Coda: Transonic Airfoils and SLE

We close by describing two rather unusual applications of complex variables. The details are beyond the scope of this book, but the ideas involved definitely deserve mention.

The first area of application is fluid dynamics. It was observed already in the nineteenth century that the equations describing the incompressibility and irrotationality of fluids are just the Cauchy-Riemann equations for the velocity components in two-dimensional flow. Since low velocity flow is nearly incompressible, this made it possible to use analytic functions (more specifically, the theory of conformal mapping) to describe such flows around airfoils and to determine lift and drag. However, for high speed flows, which are compressible, this approach is not available.

In high speed flows over airfoils, the flow becomes supersonic over parts of the airfoil. This leads to the formation of shock waves, an undesirable effect since shocks increase drag. Although Cathleen Morawetz proved mathematically that, in general, shock waves occur in partially supersonic flows [**M1**], [**M2**], this did not rule out the existence of special airfoils for which shockless flows are possible. In fact, Paul Garabedian and his student David Korn developed a hodograph method based on complex characteristics that enabled them to calculate supercritical wing sections free of shocks at a specified speed and angle of attack [**K**], [**GK1**]. However, the extensive trial and error involved in the selection of parameters defining the flow rendered this method impractical. After the preliminary results of [**BGK**], a completely satisfactory solution of the problem was obtained by Garabedian and Korn in [**GK2**]. They solve the partial differential equations of two-dimensional inviscid gas dynamics by analytic continuation into the domain of two independent complex characteristic coordinates. After mapping the domain of integration conformally onto the unit disk in the plane of one of these coordinates, they formulate a boundary value problem on that disk for the stream function which is well-posed even in the case of transonic flow. This enables them to give a procedure for calculating an airfoil on which the speed is prescribed as a function of arclength, leading to an exact solution of the problem in the case of subsonic flow and, in the transonic case, generally to a shockless flow which assumes the assigned subsonic values of the speed and approximates the given supersonic values. Truly a tour de force of applied complex analysis.

The second area of application is statistical mechanics, and the mathematics has its origin in Charles Loewner's study of univalent (i.e., one-to-one) analytic functions defined on the unit disk. Based on certain known extremal properties of the function

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n,$$

Bieberbach conjectured that for any univalent analytic function on the unit disk satisfying the normalization

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

the coefficient inequality $|a_n| \le n$ holds, with equality only for $k(z)$ and its rotates $k(\alpha z)/\alpha$, $|\alpha| = 1$. For $n = 2$, this can be demonstrated easily, but for $n \ge 3$ it remained a challenge.

Loewner was able to prove that $|a_3| \le 3$ by embedding the function f into a one-parameter family of mappings, constructed as follows. Suppose f maps the unit disk onto the exterior of a curve connecting some point p to ∞. Moving the point p along the curve gives a one-parameter family of exterior domains; denote by $f(z;p)$ the (normalized) analytic function mapping the open unit disk onto the exterior of the curve. Loewner [**Lo**] derived a differential equation for f as a function of p and used it successfully to estimate a_3. Loewner's method found significant applications to several other problems in the theory of univalent functions [**D**, pp. 95-117], but efforts to apply it to higher coefficients met with little success; and for the next 60 years, attention was focused on a variety of other approaches to the problem. However, when the Bieberbach Conjecture was finally proved (by Louis de Branges [**Br**]), it was via Loewner's approach; cf. [**FP**].

More recently, Oded Schramm [**S**] discovered a conformally invariant stochastic process, obtained by solving Loewner's equation with Brownian motion as input, which describes scaling limits in statistical mechanics. SLE, the stochastic Loewner evolution (or Schramm-Loewner evolution), was used subsequently to solve many two-dimensional problems in statistical mechanics. To cite but a single example, Lawler, Schramm and Werner [**LSW**] used it to prove Mandelbrot's conjecture that the dimension of the planar Brownian frontier (i.e., the boundary of the infinite connected component of the complement of a planar Brownian path) is 4/3. SLE has led to a major leap in our understanding of the random fractal geometry of such two-dimensional systems as critical percolation and critical Ising models [**Sm1**], [**Sm2**]. It also has close connections with two-dimensional conformal field theory, two-dimensional quantum gravity, and random matrix theory. Surely this work, which figures prominently in two recent Fields Medal citations,[1] is a most striking example of an idea which, originating in the purest mathematics, has turned out to be instrumental in theoretical physics.

[1]To Wendelin Werner (2006) "For his contributions to the development of stochastic Loewner evolution, the geometry of two-dimensional Brownian motion, and conformal field theory" and to Stanislav Smirnov (2010) "For the proof of conformal invariance of percolation and the planar Ising model in statistical physics." Moreover, according to the obituary for Oded Schramm published in the *New York Times* on September 10, 2008, "If Dr. Schramm had been born three weeks and a day later, he would almost certainly have been one of the winners of the Fields Medal ... in 2002."

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