Preface

Regularisation techniques, implemented in quantum field theory, number theory, and geometry to make sense of divergent integrals, discrete sums, or traces, might seem very arbitrary and uncanonical at first glance. They nevertheless conceal *canonical* concepts, namely canonical integrals, sums, and traces, which we want to bring to the forefront in these lectures.

Cut-off and dimensional regularisation are prototypes of regularisation techniques used in quantum field theory.[1] But we also have in mind Riesz[2] and Hadamard finite parts[3] methods used in number theory.[4] Regularisation techniques also comprise zeta regularisation used in physics in the form of zeta determinants to compute effective actions,[5] or in geometry[6] and particularly in the context of infinite dimensional manifolds[7] and index theory as a substitute for the equivalent heat-kernel methods.[8]

Regularised integrals, discrete sums, and traces[9] obtained by means of a regularisation procedure present many discrepancies responsible for various anomalies.[10] In contrast, the underlying canonical integrals, discrete sums, and traces are well behaved. Canonical integrals are indeed covariant, translation invariant, and obey

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[1] Just to quote a few books amongst the vast literature on the subject, see e.g. [**Col**], [**CMa**], [**D**], [**Sm1**], [**Sm2**] as well as more specific references in the context of renormalisation, such as [**Et**], [**CMa**], [**He**], [**HV**], [**Sp**], [**Zi**].

[2] Also called modified dimensional regularisation.

[3] Which amounts to cut-off regularisation.

[4] See e.g. [**Ca**] for an introductory presentation.

[5] Starting with pioneering work by Hawkins [**Haw**], see other applications in [**El**], [**EORZ**], and further developments in string theory, see e.g. [**D**] and [**AJPS**] for a mathematical presentation.

[6] With the work of Ray and Singer [**RaSi**] on analytic torsion where the zeta determinant was first introduced in mathematics.

[7] E.g. for the geometry of loop groups, see [**Fr**].

[8] Starting with pioneering work by Atiyah and Singer [**APS1, APS2, APS3**] and later by Quillen [**Q1**] and Bismut and Freed [**BF**], see also more recent work by Scott [**Sc1**] in the context of the family index theorem.

[9] We use the terminology regularised when a physicist might call this a **renormalised** integral, discrete sum, or trace since it is the result of a regularisation procedure combined with a subtraction scheme used to extract a finite part. We choose not to use of the word "renormalisation" because in physics this concept involves much more than merely evaluating divergent integrals, divergent discrete sums in one variable, or divergent traces that we are concerned with here.

[10] For a treatment of anomalies in physics (see e.g. [**D**] and [**N**] for a mathematical presentation) from the point of view of discrepancies also called trace anomalies, see e.g. [**CDP**] and [**Mi**].

Stokes' property; canonical discrete sums are $\mathbb{Z}^d$-translation invariant; and canonical traces vanish on commutators. So all would be well were these canonical integrals, discrete sums, and traces defined on a class of functions and operators appropriate for applications; unfortunately most functions and operators arising in most number theory, geometry, or physics do not fall in the class on which the canonical functionals have the desired invariance properties. However, one can approximate any of the functions or operators under consideration by a family of functions or operators in the class on which canonical functionals naturally live; this fact is the basic principle which underlies many regularisation procedures. To make this statement more precise, we need to specify the type of functions and operators one comes across.

Since we focus on *ultraviolet divergences*, namely divergences for large values of the momentum, it seems reasonable to pick out a specific class of functions whose controllable behaviour in the large will enable us to integrate and sum them up using appropriate regularisation methods. It turns out that functions of the form $\sigma_s(\xi) = (1+|\xi|^2)^{-\frac{s}{2}}$ which arise in Feynman integrals for $s = 2$, functions of the form $\tau_s(\xi) = |\xi|^{-s}\,\chi(\xi)$ where χ is a smooth cut-off function that gets rid of infrared divergences, which arise in number theory for negative integer values of s, and operators of the form $A_s = (\Delta+1)^{-\frac{s}{2}}$ (whose symbol is σ_s) for a generalised Laplacian Δ and some integer s, which arise in infinite dimensional geometry and index theory for integer values of s, are all of pseudodifferential nature. Classical and more generally, log-polyhomogeneous *pseudodifferential symbols and operators* form a natural class to consider in the framework of regularisation.

The pseudodifferential symbols and operators that one encounters typically have *integer order* ($-s$ in the above examples), a feature which is the main source of anomalies in physics and the cause of many a discrepancy. These obstacles disappear when working with noninteger order symbols and operators, for which integrals, sums, and traces are canonically defined. The basic idea behind dimensional, Riesz, or zeta regularisation is to embed integer order symbols σ or operators A inside holomorphic families of symbols $\sigma(z)$ or operators $A(z)$ so as to perturb the order of the symbol or the operator away from integers. In the examples mentioned above, natural holomorphic extensions are $\sigma_s(z) = (1+|\xi|^2)^{-\frac{s+z}{2}}$, $\tau_s(z) = |\xi|^{-(s+z)}\,\chi(\xi)$, and $A_s(z) = (\Delta+1)^{-\frac{s+z}{2}}$, which coincide with the original symbols σ_s, τ_s, and operator A_s at $z = 0$.

Away from integer order valued symbols (resp. operators) ordinary manipulations can be carried out on integrals and sums (resp. traces) which legitimise physicists' heuristic computations. Borrowing the physicists' metaphorical language, this amounts to (holomorphically) embedding the integer[11] dimensional world into a complex dimensional one where the canonical functionals mentioned previously have the desired invariance properties, away from an integer dimensional dimensional world. Having left integer dimensions using a holomorphic perturbation, the problem remains to get back to integer dimensions or integer orders by means of regularised evaluators at $z = 0$ which pick up a finite part in a Laurent expansion. The freedom of choice left at this stage is responsible for the one parameter renormalisation group which plays a central role in quantum field theory. Since we are concerned here with evaluating divergent integrals, discrete sums in one variable,

[11] This is 4 for usual space-time.

the renormalisation group physicists use to make sense of Feynman integrals which involve multivariables is beyond the scope of this book.

In these lectures, we hope to modestly help clarify a few aspects of this vast picture in setting some of these heuristic considerations on firm mathematical ground by providing analytic tools to describe regularisation techniques, whether those used in physics, number theory, or geometry, in a common framework. The focus is set on the underlying canonical integral, discrete sum, and trace which are characterised by natural properties such as Stokes' property, covariance, translation invariance, or cyclicity. Various anomalies/discrepancies are investigated, all of which turn out to be local insofar as they can be expressed in terms of the noncommutative residue, another central figure in these lectures.

We do not claim to present breakthrough results but rather a unified outlook with pedestrian proofs on results scattered in the physics and mathematics literature, which we try to bring to the forefront and to make accessible to the nonspecialist. Along the way we nevertheless prove yet unpublished original results such as

- a characterisation of the noncommutative residue on classical symbols (Proposition 2.60 and Theorem 3.39) and of the canonical integral on noninteger order symbols (Theorem 2.61) in terms of their translation invariance;
- a characterisation of the noncommutative residue on classical symbols (Theorem 4.21) and of the canonical integral on noninteger order symbols (Theorem 3.43) in terms of their covariance;
- a characterisation of the noncommutative residue (Proposition 5.40) and the canonical discrete sum (Theorem 5.41) in terms of their $\mathbb{Z}^d$-translation invariance;
- a regularised Euler-Maclaurin formula on symbols (Theorem 5.29);
- Taylor expansions (Theorem 4.16 part (2)) for integrals of holomorphic families extended to log-polyhomogeneous symbols (this is based on an unpublished joint work with Simon Scott);
- a (local) conformal anomaly formula for the ζ-function at zero of a conformally covariant operator in terms of noncommutative residues (Proposition 9.19).

We hope in this way to open new perspectives on and further expand openings to concepts such as regularised integrals, sums, and traces. Far from being exhaustive, these lectures leave out various important regularisation techniques such as Epstein-Glaser [**EG**], Pauli-Villars [**PV**], and lattice regularisation techniques, as well as other regularisation artefacts such as b-integrals [**Mel**] and relative determinants [**Mu**]. Regularisation procedures on manifolds with boundaries or singularities are further vast topics we do not touch upon in spite of the variety of applications and extensions they offer. We also leave aside the realm of noncommutative geometry where zeta-type regularisation procedures are extended to abstract pseudodifferential calculus as well as the ambitious renormalisation issue, which would be needed to make sense of multiple divergent integrals, such as multiloop Feynman diagrams in physics, multiple discrete sums, such as multiple zeta values in number theory, or to count lattice points on convex cones. Here we only tackle simple integrals, and discrete sums. Also, to keep this presentation down to a reasonable size, we chose not to report on regularisation methods implemented in infinite dimensional geometry initiated by the work of Quillen [**Q1, Q2**] and later Bismut and Freed [**BF**] on the geometry of families of operators, of Freed on loop groups [**Fr**], and Maeda, Rosenberg, and Tondeur on the geometry of gauge orbits [**MRT1, MRT2**], which offer interesting insights into the geometry and topology of infinite dimensional manifolds and bundles (see e.g. [**PayR1**], [**LRST**]).

These lectures, which are essentially self-contained, are based on joint work (which we refer to with precise references) with various collaborators, among whom Dominique Manchon, Jouko Mickelsson, Steven Rosenberg, Simon Scott, and former Ph.D. students Alexander Cardona, Catherine Ducourtioux, Jean-Pierre Magnot, Carolina Neira, and Marie-Françoise Ouedraogo, I would like to thank most warmly. I am also grateful to many students and colleagues in France (Clermont-Ferrand), Burkina Faso (Ouagadougou), Germany (Göttingen, Hannover, Regensburg and Potsdam[12]), Colombia (Bogotà and Villa de Leyva), and Lebanon (Beyrouth), who attended my various courses on regularisation techniques[13] which triggered this manuscript, for they all contributed in improving this presentation. Let me address my thanks to Ina Kersten in Göttingen, Elmar Schrohe in Hannover, and Bernd Ammann in Regensburg for inviting me to deliver a series of lectures on regularisation techniques. I am deeply thankful to Christian Brouder, Nicolas Ginoux, Florian Hanisch, and Carolina Neira for their valuable help in thoroughly reading a previous version of the manuscript.

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The lectures are organised into nine chapters, the first of which reviews extended homogeneous distributions as a preparation for similar techniques introduced in the subsequent chapters.

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[12]I would like to thank Christian Becker, David Hansen, Florian Hanisch, and Tobias Jürgens in Potsdam for their very constructive comments.

[13]For some lecture notes and review articles see [**Pa1**], [**Pa2**], [**Pa3**].