CHAPTER 0

Three basic definitions and three principal theorems

Before getting to the detailed presentation of the theory, we would like to distinguish a few of its most important concepts and results, forming the very core of the subject. We believe that such a preliminary knowledge helps our reader in the real understanding of the proportions of the building he/she enters. Fortunately, the main definitions and the formulations of the main theorems are sufficiently simple and transparent. They can be presented, with all proper accuracy, without taking much time and space.

Every[1] self-respecting mathematical science deals with a few special categories that are typical for the area in question. As to quantum functional analysis, its most important categories are what will be denoted by **QNor** and $\mathbf{QNor}_1$. Accordingly, the most important definitions in the area are those concerning objects and morphisms of these particular categories.

The objects of both categories are the same: the so-called abstract operator spaces. To define them, we need, first of all, a fixed separable infinite-dimensional Hilbert space. So *we take such a space, denote it by L and fix it throughout the whole text.* (It seems to be more convenient to make an "abstract" choice, and not to be tied, say, to l_2 or $L^2(\cdot)$.) In what follows, the symbol $\mathcal{B}(\cdot)$ denotes the algebra of all bounded operators on a given normed space, equipped with the operator norm. For brevity, we denote $\mathcal{B}(L)$ just by $\mathcal{B}$, and the two-sided ideal in $\mathcal{B}$ consisting of all bounded finite rank operators, by $\mathcal{F}$.

Let E be a linear space. We take the algebraic tensor product $\mathcal{F} \otimes E$ and briefly denote it by $\mathcal{F}E$. We call this space the *amplification of E*. This is, speaking informally, "the space of formal linear combinations of vectors in E with operator coefficients from $\mathcal{F}$" (cf. the Introduction). Similarly, an elementary tensor $a \otimes x$; $a \in \mathcal{F}$, $x \in E$ will be denoted just by ax.

Since the left tensor factor in $\mathcal{F}E$, being a two-sided ideal in $\mathcal{B}$, is a bimodule over the latter algebra, the space $\mathcal{F}E$ is a $\mathcal{B}$-bimodule as well. Accordingly, outer multiplications are well defined by the equalities $a \cdot bx = (ab)x$ and $bx \cdot a = (ba)x$; $a \in \mathcal{B}$, $b \in \mathcal{F}$, $x \in E$. This simple observation is very important through all the theory that follows.

Definition 0.1 (cf. 2.2.3[2]). A *quantum norm on E* is a norm on $\mathcal{F}E$, satisfying the following two conditions ("Ruan's axioms"):

[1] Well, almost every...

[2] We put in parenthesis the numbers, under which definitions and statements appear in the subsequent chapters.

(RI) For every $a \in \mathcal{B}$ and $u \in \mathcal{F}E$, we have

$$\|a \cdot u\|, \|u \cdot a\| \leq \|a\|\|u\|.$$

(RII) If, for $u, v \in \mathcal{F}E$, there exist projections (i.e., self-adjoint idempotents) $P, Q \in \mathcal{B}$ such that $P \cdot u \cdot P = u$, $Q \cdot v \cdot Q = v$ and $PQ = 0$, then we have

$$\|u + v\| = \max\{\|u\|, \|v\|\}.$$

An *abstract operator space* is a linear space, equipped with a quantum norm (or, if we want to play a precisian, it is a pair $(E, \|\cdot\|)$, consisting of a linear space and a quantum norm on it). Rather often, when it seems to be convenient, we use the term *quantum space* instead of "abstract operator space". We emphasize that both terms have absolutely the same meaning.

We note that a quantum norm on E is a (usual) norm not on E itself, but on the "larger" space $\mathcal{F}E$.

What makes our two categories different, is the choice of their morphisms. The morphisms of **QNor** are the so-called completely bounded operators, and the morphisms of $\mathbf{QNor}_1$ form the lesser class of the so-called completely contractive operators. To define them, suppose we are given two linear spaces E and F and a linear operator $\varphi\colon E \to F$. Consider the operator $\mathbf{1}_{\mathcal{F}} \otimes \varphi\colon \mathcal{F}E \to \mathcal{F}F$, well defined by $ax \mapsto a\varphi(x)$; $x \in E$, $a \in \mathcal{F}$. We denote it, for brevity, by φ_∞ and call it the *amplification of the operator* φ.

Definition 0.2 (cf. 3.1.1). A *completely bounded operator* between two abstract operator spaces E and F is a linear operator $\varphi\colon E \to F$ such that its amplification $\varphi_\infty\colon \mathcal{F}E \to \mathcal{F}F$ is a bounded operator with respect to the relevant quantum norms. The operator norm of φ_∞ is called the *completely bounded norm* of φ and is denoted by $\|\varphi\|_{cb}$.

Definition 0.3 (cf. *idem*). A *completely contractive operator* is a completely bounded operator φ with $\|\varphi\|_{cb} \leq 1$, or, in other words, an operator φ such that its amplification φ_∞ is a contractive operator.

Thus both categories **QNor** and $\mathbf{QNor}_1$ are defined. In particular, we can speak about their isomorphisms, specified as *complete topological isomorphisms* and *complete isometric isomorphisms*, respectively. Obviously, a complete topological isomorphism is exactly an operator, the amplification of which is (just) a topological isomorphism of normed spaces, and the same is true if we replace the word "topological" by "isometric". Thus we have two kinds of identifications of our objects: more tolerant and more rigid, the latter taking into account the exact values of completely bounded norms. The second kind of identifications of quantum spaces provides actually the most rigid of their possible identifications. It is this identification that participates in the first of the promised principal results.

To formulate it, let us consider what appears at first sight as just a particular class of examples of abstract operator spaces. Suppose that our given linear space E is a subspace of the space $\mathcal{B}(H)$ for some Hilbert space H. Then we can identify $\mathcal{F}E$ with a subspace of $\mathcal{B}(L \mathbin{\dot\otimes} H)$, where $L \mathbin{\dot\otimes} H$ is the Hilbert tensor product of L and H: the respective embedding takes an elementary tensor $a \otimes b$; $a \in \mathcal{F}$, $b \in \mathcal{B}(H)$ to the operator $a \mathbin{\dot\otimes} b$, the Hilbert tensor product of the operators a and b. Thus $\mathcal{F}E$ becomes a normed space with respect to the norm, induced by the operator norm on $\mathcal{B}(L \dot{\otimes} H)$. It is an easy matter to check that in this way we give rise to a quantum norm on E.

Thus subspaces of $\mathcal{B}(H)$ for various Hilbert spaces H automatically become what we call abstract operator (= quantum) spaces. We shall call these objects *concrete operator spaces*, and refer to their quantum norm as a *concrete quantum norm.*

The meaning of the first principal theorem is that, as a matter of fact, the presented example provides all the existing abstract operator spaces.

THEOREM 0.1 (cf. 10.1.4 (*Ruan Representation Theorem*, proved in [**146**])[3]). *Every quantum (= abstract operator) space coincides, up to a complete isometric isomorphism, with some concrete operator space.*

Thus we have a maxim "there are no other quantum spaces than the concrete operator spaces". A striking similarity to the message of the Gelfand/Naimark Theorem, claiming that "there are no other C^*-algebras than the concrete C^*-algebras". The very word "concrete" in Theorem 0.1 is apparently inherited from the theory of operator algebras, where "concrete C^*-algebra" means "self-adjoint uniformly closed operator algebra". Such an adopting is by no means surprising. The roles of both theorems, that of Ruan in the younger area and that of Gelfand/Naimark in the older one, closely resemble each other. Indeed, both of them are distinguished representatives of the family of the so-called realization theorems in the mathematical science. In that capacity they provide a binocular view on objects of a relevant theory and therefore have a great practical, as well as aesthetic, value. We mean that, working in the respective area, one can either use the axiomatic, space-free approach to an object in question or treat the latter as a set, consisting of operators. This can be done alternatively, and every time one can choose what is more convenient at the moment. In our book most of samples of such a synthetic approach are presented in Section 10.2 (cf. also the very end of this chapter).

In this connection, we want to distinguish, as Theorem 0.1a below, the statement, which is actually an equivalent form of the Ruan Theorem. (It is not hard to show this equivalence, and this will be done on the comparatively early stage of our presentation; see Theorem 3.2.13.) As we shall see, in quite a few important cases this assertion can be used independently, without applying to operator representation of given "abstract" spaces.

THEOREM 0.1a (cf. 10.1.5). *Let E be an arbitrary abstract operator space. Then, for every $u \in \mathcal{F}E$, we have*

$$\|u\| = \sup\{\|\varphi(u)\|\},$$

where the supremum is taken over all completely contractive finite rank operators $\varphi\colon E \to \mathcal{F}$ or (as an immediate corollary) over all possible completely contractive operators $\varphi\colon E \to \mathcal{F}$. Here $\mathcal{F} := \mathcal{F}(L)$ is considered as a concrete quantum space.

The property of abstract operator spaces, expressed in this theorem, will be called *attainability.*

As a matter of fact, one could replace here supremum by maximum, and also, if he/she wishes, replace $\mathcal{F}$ by the larger algebra $\mathcal{B}$ or, in the opposite direction, by

[3]Here and thereafter, giving a reference, we indicate the paper which contained, up to our knowledge, the original formulation and proof of the relevant result.

the whole family of algebras $\mathcal{B}(\mathbb{C}^n)$; $n = 1, 2, \ldots$. But for most applications it is sufficient to use this theorem in the presented formulation.

Both Theorems 0.1 and 0.1a will be obtained as rather straightforward corollaries of a certain property of abstract operator spaces, expressed in Theorem 10.1.6 below. But the proof of this property is by no means an easy matter. It requires a considerable preparation as well as rather subtle subsequent argument.

We proceed to the second principal theorem. This one plays the role of a "quantum" version of the classical Hahn/Banach Theorem on the extension of bounded functionals. To formulate it, suppose that F is a subspace of a linear space E, and the latter is endowed with a quantum norm. Then it is easy to verify that F has itself a quantum norm defined by $\|u\| := \|i_\infty(u)\|$, where $i\colon F \to E$ is the natural embedding. In this situation we say that the respective quantum (= abstract operator) space F is a *quantum subspace* of the quantum space E.

THEOREM 0.2 (cf. 9.4.3 (the *Arveson/Wittstock Extension Theorem*, proved in [**170**] after an important preparatory step in [**5**])). *Let F be a quantum subspace of a quantum space E, and H be an arbitrary Hilbert space. Then every completely bounded operator φ from F into $\mathcal{B}(H)$, the latter being considered as a concrete operator space, can be extended to a completely bounded operator $\psi\colon E \to \mathcal{B}(H)$ such that $\|\psi\|_{cb} = \|\varphi\|_{cb}$.*

Note that the Hahn/Banach Theorem is in fact but a particular case of the Arveson/Wittstock Theorem where H and hence $\mathcal{B}(H)$ is $\mathbb{C}$ (cf. Remark 9.4.6 given hereinafter).

The third principal result outwardly deals with "very concrete" operator spaces, but, as we shall see very soon, actually concerns all abstract operator spaces.

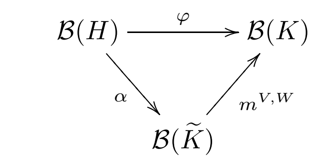
Let H, K be Hilbert spaces, and $\mathcal{B}(H), \mathcal{B}(K)$ the respective concrete operator spaces. To begin with, we distinguish two classes of operators between these spaces that are automatically completely bounded.

The first class consists of all *-homomorphisms from $\mathcal{B}(H)$ into $\mathcal{B}(K)$. To indicate the second class, take an arbitrary pair of bounded operators $S\colon H \to K$ and $T\colon K \to H$. They give rise to the map $m^{S,T}\colon \mathcal{B}(H) \to \mathcal{B}(K)$ taking an operator a on H to the operator SaT on K. These maps, the so-called *double multiplication operators*, form another class of automatically completely bounded operators that we need.

Note that in both cases it is an easy matter to verify the complete boundedness. Indeed, the amplification of a *-homomorphism is itself a *-homomorphism between *-algebras that are, in a reasonable sense, "almost C^*-algebras", and the desired property follows from the automatic boundedness of *-homomorphisms between "genuine" C^*-algebras. At the same time, the amplification of $m^{S,T}$ is the restriction of another double multiplication operator, acting between $\mathcal{B}(L \dot{\otimes} H)$ and $\mathcal{B}(L \dot{\otimes} K)$, and therefore it is also bounded. See Theorems 3.2.10 and 3.2.11 for details of the respective proofs.

THEOREM 0.3 (cf. 11.3.3 (the *Decomposition Theorem* [**121**], first explicitly proved in [**121, 170, 171, 72**] after an important preparatory step in [**160**])). *Let $\varphi\colon \mathcal{B}(H) \to \mathcal{B}(K)$ be a completely bounded operator. Then there exist a Hilbert space $\widetilde{K}$, a *-homomorphism $\alpha\colon \mathcal{B}(H) \to \mathcal{B}(\widetilde{K})$ and bounded operators $V\colon \widetilde{K} \to K$*

and $W: K \to \widetilde{K}$ *such that the diagram*



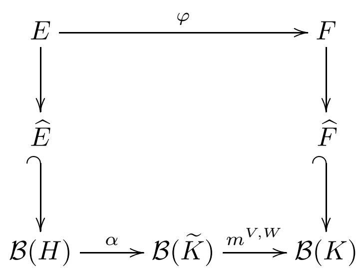
is commutative.

Thus an arbitrary completely bounded operator between $\mathcal{B}(H)$ and $\mathcal{B}(K)$ can be represented as the composition of a *-homomorphism and a double multiplication operator.

This theorem may be considered as a far going generalization of the structure theorem, describing bounded functionals on C^*-algebras and connected with the names of Gelfand, Naimark and Segal (cf. Section 1.8).

Now observe that, combining all three principal results, one can easily obtain a powerful theorem about the nature of completely bounded operators between arbitrary abstract operator spaces as well as about the nature of the latter objects themselves. In the spirit of [**140**, p. 23], one could call it "the fundamental representation/factorization/extension theorem".

THEOREM 0.3a. *Let* E *and* F *be arbitrary abstract operator spaces, and* $\varphi: E \to F$ *an arbitrary completely bounded operator. Then there exist Hilbert spaces* H, K *and* $\widetilde{K}$*, concrete operator spaces* $\widehat{E} \subseteq \mathcal{B}(H)$ *and* $\widehat{F} \subseteq \mathcal{B}(K)$*, a* **-homomorphism* $\alpha: \mathcal{B}(H) \to \mathcal{B}(\widetilde{K})$ *and, finally, bounded operators* $V: \widetilde{K} \to K$ *and* $W: K \to \widetilde{K}$ *such that there is a commutative diagram*



where the upper vertical arrows depict complete isometric isomorphisms, and the lower vertical arrows depict natural embeddings.

Thus we see that an arbitrary completely bounded operator can be represented, up to identifications of its domain and range with some concrete quantum spaces, as a birestriction of the composition of a *-homomorphism and a double multiplication operator.

Indeed, Theorem 0.1 provides Hilbert spaces H, K and isometric isomorphisms of E and F onto some concrete operator spaces $\widehat{E} \subseteq \mathcal{B}(H)$ and $\widehat{F} \subseteq \mathcal{B}(K)$, respectively. Then, identifying F with $\widetilde{F}$, we can speak about the coextension $\widetilde{\varphi}: E \to \mathcal{B}(K)$ of the operator φ. After this, identifying E with $\widetilde{E}$, we set in Theorem 0.2, instead of E, F and φ, our $\mathcal{B}(H)$, E and $\widetilde{\varphi}$ in the present case, respectively. The theorem immediately provides an extension $\widehat{\varphi}: \mathcal{B}(H) \to \mathcal{B}(K)$ of $\widetilde{\varphi}$ such that $\|\widehat{\varphi}\| = \|\widetilde{\varphi}\|$ and hence $\|\widehat{\varphi}\| = \|\varphi\|$. It remains to apply Theorem 0.3 to the operator $\widehat{\varphi}$.

Needless to say, each of Theorems 0.1–0.3 is contained, as a particular case, in this united theorem.

We have seen that the principal results of the theory were formulated in rather elementary terms. But their proofs, at least those we have at the moment, are rather long, and they are by no means elementary. To gain their complete understanding, the reader must make rather considerable efforts. The proofs of Theorems 0.1 and 0.3, despite that the original argument was later simplified, still use heavy techniques needed in the work with positive functionals and operators. Some of their fragments are rather tricky. As to the proof of Theorem 0.2, it is much less technical, but it requires some knowledge of module tensor products. This preparatory staff, concentrated in Section 9.0, is a sheer banality for an algebraist, but might be a novelty for, say, a specialist in operator theory.

At the same time, plenty of a valuable material can be presented independently of these advanced theorems. Mostly it is because the relevant construction and facts can be fairly expounded in the framework of the axiomatic approach. This is true for greater parts of such important topics as quantum tensor products and the duality theory. The same is true for instructive and illuminative examples, "abstract" and "concrete" alike, that are indispensable for the real understanding of the theory: maximal and minimal quantizations, column and row Hilbertian spaces, the self-dual quantum space of Pisier, etc.

For these reasons we tried to postpone the whole story of the principal theorems, including their complete proofs, as much as possible. We do not speak about them, save some informal remarks, so to say, promises for the future, up to the moment when the usage of these theorems becomes really necessary for the subsequent development of the theory. But—may we repeat it again?—we believe that the knowledge, from the very beginning, of what they tell us will orientate our reader in a right way and essentially help him to master the subject.