions of quasi-isometries of hyperbolic groups are quasisymmetric. Quasisymmetric maps of the boundaries of strictly pseudoconvex domains in $\mathbb{C}^n$, equipped with the standard Carnot-Carathéodory metric adapted to their CR structures, control the behavior of proper holomorphic maps near the boundary. Julia sets of rational maps, parametrized by hyperbolic components of the moduli space, are quasisymmetrically equivalent. Other examples could be given.

Determining the quasisymmetry type of a metric space is one of the premier open problems in geometric function theory. One of the most well-studied quasisymmetric invariants is *conformal dimension.* This invariant was introduced by Pansu in the late 1980's as a tool for the quasi-isometric classification of the classical rank one symmetric spaces and lattices in such spaces. In recent years a significant body of mathematics has accumulated concerning conformal dimensions of metric spaces which may not necessarily arise as group boundaries. Other applications are beginning to be identified, in areas as diverse as first-order analysis in metric spaces, dynamics of rational maps, Gromov hyperbolic geometry, and self-affine fractal geometry. In our view, the theory has developed and expanded to a point which justifies a presentation emphasizing its scope and maturity. We have attempted to provide such a presentation in the present survey.

We provide a comprehensive picture of the current state of the art regarding both computations and estimates of conformal dimension. Our presentation emphasizes both pure theory—computation of the conformal dimensions of metric spaces for its own sake, as an interesting exercise in geometric function theory—as well as applications of such computations in related fields. We discuss the relevance of conformal dimension for the quasi-isometric classification of Gromov hyperbolic groups, for the investigation of post-critically finite branched covers of the Riemann sphere arising from the iteration of rational maps, and for the study of self-similar and self-affine subsets of Euclidean space. These applications serve both as motivation for the development of a rich theory of conformal dimensions of metric spaces and as a source for new questions and problems.

Outline. We begin with a chapter of background material (Chapter 1) reviewing basic terminology of analysis in metric spaces. We recall several important classes of mappings between metric spaces (Lipschitz and bi-Lipschitz, Hölder, and quasisymmetric maps) and several quasisymmetric invariants of metric spaces. In Section 1.4 we recall several key metrically defined notions of dimension for metric spaces: Hausdorff dimension, Assouad dimension and Ahlfors regular dimension. We also review some of the primary tools used to compute or estimate these dimensions.

In Chapter 2 we introduce the principal actors of this story: the *conformal gauge* of a metric space and its *conformal dimension*. More precisely, we introduce two distinct yet related notions of conformal dimension, defined using two different metrically natural notions of dimension. The former, *conformal Hausdorff dimension*, was Pansu's original concept, while the latter, *Ahlfors regular conformal dimension*, arises in many of the most striking applications.

In Chapter 3 we recall the definition of Gromov hyperbolicity for metric spaces and groups. We illustrate with examples of group boundaries, including spheres at infinity for rank one symmetric spaces and hyperbolic buildings. The calculations by Pansu and Bourdon of the conformal dimensions of these boundaries were an early impetus for the development of the theory of conformal dimension.

The heart of this survey resides in Chapter 4, where we indicate the main tools and techniques used for establishing lower bounds on conformal dimension. The typical approach is to exhibit a large, diffuse family of rectifiable curves. These techniques can be used to show that many spaces (Carnot groups, boundaries of hyperbolic buildings, Ahlfors regular Loewner spaces, etc.) are minimal for conformal dimension.

Sets and spaces of conformal dimension zero are well understood. Chapter 5 describes what is known. The most important result is Kovalev's theorem (Theorem 5.1.12) on spaces of conformal dimension less than one. We indicate some ideas from the proof of Kovalev's theorem in Section 8.5.

Chapter 6 discusses the relationship between conformal dimension and Gromov–Hausdorff weak tangents. The highlight is a theorem of Keith and Laakso (Theorem 6.2.1) characterizing certain spaces which are minimal for Ahlfors regular conformal dimension in terms of weak tangents.

In Chapter 7, we review various applications of Ahlfors regular conformal dimension to recognition and uniformization problems for metric 2-spheres arising as boundaries of hyperbolic groups or through the dynamics of post-critically finite rational and branched covering maps. We also describe the connection between Ahlfors regular conformal dimension and ℓ^p cohomology of hyperbolic graphs.

Conformal dimension is an abstract concept, defined in terms of arbitrary quasisymmetric images of a given metric space. For subsets of a fixed space, we introduce a related invariant, the *global quasiconformal dimension*. Chapter 8 outlines known results on the global quasiconformal dimensions of subsets of Euclidean space. Of particular interest are constructions which reduce dimension by global quasiconformal maps. Such constructions are feasible for sets which are totally disconnected or almost totally disconnected: Cantor sets and post-critically finite self-similar fractals. We illustrate the techniques developed in Chapter 4 through a discussion of the conformal dimensions of self-affine fractals which are not self-similar. We conclude with Kovalev's theorem (Theorem 8.5.1) on sets of global quasiconformal dimension less than one.

We have oriented this survey towards a broad audience with interests in analysis in metric spaces, geometric group theory, dynamics, rigidity, and related subjects. No significant prior background in these areas is required. We have included proofs for results of particular interest or importance. For other results, we only provide a brief sketch of the proof or refer the interested reader to the literature. Our bibliography is extensive but by no means complete; it should be treated merely as an invitation to explore this rich subject in greater detail. Likewise, the choice of specific results which we present is greatly influenced by our own research interests and expertise.

Finally, a warning to the reader. The term *conformal dimension* is also used in mathematical physics with an entirely different meaning, where it refers to a feature of Hilbert space operators which describes their scaling behavior under change of variables. This conflict in terminology is unfortunate; however, both uses of the term are now firmly entrenched in the literature.

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The notion of conformal dimension was introduced by Pierre Pansu in his 1989 *Ann. Acad. Sci. Fenn.* paper "Dimension conforme et sphère à l'infini des variétés à courbure négative" [**133**]. The influence of this notion on the fields of geometric group theory and analysis in metric spaces has been extensive. We have attempted to do justice to the intricate theory

that has developed in response to Pansu's original paper. We dedicate this survey to him with great admiration on the occasion of his fiftieth birthday and the twentieth anniversary of the publication of [**133**].

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