Introduction

> It seems to me that the term "quantized functional analysis" ... has no deep physical meaning. It just refers to the fact that inspired by Heisenberg's uncertainty principle, one deals with possibly non-commutative operations.
>
> Albrecht Pietsch [**133**, §6.9.16]

> The Oedipus complex is the inability of mathematical objects to recognize their (mother) parent. ... As in the Greek myth, it is also a case of the mother not recognizing the son.
>
> Carl Faith [**63**, Notes to Ch. 4]

This text contains a systematic presentation of the ABC's of quantum functional analysis. It is intended for the reader who belongs, as the author himself, to the so-called "pedestrians", the recent fashionable name for the bulk of mathematicians.

The names of the founding fathers of the theory are properly, as we hope, mentioned in various places of the book, in the Introduction as well as in the main text. The present author wants to stress that he is not one of these creators, and he did not participate in obtaining most fundamental and sound results in the area. He was just impressed and enchanted by the might and beauty of the theory. However, since he and quite a few other "pedestrians" experienced certain difficulties and inconveniences in learning the subject as it was taught in available sources, he attempted to present its ABC's from a slightly different angle of view. Namely, we use, in the capacity of "quantizing coefficients", not matrices, as is usually the custom, but finite rank operators.

The words "quantized functional analysis" (we use the shorter term "quantum" instead of "quantized") are in the title of the memorable lecture of Effros [**47**] that has contributed so much in the propagation of the theory. These words testify to a further step in the progress of a certain general trend in modern mathematics, the so-called "non-commutative" or "quantum" ideology and methodology. This view on the whole of mathematics had arisen in the mathematical apparatus of quantum mechanics, had penetrated deeply into modern algebra and topology, and then, trespassing the boundaries of functional analysis, rolled into the theory of operator algebras. Now we witness its next conquest. The very foundations of functional analysis, including the notion of norm itself, underwent quantization.

Very roughly speaking, quantization of a mathematical science could be described as follows. You take crucial notions of the theory in question and, in the very core of their definitions, you replace the participating "commutative" objects, mostly functions or even just scalars, by things "non-commutative", such as matrices and especially operators. This sounds vague, and indeed, in practice there is no general recipe for performing such a procedure. Nevertheless one thing is certain: if you manage to do this in the "right" place and in the "right" way (as it is seen, of course, a posteriori), your reward will be great.

As to our area, the initial objects of the quantization are normed spaces and bounded operators. Take a linear space E, rewrite it as $\mathbb{C}\otimes E$ and then replace this field of scalars by some "good" operator algebra. It is this enlarged object, the so-called *amplification* of E, and not E itself that must be properly normed. In other words, in this context we replace "commutative" scalars by "non-commutative" operators in the capacity of coefficients of our vectors. Then we demand from operators, connecting our initial spaces, that they respect, in a certain proper sense, this passing to amplifications, and thus a start is made.

This is why we say "quantum functional analysis". However, more often than not, authors of books and papers (including one of the founding fathers who was mentioned before) prefer to call the area "operator space theory". It is because all of its objects can be realized as spaces (algebras, modules, ...), consisting of operators. It is difficult to judge which name is preferable. We choose the former, and not only because it pays a tribute to the general philosophy of the subject briefly discussed above. In our presentation we try to prove as much as possible using the axiomatic approach and postponing the fundamental realization theorem of Ruan to comparatively late stage. In this connection, in order to avoid confusion, we use the protean words "operator space", when there is no adjective "abstract", only in its classical meaning, that is, as a space consisting of genuine operators. As to axiomatically defined "abstract operator spaces", we use this term on equal footing with the more convenient for our aims, but much less widespread term *quantum spaces*; cf. [**141**, p. 1427] or [**110**] (and we shall try our best to avoid a misunderstanding). Recall that in classical functional analysis people say "normed space", not "abstract function space", and this is despite the well-known realization of the former as a subspace of $l_\infty(\cdot)$ or $C(\cdot)$. But again, it is our subjective preference, nothing more.

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Needless to say, however, it was not just the general trend to quantize "whatever is moving" that called into being the concepts and theorems presented in this text. To a much greater extent, they appeared as a result of the inner development of modern functional analysis. Of course, the most fundamental notion of the whole of functional analysis was and is a norm, given on a linear space. Nevertheless, in the last 20–25 years people working in various areas of this vast and ramified part of mathematics have come across a certain phenomenon. Studying this or that circle of questions, they noticed that behind a norm given on a space, algebra or module, an essentially richer structure of these objects is hidden. This is what many people call "operator space structure", for which we shall use the term *quantum norm.*

It was very important to realize that here and there, trying to understand the core of a problem under consideration, one must take into account not just a norm, but a quantum norm of relevant objects. Still, to a larger degree, this concerns the maps between objects: they must duly respect this additional structure. (As is usual in modern mathematics, "morphisms are more important than objects".) These maps are now the celebrated completely bounded operators. Historically they appeared (as special maps in the context of some classes of "genuine" operator spaces) much earlier than the abstract operator spaces were introduced.

The appeal to quantum norms and completely bounded operators bore fruit. Being duly reformulated, the problems became considerably clarified, and they often acquired a "right" way to be put. As a result, new approaches appeared, leading to big advances. Sometimes such a rethinking gave a key to a complete and spectacular solution of a known and long-standing open problem.

We now present some illustrations.

1. Let E_k and F_k, $k = 1, 2$, be operator spaces with their operator norms, and let $\varphi\colon E_1 \to F_1$, $\psi\colon E_2 \to F_2$ be bounded operators between these spaces. Naturally, interest arises in the tensor product $\varphi \otimes \psi$. This is the operator, acting between $E_1 \dot{\otimes} E_2$ and $F_1 \dot{\otimes} F_2$, the spatial tensor products of our operator spaces (cf. Section 1.7). But what a nuisance: despite that the operators φ and ψ are bounded, their tensor product is by no means "bound to be bounded" (see Section 3.1 for counterexamples). Now we know where the roots of this phenomenon are. We work within a large area in operator theory where the proper structure on an operator space is not the norm but the quantum norm. Accordingly, we must work not with just bounded operators between operator spaces but with completely bounded operators (cf. above) equipped with a special norm "$\| \cdot \|_{cb}$".

After we realize and accept this, things begin to behave perfectly. Namely, if our φ and ψ are *completely* bounded, then the same is true for $\varphi \otimes \psi$ and, moreover, we have $\|\varphi \otimes \psi\|_{cb} = \|\varphi\|_{cb} \|\psi\|_{cb}$. This is the de Cannière/Haagerup Theorem [**42**]; also see, e.g., Pisier [**140**, Prop. 2.1.1] and Theorem 3.4.2 of the present text.

2. We turn to a very natural and old question in the theory of Banach and operator algebras. Let A be a Banach algebra. What kind of necessary and sufficient condition must we impose on A in order to know that our algebra is actually an operator algebra in disguise? We mean by this that our A is topologically isomorphic to a (not necessarily self-adjoint) subalgebra of $\mathcal{B}(H)$ equipped with the operator norm.

In fact, such a kind of criterion was given by Varopoulos [**164**]. However, the formulation of his undoubtedly strong and non-trivial result seems to be not sufficiently transparent and not easy to verify (cf. also the criterion of Dixon [**45**]). Afterwards several people tried to find a transparent and "manageable" criterion. Most naturally, they desired to formulate it in terms of the bilinear operator of multiplication $\mathcal{M}\colon A \times A \to A$ or, equivalently, of the linearization $M\colon A \otimes A \to A$ of $\mathcal{M}$. Gradually the experience has shown that conditions, presented in "classical" terms of functional analysis, fail to give an answer. For example, one could ask whether there exists a reasonable tensor product norm on $A \otimes A$ such that the boundedness of M with respect to this norm means exactly that A can be realized as an operator algebra. However, one by one all the norms hitherto known on various types of classical functional-analytic tensor products were discarded (cf. Carne [**26**]).

Again, to get a sound theorem that would answer such a question, one should make A into a quantum algebra—that is, endow it with a quantum norm—and compare it with quantum operator algebras. (All operator spaces, in particular operator algebras, have the standard so-called concrete quantum norms; cf. Section 2.3.) Then the *Blecher Theorem* [**11**] came to assert that A is *completely* topologically isomorphic to an operator algebra if and only if the bilinear operator $\mathcal{M}$ is, in our terms, strongly completely bounded (see Definition 5.1.1). Equivalently, this means that the linear operator M is completely bounded with respect to the so-called Haagerup quantum norm on $A \otimes A$, probably the best known among the quantum norms on tensor products of quantum spaces. Even more, if A is unital, then we can speak about its completely isometric (and not just topological) isomorphism onto an operator algebra; this happens exactly when $\mathcal{M}$ satisfies the condition of the so-called strong complete contractiveness (cf. *idem*). Such a theorem was proved earlier by Blecher/Ruan/Sinclair [**16**].

Note that the Blecher Theorem has quite surprising consequences even within the realm of "classical" functional analysis. Take the algebra $\mathcal{N}(H)$ of nuclear (= trace class) operators on H with its proper, nuclear norm. Since the latter is essentially stronger than the operator norm, this algebra outwardly has nothing to do with what we call operator algebras. But Blecher/Le Merdy [**13**] observed that $\mathcal{N}(H)$ can be made a quantum algebra, satisfying the hypothesis of the Blecher Theorem. (Actually, the so-called maximal quantum norm fits; cf. Section 2.3.) Therefore $\mathcal{N}(H)$ is, up to a complete and hence up to a "classical" topological isomorphism, an operator algebra, and its norm, accordingly, is equivalent to the operator norm. It is true that this new guise of $\mathcal{N}(H)$ is no more self-adjoint and acts on another Hilbert space, but the aim is achieved.

3. Let A and B be two von Neumann algebras, and $A\overline{\otimes}B$ their von Neumann tensor product. By virtue of the famous Sakai Theorem (cf. [**152**]), the latter, being itself a von Neumann algebra, has a unique predual Banach space. One can feel that this predual, $(A\overline{\otimes}B)_*$, must be somehow expressed in terms of the preduals, A_* and B_*, of our tensor factors, but how? As a natural suggestion, can we obtain $(A\overline{\otimes}B)_*$ in the form $A_* \otimes_\alpha B_*$, where $\otimes_\alpha$ would be a reasonable kind of tensor product of Banach spaces?

And again, neither projective, nor injective, nor any other known "classical" tensor product fits. But if we turn to quantum tensor products, we find one, and indeed, a very good one. It is the so-called (completed) operator-projective tensor product, discovered independently and simultaneously by Effros/Ruan [**54**] and Blecher/Paulsen [**15**]. In our text this important tensor product is denoted by $\otimes^{op}$. Namely, and this is the *Effros/Ruan Theorem* [**53**], we have the identification $(A\overline{\otimes}B)_* = A_* \otimes^{op} B_*$ by means of an isometric isomorphism which is actually a completely isometric isomorphism with respect to proper quantum norms in both spaces.

4. What we present now is historically connected with the long-standing—several decades long—quest for a generalization of the Pontrjagin duality theory that would embrace all, and not only Abelian, locally compact groups. To speak more formally, we recall that, from the categorical point of view, the Pontrjagin Duality Theorem shows that the category **LCA** of locally compact Abelian groups is self-dual, and this self-duality is provided by a specific anti-equivalence functor $(\widehat{\cdot})\colon \mathbf{LCA} \to \mathbf{LCA}$. The problem was to extend this functor to an anti-equivalence

functor, acting on a larger category that would contain (as an only part, if necessary) the category of *all* locally compact groups.

Finally, a sample of such a large self-dual category was found. This is the category of the so-called Kac algebras, discovered independently by G. I. Kac/Vainerman and Enock/Schwartz (see [**61**] for a detailed exposition). Despite the existing complaints that the axioms of a Kac algebra, involving some advanced parts of the theory of von Neumann algebras, are too complicated, there is no doubt that it was a great achievement.[1] Our concern now is the action of the anti-equivalence functor, extending Pontrjagin's $(\widehat{\cdot})$ and acting on the category of Kac algebras. What does it do with the object identified with a locally compact group G or, equivalently, with its traditional group algebra $L_1(G)$? It was shown that this functor takes it to an object that can be identified with another kind of Banach algebra, connected with G, the so-called Fourier algebra $A(G)$. Such a class of group algebras, this time always commutative, was introduced much earlier by Eymard [**62**].

Thus, roughly speaking, one can consider Fourier algebras as a sort of dual objects to L_1-algebras. Not surprisingly, the interest in Fourier algebras sharply increased, and they became very fashionable objects of study. It was shown that many existing results about the behaviour of the traditional group algebras $L_1(G)$ have interesting and substantial counterparts for Fourier algebras.

However, one thing somehow disturbed harmony. We recall that one of the main concepts in homological theory of "algebras in analysis" is that of an amenable Banach algebra. According to the well-known *Johnson Theorem*, a group algebra $L_1(G)$ is amenable if and only if G is amenable in the classical group-theoretic sense; that is, it has an invariant mean. (See the original proof in [**89**] and another proof in [**77**].) One could suggest that a similar criterion holds for Fourier algebras. However, it is not the case: G can be a fairly amenable locally compact group, and at the same time $A(G)$ can fail to be an amenable Banach algebra (Johnson [**90**]; see also [**149**]).

Now it is Ruan who shows the way to restore harmony. For this aim, one should turn from the "classical" notion of amenability to its natural "quantum" version. Indeed, we notice that Fourier algebras, being predual spaces of operator spaces, have a kind of standard quantum norm (see [**60**] or Example 2.3.9 and Definition 8.2.1 below for the details). If we take into account that hitherto hidden structure and consider $A(G)$ as a quantum algebra, we get the desired counterpart of the Johnson Theorem, the *Ruan Theorem* [**147**]. Namely, the Fourier algebra of a locally compact group is amenable as a quantum algebra ("operator amenable" in terms of the cited paper) if and only if G is amenable in the group-theoretic sense.

For the development of the "quantum Banach homology" in the context of biprojective Fourier algebras, see Aristov [**4**], Wood [**176**], and in the context of more general Kac algebras, see Ruan/Xu [**148**].

[1] Later some larger categories, providing a generalization of the Pontrjagin duality theory, were suggested. This was done by Masuda/Nakagami [**108**] and afterwards by Kustermans/Vaes [**96, 97**]. The choice of the latter category, consisting of the so-called *locally compact quantum groups*, is apparently the most successful. Being defined in relatively simple terms, it contains, after relevant identifications, all Kac algebras and also some other important objects such as the "quantum $SU(2)$ group" of Woronowicz [**177**]. An elegant alternative approach to the notion of locally compact quantum group is developed by A. Van Daele; see `arXiv:math/0602212vl [math.OA]`.

5. We now present one more example from topological homology, this time from homology of operator algebras.

What are the best operator algebras, acting on a Hilbert space H? A possible answer is those that could be viewed as a "right" infinite-dimensional (or, equivalently, functional-analytic) analogue of the algebras described in the classical Wedderburn Theorem in 1905. In this connection we recall that the interest in possible infinite-dimensional versions of Wedderburn type theorems served as one of the main stimuli for von Neumann's introduction of the algebras now carrying his name (cf., e.g., [**154, 91**]).

These "best" algebras can be defined in several ways. We choose here the simple language of matrices. Let us say that an algebra $A \subseteq \mathcal{B}(H)$ is a Wedderburn algebra if, for some orthonormal basis in H, A consists of all operators that are given by diagonal block-matrices such that every block is a scalar block-matrix. In other words, an operator in A is represented by a block-matrix of the form

$$\begin{pmatrix} a & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & a & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \dots \\ 0 & 0 & \dots & a & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & b & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & b & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus the picture is the same as in the case of finite-dimensional self-adjoint operator algebras, with the only difference being that now the number of big blocks, their sizes as block-matrices, as well as the sizes of the matrices $a, b, c, \dots,$ can be of arbitrary cardinality.

It is obvious that a Wedderburn algebra must be a von Neumann algebra, and its center must be discrete (= isomorphic to $l_\infty(\cdot)$). But this condition is not sufficient: one must add that our algebra belongs to type I; see, e.g., [**162**], [**93**]. We recall that the very existence of relevant counterexamples, even in the case of a one-dimensional center, represents one of the major discoveries of 20-th century mathematics. We mean, of course, the celebrated continuous factors, the first constructed by Murrey/von Neumann in [**113**].

So, what kind of "outer" additional conditions, imposed on a von Neumann algebra, distinguish Wedderburn algebras? A venerable algebraic tradition suggests looking for such conditions in the realm of homology, especially among the suitable versions of the fundamental concept of projectivity. (Recall, for example, one of the best-known pure algebraic theorems in this area: a complex associative algebra has a Wedderburn structure if and only if all modules and bimodules over this algebra are projective; cf., e.g., [**132**].)

The first condition of that kind was tried in [**79**]; see also [**81**]. An operator algebra A (so far arbitrary), acting on H, was called *spatially projective*, SP for brevity, if its spatial module, that is, the very H with outer multiplication $a \cdot x := a(x)$; $a \in A$, $x \in H$, is a projective left Banach module over A (we still mean the "classical" context; see, e.g., [**77**]). Indeed, it was proved in [**79**] that an SP von Neumann algebra is bound to be Wedderburn. However, it turned out that in this way we get only part of the whole class of Wedderburn algebras. Namely, a Wedderburn algebra A is SP (looking ahead, we should now say "classically SP") if and only if it satisfies the additional and at first sight rather exotic condition of so-called essential finiteness. In the language of matrices, which we have already used, this means the following. Look at the matrix depicted above as well as at its "big" blocks with the same "small" matrix (in our picture, a, b, etc.) on the diagonal. Then, in each of these big blocks, at least one of two things (or both) must be fulfilled: either the matrix on its diagonal (say, a) is of a finite size or the number of these matrices on the diagonal (= the size of the respective scalar block-matrix) is finite; which condition is satisfied depends on the big block taken. Thus we see, for example, that $\mathcal{B}(H)$ itself is SP (this is, of course, trivial), whereas the same algebra presented in the standard form (that is, $\mathcal{B}(H) \mathbin{\dot\otimes} \mathbf{1}_H \subset \mathcal{B}(H \mathbin{\dot\otimes} H)$) is not SP.

The reader has already guessed that, to achieve our aim completely, we must turn to the quantum version of projectivity. The notion of an operator projective (we say "quantum projective") module was introduced by Paulsen in [**124**]; see also [**125**]. Accordingly, one can say that a given operator algebra is *quantum SP* if its spatial module is quantum projective. After this we can close the question: *a von Neumann algebra is Wedderburn if and only if it is quantum SP* [**81**].

6. Last but not least is Pisier's negative solution of the Halmos Similarity Problem.

Let T be a contractive operator on H, and p a polynomial. Then we have the famous von Neumann inequality: $\|p(T)\| \le \max\{|p(z)|\colon |z| \le 1\}$. As an easy corollary, every operator T which is similar (= topologically equivalent) to a contractive operator, is polynomially bounded. This means that there is a constant C such that, for all polynomials p, we have $\|p(T)\| \le C \max\{|p(z)|\colon |z| \le 1\}$. Long ago Halmos asked, whether the converse is true. In other words, is every polynomially bounded operator similar to a contraction?

The history (and prehistory) of this problem is very interesting in itself, being full of rather dramatic turns. The reader can enjoy it in [**139**]; see also [**40**]. What is essential for us is that the Halmos Problem is typical for a wide variety of problems that, as we know from the solver himself,

> *can be formulated as asking whether "boundedness" implies "complete boundedness" for linear maps satisfying certain additional algebraic conditions.*[2] [**139**, Foreword]

Such a reduction of the Halmos Problem to a question of the above-mentioned type, which happened to be an important step to its solution, was done by Paulsen [**121**]. Namely, one should consider the algebra $\mathbb{C}[t]$ of polynomials of one variable and the algebra $\mathcal{B}(H)$, both equipped with some standard quantum norms. Then, according to Paulsen, the Halmos Problem is equivalent to the following question: Is every bounded homomorphism from $\mathbb{C}[t]$ into $\mathcal{B}(H)$ automatically completely bounded?

[2] Actually, Pisier writes about three outwardly disconnected "similarity problems": that of Halmos and two others, the first concerning representations of groups and the second dealing with representations of C^*-algebras. But we do not discuss these two others here: our list of examples is already rather long.

There was a moment (after the appearance of the preprint of [**1**]) when it seemed that such a question was near to being solved in the positive. But eventually Pisier [**138**] managed to find an operator T such that the homomorphism $\mathbb{C}[t] \to \mathcal{B}(H)\colon p \mapsto p(T)$ turned out to be "bad". Shortly afterwards his proof was considerably modified in [**41**]. Again, see the cited papers for the details.

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No doubt, by this series of examples we convinced our reader that mankind cannot survive without quantum norms and completely bounded operators.

But what is this quantum norm?

The great majority of mathematicians, writing on this subject, take a given linear space E and consider simultaneously the matrix spaces $M_n(E)$ of all sizes with entries from E. Then, again simultaneously, they supply each of these matrix spaces with its own norm, say $\|\cdot\|_n$. If this is, in a sense, well done, the resulting sequence of norms satisfies some properly chosen conditions, the so-called Ruan axioms; notably, it behaves well under the operation of taking direct sums of matrices. In this case this sequence $\|\cdot\|_n$; $n = 1, 2, \dots$ of norms on these matrix spaces of different sizes is called a quantum norm or, more often, an operator space structure. One can learn the subject in the framework of the matrix approach using carefully written monographs [**60, 126, 14**] of the founding fathers of the theory.

However, there exists another way to "quantize" a norm. Instead of the sequence of matrix spaces, one may consider a single space, consisting, roughly speaking, of vectors from the initial linear space E but equipped with coefficients taken from some good operator algebra. We mean what was called above the amplification of E. Now it is a (single) norm on the amplification that must satisfy reasonable conditions, that is, a version of the just-mentioned Ruan axioms for matrix spaces.

The very fact that both approaches, matricial (= coordinate) and "operator" (= non-coordinate), being formally different, give essentially equivalent results, is known. To begin with, this is clearly indicated in the seminal book by Pisier [**140**]. It seems to us that in his book, as a whole, the matricial approach prevails, but the author demonstrates the virtues of the non-matricial approach in a number of basic questions (see, e.g., *idem*, p. 40). Note that Pisier's "good operator algebra" is $\mathcal{K}(l_2)$, which slightly differs from what we shall use. Besides, the very existence and some advantages of the second approach were well realized by Barry Johnson, as one may judge from his unpublished notes. The same fact is reflected in the form of some theorems on the equivalence of various categories [**168, 114**]. Finally, no doubt, it was on the minds of the authors of fairly advanced results in representation theory of bimodules over operator algebras and tensor products of these bimodules; see, for example, [**106, 142, 2**].

Nevertheless, it seems that there was no systematic exposition of quantum functional analysis, presenting the main notions and results of the theory exclusively in the framework of the non-coordinate approach, without an appeal to matrix spaces.

Does one need such a presentation? Of course, the choice between the two indicated approaches is a matter of taste: one person prefers to work with tensor products of linear operators, another with Kronecker products of matrices, and "both are right". However, the present author does believe that, as a whole, the non-coordinate approach gives a better picture of the subject, not only more elegant but also more transparent and instructive. The only thing required from the reader is the acquaintance with some basic facts concerning normed modules and, on several occasions, their projective tensor products (cf. Sections 6.2 and 9.0). Anyhow, for modern functional analysts, this is a part of their bag of tricks.

Take, for example, one of the cornerstones of the theory, the Arveson–Wittstock Extension Theorem. From the non-coordinate point of view, it is essentially an extension theorem for morphisms of bimodules over a standard operator algebra, say $\mathcal{B}(\cdot)$. Even more: it is a rather straightforward corollary of an apparently simpler observation, concerning certain one-sided Hilbert modules over $\mathcal{B}(\cdot)$. We mean their homological property called extreme flatness, namely, the preservation of isometric morphisms under module tensor products (see Theorem 9.3.6 below).

But especially the virtues of the non-matricial approach reveal themselves in questions that are apparently "non-coordinate" in their essence. First of all, we mean quantum tensor products, one of the most important topics in the whole area. It seems that it is the non-coordinate language that provides a more complete understanding of the essence of both quantum versions of the classical (Grothendieck) projective tensor product of normed spaces. These are usually called the Haagerup and operator-projective tensor products. Their explicit constructions sound simpler and more natural since we clearly see that both tensor products mentioned are actually quotient spaces of certain "genuine" (= classical) projective tensor products. Moreover, the Haagerup tensor product is itself a "genuine" projective tensor product, however not of just normed spaces but of some normed modules (see Theorem 7.1.11 below). This seems to be a valuable addition to its better understanding.

Among other occasions where the non-matricial approach gives a more transparent picture of what is happening, we would mention the proof of the injective property of the Haagerup tensor product, some aspects of the duality theory (including the very definition of the quantum dual space), the commendable behaviour, under some basic constructions, of the column and the row Hilbertian spaces.

But again, all this is no more than our subjective opinion. We just believe that a systematic non-matricial presentation of the subject, at least as a complement to the already existing "matricial" presentations, could be useful.

Finally, let us make a remark of a more technical character. To present the area in the non-coordinate way, one must choose, as a first preparatory step, some space in the capacity of the "main" or "canonical" Hilbert space in the whole exposition. This can be l_2 (cf. [**140**]) or, as we prefer, just an arbitrarily chosen but fixed "forever", infinite-dimensional separable Hilbert space L. Then, as a more important step, one must distinguish, among several natural candidates with their own advantages and disadvantages, two operator algebras acting on this L. The role of the first algebra is that it "amplifies" a given linear space, providing operator coefficients to its vectors, instead of scalar coefficients. The other algebra (a priori this can be the same or a larger one) serves as a basic algebra with respect to bimodules, naturally arising after such an amplification. Gradually, we come across a pair of algebras that seems to provide a comparatively smooth and transparent presentation. It consists of $\mathcal{F}(L)$, the algebra of bounded finite rank operators on L, in the capacity of the amplifying algebra, and $\mathcal{B}(L)$, the algebra of all bounded operators on L, as a basic algebra of bimodules. The main virtue of $\mathcal{F}(L)$ is that it has the guise $L \otimes L^{cc}$, the algebraic tensor product of L with its complex conjugate space; this is why it is so convenient to work with it. (Yes, it is not complete, but, as experience shows, this circumstance does not create any harm.) And by taking $\mathcal{B}(L)$ as a basic algebra, we get the opportunity to have in store, as outer multiplication factors, many operators that turn out to be very useful. Their supply includes, in particular, isometries and coisometries (= quotients) acting on our L.

In conclusion, we would like to stress that the subject of our book is what could be called *linear* quantum functional analysis. We mean the initial and apparently most developed part of quantum functional analysis, dealing with quantum spaces still without any additional algebraic structure. But, similar to what we see in "classical" functional analysis, there is a younger chapter, dedicated to algebras and modules. It has its own distinct face, best represented by its own principal theorems (cf. Chapter 0 below), like the theorems of Blecher and Blecher/Ruan/Sinclair we have already mentioned or, say, the representation theorem for "quantized" modules; the latter was essentially proved by Christensen/Effros/Sinclair [**28**] as early as in 1988. A rich material, concerning these representation theorems as well as some advanced topics, is contained in the monograph [**14**]; see also the references therein.

Besides, together with the investigation of algebras, endowed with a quantum norm, interest in homological properties of such algebras inevitably arose. The first sign was the paper [**28**] just cited. Now we have a long and constantly increasing list of publications, entirely or partially dedicated to various aspects of this "quantum homology". Apart from the already cited papers [**147, 4, 176, 148, 124, 125, 81**], it includes [**156, 82, 31, 128, 150, 3, 167**] and many other items.

But we leave these further parts of quantum functional analysis, dealing with algebras, modules and homology, outside the scope of our presentation.

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We assume that our reader is familiar with standard university courses of functional analysis and algebra, for example, with the contents of what is presented by ordinary print in [**83**], together with the required background. Apart from this, the reader is supposed to know what a module and a bimodule over an algebra are, and what a (bi)module morphism is (see, e.g., the first chapter in [**38**]). As to basic facts on C^*-algebras, the way they are given, say, in the first three chapters of [**112**], is very desirable for deeper understanding of the subject. But, as a matter of fact, we come across general C^*-algebras in very few places of the book, and when this happens, the reader can assume without much loss that he is dealing with the algebra of all operators on a Hilbert space. Other preparatory things that we need will be explained in the text.

The symbol $\Longleftrightarrow$ means "if and only if". The combination $:=$ means "equality by definition".

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and notation that were gratefully accepted. Finally, their recommendations led to the enrichment of the manuscript with a number of interesting examples and profound results. This valuable material had remained outside the framework of the manuscript before.

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