CHAPTER 1

The Gamma function extended to nonpositive integer points

This first chapter discusses how to extend the Gamma function to nonpositive integer points and serves as a preparation for the more general issue as how to make sense of certain types of divergent integrals. Whereas here the divergence is at zero, later in the notes the divergences will take place at infinity, but the way one cures these divergences is similar. The Gamma function offers a good toy model to compare regularisation methods mentioned in the Preface. Extending the Gamma function to nonpositive integers arises as an instance of the more general problem of extending homogeneous distributions at negative integers. We show that Riesz and Hadamard's "finite part" regularisation methods lead to the same extended homogeneous distributions (see Theorem 1.24) and hence to the same extended Gamma function, a feature which arises again later in these notes. We discuss discrepancies induced by the regularisation procedure, which are a first hint to further obstructions we will encounter while working with regularised integrals.

1.1. Homogeneous distributions

Let

$$\begin{aligned}\mathcal{S}(\mathbb{R}^+) &= \{f \in C^\infty(\mathbb{R}^+), \forall (\alpha,\beta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0},\\ &\quad \exists C_{\alpha,\beta}, \text{ s.t. } |x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta} \quad \forall x \in \mathbb{R}^+\}\end{aligned}$$

denote the space of **Schwartz functions on** $\mathbb{R}^+ :=]0,+\infty[$. The following exercise shows that Schwartz functions on $\mathbb{R}^+$ are smooth functions f on $\mathbb{R}^+$ whose derivatives $\partial^\beta f(x)$ go faster to zero as x tends to infinity than any inverse power $x^{-\alpha}$.

Exercise 1.1. *Let $f \in C^\infty(\mathbb{R}^+)$. Show that*

$$f \in \mathcal{S}(\mathbb{R}^+) \Longleftrightarrow \lim_{x\to+\infty} \left(x^\alpha \partial^\beta f(x)\right) = 0 \quad \forall (\alpha,\beta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}.$$

Decreasing exponentials are typical Schwartz functions.

Example 1.2. Show that the map $\phi : x \mapsto e^{-x}$ defines a Schwartz function on $\mathbb{R}^+$.

One can build homogeneous distributions given by linear forms on $\mathcal{S}(\mathbb{R}^+)$ in the following manner.

Exercise 1.3. *Given a Schwartz function ϕ on $\mathbb{R}^+$ and any complex number a with real part* $\mathrm{Re}(a)$ *larger than* -1*, show that the map $x \mapsto x^a\,\phi(x)$ lies in $L^1(\mathbb{R}^+)$.*

For a complex number a with real part $\mathrm{Re}(a)$ larger than -1, we can therefore consider the distribution

$$\begin{aligned} F_a : \mathcal{S}(\mathbb{R}^+) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \int_0^\infty x^a\, \phi(x)\, dx. \end{aligned}$$

F_a defines a **homogeneous distribution** of degree a in the following sense. Given a Schwartz function f in $\mathcal{S}(\mathbb{R}^+)$ and any positive number t, we set $\phi_t = t^{-1}\phi(t^{-1}\cdot)$. Then,

$$F_a(\phi_t) = t^a\, F_a(\phi) \quad \text{if} \quad \mathrm{Re}(a) > -1. \tag{1.1}$$

The integral

$$\Gamma(b) := \int_0^\infty x^{b-1}\, e^{-x}\, dx,$$

which is defined for $\mathrm{Re}(b) > 0$, is called the **Gamma function**.

The Gamma function $z \mapsto \Gamma(z)$ defines a holomorphic map on the half-plane $\mathrm{Re}(z) > 0$.

EXERCISE 1.4. *Check this assertion.*

Integration by parts shows that $\Gamma(b) := \frac{\Gamma(b+1)}{b}$ for $\mathrm{Re}(b) > 0$.

EXERCISE 1.5. *For any complex number b with positive real part show that:*

$$\Gamma(b) := \frac{1}{b(b+1)\cdots(b+k-1)}\Gamma(b+k) \quad \forall k \in \mathbb{N}. \tag{1.2}$$

Deduce that $\Gamma(k) = (k-1)! \quad \forall k \in \mathbb{N}$.

Extending the Gamma function to the whole complex plane is related to the problem of extending homogeneous distributions F_a considered by Hadamard and Riesz (see e.g. [**Sch**, Chapter II]) to all complex values a. One wants to assign to the poles of Γ a finite value, which amounts to assigning a finite value $\tilde{F}_{-k}(\phi)$ to negative integers $-k$.

EXERCISE 1.6. *Show that the map* $x \mapsto (\log x)\, e^{-x}$ *lies in* $L^1(\mathbb{R}^+)$.

Hence we can define the **Euler's constant**[1]

$$\gamma := -\int_0^\infty \log x\, e^{-x}\, dx.$$

The following elementary properties of the Gamma function are useful for forthcoming applications.

PROPOSITION 1.7. (1) *The Gamma function is differentiable at any positive integer k and*

$$\Gamma'(1) = -\gamma; \quad \frac{\Gamma'(k)}{\Gamma(k)} = \left(\sum_{j=1}^{k-1} \frac{1}{j} - \gamma\right) \quad \forall k \in \mathbb{N} - \{1\}. \tag{1.3}$$

[1]The Gamma constant was first introduced by Euler in 1735 as the limit $\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^N \frac{1}{n} - \int_1^N \frac{1}{t}\, dt\right)$.

(2) *The inverse of the Gamma function* $\frac{1}{\Gamma(z)}$ *defined on the half-plane* $\mathrm{Re}(z) > 0$ *extends to a holomorphic map at* $z = 0$ *and*

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + o(z^2). \tag{1.4}$$

In particular, $\left(\frac{1}{\Gamma}\right)'(0) = 1$.

PROOF. (1) The derivative of Γ at 1 reads

$$\begin{aligned} \Gamma'(1) &= \partial_z \Gamma(1+z)_{|z=0} \\ &= \partial_z \left(\int_0^\infty t^z\, e^{-t}\, dt \right)_{|z=0} \\ &= \int_0^\infty \log t\, e^{-t}\, dx \\ &= -\gamma. \end{aligned}$$

The derivative at $k \in \mathbb{N} - \{1\}$ reads

$$\begin{aligned} \Gamma'(k) &= \lim_{z\to 0} \frac{\Gamma(k+z) - \Gamma(k)}{z} \\ &= \lim_{z\to 0} \frac{(k+z-1)\cdots(z+1)\cdot\Gamma(z+1) - (k-1)!}{z} \\ &= (k-1)! \left(\sum_{j=1}^{k-1} \frac{1}{j} - \gamma \right) = \Gamma(k) \left(\sum_{j=1}^{k-1} \frac{1}{j} - \gamma \right), \end{aligned}$$

so that for $k \geq 2$,

$$\frac{\Gamma'(k)}{\Gamma(k)} = \sum_{j=1}^{k-1} \frac{1}{j} - \gamma.$$

(2) It follows from (1.2) that

$$\frac{1}{\Gamma(z)} = \frac{z}{\Gamma(z+1)} = \frac{z}{\Gamma(1) + \Gamma'(1)z + o(z)} = z + \gamma z^2 + o(z^2).$$ □

EXERCISE 1.8. *Show that*

$$\Gamma(z) \sim_0 \frac{1}{z} - \gamma, \tag{1.5}$$

where by $f(z) \sim_0 g(z)$ *we mean that* $\lim_{z\to 0} \frac{f(z)}{g(z)} = 1$. **Hint:** Formula (1.5) follows from (1.2) setting $b = z, k = 1$ or equivalently from (1.4).

EXERCISE 1.9. *Show that*[2]

$$\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

EXERCISE 1.10. *Show that for two complex numbers* a *and* b *with positive real parts, we have*

$$\Gamma(a)\Gamma(b) = B(a,b)\Gamma(a+b),$$

where $B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}dt$ *is the* **Beta function**.

[2] I thank Stephan Weinzierl for pointing out this formula to me.

1.2. A meromorphic extension of the Gamma function

Let a be a complex number, the real part of which satisfies $\mathrm{Re}(a) > -1$, and let ϕ be a Schwartz function on $\mathbb{R}^+$.

EXERCISE 1.11. *Show that* $F_a(\phi) = -\frac{F_{a+1}(\phi')}{a+1}$.

After integrating by parts k times we get

$$F_a(\phi) = \frac{(-1)^k}{(a+1)\cdots(a+k)} F_{a+k}(\phi^{(k)}). \tag{1.6}$$

Since $\phi^{(k)}$ is a Schwartz function, the expression on the right-hand side is well defined provided $a+k$ has its real part larger than -1. The expression $\frac{(-1)^k}{(a+1)\cdots(a+k)} F_{a+k}(\phi^{(k)})$ therefore defines an extension of F_a to the half-plane $\mathrm{Re}(a) > -k-1$. Given a complex number a, there is a positive integer k such that $\mathrm{Re}(a) > -k-1$, and we set

$$\tilde{F}_a(\phi) := \frac{(-1)^k}{(a+1)\cdots(a+k)} F_{a+k}(\phi^{(k)}). \tag{1.7}$$

EXERCISE 1.12. *Show that this definition does not depend on the choice of* $k > -\mathrm{Re}(a) - 1$ *in checking that* $\tilde{F}_a(\phi) = \frac{(-1)^{k+l}}{(a+1)\cdots(a+k+l)} \tilde{F}_{a+k+l}(\phi^{(k+l)})$ *for any positive integer* l*.*

Equation (1.7) therefore extends $F_a(\phi)$ to a meromorphic function $a \mapsto \tilde{F}_a(\phi)$ on the plane with simple poles at negative integers. The residue at a negative integer $-k \in -\mathbb{N}$ is given by

$$\begin{aligned} \mathrm{Res}_{a=-k} \tilde{F}_a(\phi) &= \lim_{a\to -k} \Big((a+k)\, \tilde{F}_a(\phi) \Big) \\ &= \frac{(-1)^k}{(-k+1)\cdots(-1)} \int_0^\infty \phi^{(k)}(x)\, dx \\ &= \frac{\phi^{(k-1)}(0)}{(k-1)!}, \end{aligned} \tag{1.8}$$

where for a meromorphic function f with a simple pole at z_0, we have set

$$\mathrm{Res}_{z=z_0} f := \lim_{z\to z_0} \left((z-z_0) f(z) \right).$$

When applied to $\phi : x \mapsto e^{-x}$, this construction provides an extension of the Gamma function to the whole complex plane, defined on the half-plane $\mathrm{Re}(b) > -k$ with $k \in \mathbb{N}$ by

$$\tilde{\Gamma}(b) := \frac{1}{b(b+1)\cdots(b+k-1)} \Gamma(b+k). \tag{1.9}$$

EXERCISE 1.13. *Show that* $\tilde{\Gamma}$ *has simple poles at integers* $-k \in -\mathbb{N} \cup \{0\}$ *with residue at these poles given by*

$$\mathrm{Res}_{b=-k} \tilde{\Gamma}(b) = \frac{(-1)^k}{k!}. \tag{1.10}$$

From now on we use the same notation Γ for the extension $\tilde{\Gamma}$. By (1.9) we have the following recursive formula:

$$\Gamma(b+k) = b(b+1)\cdots(b+k-1)\,\Gamma(b) \quad \text{if} \quad \mathrm{Re}(b) > 0. \tag{1.11}$$

The following proposition gives the asymptotics at negative integers.

PROPOSITION 1.14. *At negative integers $-k \in -\mathbb{N}$, we have*

$$\tilde{\Gamma}(-k+z) \sim_0 \frac{(-1)^k}{k!}\left(\frac{1}{z} - \gamma + \sum_{j=1}^{k}\frac{1}{j}\right). \tag{1.12}$$

PROOF. By formula (1.9) we have

$$\begin{aligned}\tilde{\Gamma}(-k+z) &= \frac{\tilde{\Gamma}(z)}{(z-k)\cdots(z-1)}\\ &\sim_0 \frac{\frac{1}{z}-\gamma}{(z-k)\cdots(z-1)}\\ &\sim_0 \frac{(-1)^k}{k!}\left(\frac{1}{z}-\gamma\right)\left(1+\frac{z}{k}\right)\cdots(1+z),\end{aligned}$$

from which (1.12) follows. □

REMARK 1.15. Formula (1.12) yields back the residue at $z = -k$ given by formula (1.10).

1.3. Riesz regularisation

We now want to make sense of $\tilde{F}_{-k}(\phi)$ in spite of the fact that $a = -k$ arises as a pole in (1.7). Following Riesz (see e.g. [**Sch**]), we pick the constant term at $z = 0$ in the Laurent expansion of the map $z \mapsto \tilde{F}_{-k+z}(\phi)$.

PROPOSITION 1.16. *Given a Schwartz function ϕ on $\mathbb{R}^+$ and a complex number a, the map $z \mapsto F_{a+z}(\phi)$ is meromorphic on the plane with simple poles in $-a-\mathbb{N}$. The constant term in the Laurent expansion*[3]

$$\begin{aligned}\int_0^{\infty,\mathrm{Riesz}} x^a\,\phi(x)\,dx &:= \mathrm{fp}_{z=0}\tilde{F}_{a+z}(\phi)\\ &:= \lim_{z\to 0}\left(\tilde{F}_{a+z}(\phi) - \frac{1}{z}\mathrm{Res}_{z=0}\tilde{F}_{a+z}(\phi)\right)\end{aligned} \tag{1.13}$$

coincides with the ordinary integral $\int_0^\infty x^a\,\phi(x)\,dx$ whenever $\mathrm{Re}(a) > -1$.

(1) *If $a \notin -\mathbb{N}$, then*

$$\int_0^{\infty,\mathrm{Riesz}} x^a\,\phi(x)\,dx = \tilde{F}_a(\phi) = \frac{(-1)^k}{(a+1)\cdots(a+k)}F_{a+k}(\phi^{(k)}),$$

where k is any integer such that $\mathrm{Re}(a)+k > -1$.

(2) *If $a = -k \in -\mathbb{N}$, then*

$$\int_0^{\infty,\mathrm{Riesz}} x^{-k}\,\phi(x)\,dx = -\frac{1}{(k-1)!}\int_0^\infty \log x\,\phi^{(k)}(x)dx + \mathrm{Res}_{a=-k}\left(\tilde{F}_a(\phi)\right)\sum_{j=1}^{k-1}\frac{1}{j},$$

setting the sum over j equal to zero if $k = 1$.

PROOF. The case $a \notin -\mathbb{N}$ follows from the previous discussion. We therefore prove the proposition when $a = -k$ for some integer $k \in \mathbb{N}$.

[3] Here fp stands for the finite part.

(1) Let us start with $k=1$. Integrating by parts for $\mathrm{Re}(z)>-1$, we have

$$\tilde{F}_{-1+z}(\phi) = -\frac{1}{z}\int_0^\infty x^z \phi'(x)dx.$$

Since

$$\mathrm{Res}_{z=0}\tilde{F}_{-1+z}(\phi) = \phi(0),$$

we have

$$\begin{aligned}
&\lim_{z\to 0}\left(\tilde{F}_{-1+z}(\phi) - \frac{1}{z}\mathrm{Res}_{z=0}\tilde{F}_{-1+z}(\phi)\right) \\
&= \lim_{z\to 0}\left(-\frac{1}{z}\int_0^\infty x^z\phi'(x)dx - \frac{\phi(0)}{z}\right) \\
&= \lim_{z\to 0}\left(-\int_0^\infty \frac{x^z-1}{z}\,\phi'(x)\,dx\right) \\
&= -\int_0^\infty \log x\,\phi'(x)dx.
\end{aligned}$$

(2) When $k>1$, integrating by parts k times, for $\mathrm{Re}(z)>-1$ we find that

$$\tilde{F}_{-k+z}(\phi) = \frac{(-1)^k}{(-k+1+z)\cdots(z-1)z}\int_0^\infty x^z\phi^{(k)}(x)dx.$$

Since $\mathrm{Res}_{z=0}\left(\tilde{F}_{-k+z}(\phi)\right) = \frac{\phi^{(k-1)}(0)}{(k-1)!}$, we have

$$\begin{aligned}
&\lim_{z\to 0}\left(\tilde{F}_{-k+z}(\phi) - \frac{1}{z}\mathrm{Res}_{z=0}\left(\tilde{F}_{-k+z}(\phi)\right)\right) \\
&= \lim_{z\to 0}\left(\frac{(-1)^k}{(-k+1+z)\cdots(z-1)z}\int_0^\infty x^z\phi^{(k)}(x)dx - \frac{1}{z}\frac{\phi^{(k-1)}(0)}{(k-1)!}\right) \\
&= \lim_{z\to 0}\left(\frac{(-1)^k}{(-k+z+1)\cdots(z-1)}\int_0^\infty \frac{x^z-1}{z}\phi^{(k)}(x)dx\right. \\
&\quad \left. + \frac{1}{z}\left[\frac{(-1)^k}{(-k+z+1)\cdots(z-1)}\int_0^\infty \phi^{(k)}(x)dx - \frac{\phi^{(k-1)}(0)}{(k-1)!}\right]\right) \\
&= -\frac{1}{(k-1)!}\int_0^\infty \log x\,\phi^{(k)}(x)dx \\
&\quad + \lim_{z\to 0}\left(\frac{\phi^{(k-1)}(0)}{z}\left[\prod_{j=1}^{k-1}\frac{1}{j-z} - \frac{1}{(k-1)!}\right]\right) \\
&= -\frac{1}{(k-1)!}\int_0^\infty \log x\,\phi^{(k)}(x)dx + \lim_{z\to 0}\left(\phi^{(k-1)}(0)\left[\frac{\Psi_k(z)-\Psi_k(0)}{z}\right]\right) \\
&= -\frac{1}{(k-1)!}\int_0^\infty \log x\,\phi^{(k)}(x)dx + \Psi_k'(0)\,\phi^{(k-1)}(0) \\
&= -\frac{1}{(k-1)!}\int_0^\infty \log x\,\phi^{(k)}(x)dx + \frac{\phi^{(k-1)}(0)}{(k-1)!}\sum_{j=1}^{k-1}\frac{1}{j},
\end{aligned}$$

where we have set $\Psi_k(z) := \prod_{j=1}^{k-1}\frac{1}{j-z}$ and used (1.10). ∎

EXERCISE 1.17. *Show that for any real numbers $\lambda \neq 0$ and μ and any holomorphic function $f(z) = \lambda z + \mu z^2 + o(z^2)$ in a neighborhood of zero, with the notation of* (1.13) *and for any Schwartz function ϕ on $\mathbb{R}^+$ we have*

$$\mathrm{fp}_{z=0}\tilde{F}_{a+z}(\phi) = \int_0^{\infty,\mathrm{Riesz}} x^a\, \phi(x)\, dx - \delta_{a+k}\, \mu \frac{\phi^{(k-1)}(0)}{(k-1)!}.$$

Hint: Notice that $\frac{1}{f(z)} = \frac{1}{\lambda z} - \mu z + o(z)$.

Applying the previous proposition to $\phi(x) = e^{-x}$ leads to the following **Riesz extension** of the Gamma function.

Given a complex number b, the map $z \mapsto \tilde{\Gamma}(b+z)$ is meromorphic on the plane with simple poles in $-b - \mathbb{N} \cup \{0\}$. The finite part of the integral $\tilde{\Gamma}(b+z)$, which is defined by the constant term in the Laurent expansion

$$\Gamma^{\mathrm{Riesz}}(b) := \lim_{z\to 0}\left(\tilde{\Gamma}(b+z) - \frac{1}{z}\mathrm{Res}_{z=0}\tilde{\Gamma}(b+z)\right), \tag{1.14}$$

coincides with the ordinary Gamma function $\Gamma(b)$ whenever $\mathrm{Re}(b) > 0$.

EXERCISE 1.18. *Show the following:*

(1) *If $b \notin -\mathbb{N} \cup \{0\}$, then*

$$\Gamma^{\mathrm{Riesz}}(b) = \tilde{\Gamma}(b) = \frac{\Gamma(b+k)}{b\,(b+1)\cdots(b+k-1)}$$

for any k such that $\mathrm{Re}(b) + k > 0$.

(2) *For any nonpositive integer $b = -k$ show that*

$$\Gamma^{\mathrm{Riesz}}(0) = -\gamma; \quad \Gamma^{\mathrm{Riesz}}(-k) = \mathrm{Res}_{b=-k}\tilde{\Gamma}(b)\sum_{j=1}^{k}\frac{1}{j} - \frac{(-1)^k}{k!}\gamma \quad \text{if} \quad k > 0.$$

Hint: Apply Proposition 1.16 to $a = -(k+1)$ and $\phi(t) = e^{-t}$.

1.4. Hadamard's "finite part" method

As before, ϕ denotes a Schwartz function on $\mathbb{R}^+$. Clearly, for $\mathrm{Re}(a) > -1$,

$$F_a(\phi) := \int_0^{\infty} x^a\, \phi(x)\, dx = \lim_{\epsilon\to 0}\int_\epsilon^\infty x^a\, \phi(x)\, dx.$$

In particular, for $\mathrm{Re}(b) > 0$, we have

$$\Gamma(b) = \lim_{\epsilon\to 0}\int_\epsilon^\infty x^{b-1}\, e^{-x}\, dx.$$

Following Hadamard (see e.g. [**Sch**]) we want to extend $F_a(\phi)$ to all complex values of a in picking out the constant term denoted by $\int_0^{\infty,\mathrm{Had}} x^a\, \phi(x)\, dx$ in the asymptotic expansion of the map $\epsilon \mapsto \int_\epsilon^\infty x^a\, \phi(x)\, dx$ as $\epsilon \to 0$, a procedure we are about to describe.

Applying this to $\phi(x) = e^{-x}$ yields an alternative extension

$$\Gamma^{\mathrm{Had}}(b) := \int_0^{\infty,\mathrm{Had}} x^{b-1}\, e^{-x}\, dx$$

of the Gamma function to the whole complex plane.

Let us introduce some notation.[4] $\mathcal{A}[\epsilon]$ denotes the set of germs of smooth functions around zero in the variable ϵ,[5] and for any complex number α and any positive integer l we set

$$\begin{aligned} \mathcal{A}[\epsilon]\,\epsilon^\alpha &:= \{f(\epsilon)\,\epsilon^\alpha,\ f\in\mathcal{A}[\epsilon]\},\\ \mathcal{A}[\epsilon]\log^l\epsilon &:= \{f(\epsilon)\,\log^l\epsilon,\ f\in\mathcal{A}[\epsilon]\}. \end{aligned} \tag{1.15}$$

Using a Taylor expansion of f at zero, a function $g(\epsilon)=f(\epsilon)\epsilon^\alpha$ in $\mathcal{A}[\epsilon]\epsilon^\alpha$ reads

$$g(\epsilon)=\sum_{k=0}^N\frac{f^k(0)}{k!}\epsilon^{\alpha+k}+o(\epsilon^{N+\alpha})\quad\forall N\in\mathbb{N}.$$

The finite part is given by the constant term in the expansion

$$\mathrm{fp}_{\epsilon=0}g(\epsilon)=\sum_{k=0}^N\frac{f^k(0)}{k!}\delta_{\alpha+k},$$

independently of the choice of N provided it is chosen large enough. Similarly, for a function $h(\epsilon)=f(\epsilon)\log^l\epsilon$ in $\mathcal{A}[\epsilon]\log^l\epsilon$ for some positive integer l, we set $\mathrm{fp}_{\epsilon=0}h(\epsilon)=0$. We need a technical lemma.

LEMMA 1.19. *Let ϕ denote a Schwartz function on $\mathbb{R}^+$. The map $\epsilon\longmapsto\int_\epsilon^\infty\log x\,\phi(x)\,dx$ lies in $\mathcal{A}[\epsilon]\bigoplus\mathcal{A}[\epsilon]\,\log\epsilon$ and*

$$\lim_{\epsilon\to0}\int_\epsilon^\infty\log x\,\phi(x)\,dx=\int_0^\infty\log x\,\phi(x)\,dx.$$

PROOF. We have

$$\begin{aligned}\int_\epsilon^\infty\log x\,\phi(x)\,dx &= \int_0^\infty\log x\,\phi(x)\,dx-\int_0^\epsilon\log x\,\phi(x)\,dx\\ &= \int_0^\infty\log x\,\phi(x)\,dx-\epsilon\int_0^1\log(\epsilon\,x)\,\phi(\epsilon\,x)\,dx\\ &= \int_0^\infty\log x\,\phi(x)\,dx-\epsilon\int_0^1\log x\,\phi(\epsilon\,x)\,dx\\ &- \epsilon\,\log\epsilon\int_0^1\phi(\epsilon\,x)\,dx.\end{aligned}$$

A Taylor expansion of ϕ at 0 shows that $\epsilon\mapsto\int_0^1\phi(\epsilon x)\,dx$ lies in $\mathcal{A}[\epsilon]$. The map $\int_0^1\log x\,\phi(\epsilon x)\,dx$ also lies in $\mathcal{A}[\epsilon]$; indeed

$$\int_0^1\log x\,\phi(\epsilon x)\,dx=-\epsilon\int_0^1(x\log x-x)\,\phi'(\epsilon\,x)\,dx-\phi(\epsilon)$$

and a Taylor expansion of ϕ and ϕ' at 0 provides the required asymptotic expansion. Hence the map $\epsilon\mapsto\int_\epsilon^\infty\log x\,\phi(x)\,dx$ lies in $\mathcal{A}[\epsilon]\bigoplus\mathcal{A}[\epsilon]\,\log\epsilon$, and we have

$$\lim_{\epsilon\to0}\int_\epsilon^\infty\log x\,\phi(x)\,dx=\int_0^\infty\log x\,\phi(x)\,dx.\qquad\square$$

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[4] I thank Bing Zhang for interesting discussions and comments concerning this paragraph and Florian Hanisch for his very constructive suggestions.

[5] That is, equivalence classes of smooth functions are defined on a neighborhood of zero for the equivalence relation $f\sim g$ if f and g coincide on some open neighborhood of zero.

EXERCISE 1.20. *Deduce from the previous lemma that the map* $\epsilon \mapsto \int_\epsilon^\infty \log x\, e^{-x}\, dx$ *lies in* $\mathcal{A}[\epsilon] \bigoplus \mathcal{A}[\epsilon]\, \log\epsilon$, *and that we have*

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty \log x\, e^{-x}\, dx = -\gamma.$$

PROPOSITION 1.21. *Let* ϕ *denote a Schwartz function on* $\mathbb{R}^+$.

(1) *If* $\mathrm{Re}(a) > -1$, *the map* $\epsilon \mapsto \int_\epsilon^\infty x^a\, \phi(x)\, dx$ *lies in* $\mathbb{C}\oplus\mathcal{A}[\epsilon]\epsilon^{a+1}$, *and we have*

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty x^a\, \phi(x)\, dx = \int_0^\infty x^a\, \phi(x)\, dx = F_a(\phi).$$

(2) *If* $a \notin -\mathbb{N}$, *the map* $\epsilon \mapsto \int_\epsilon^\infty x^a\, \phi(x)\, dx$ *lies in* $\mathbb{C}\oplus\mathcal{A}[\epsilon]\,\epsilon^{a+1}$ *with constant term given by*

$$\int_0^{\infty,\mathrm{Had}} x^a\, \phi(x)\, dx = \frac{(-1)^k}{(a+1)\cdots(a+k)}\int_0^\infty x^{a+k}\, \phi^{(k)}(x)\, dx = \tilde{F}_a(\phi)$$

for any integer k *such that* $a+k > -1$.

(3) *If* $a = -k \in -\mathbb{N}$, *the map* $\epsilon \mapsto \int_\epsilon^\infty x^a\, \phi(x)\, dx$ *lies in* $\mathcal{A}[\epsilon]\epsilon^{-k+1} \bigoplus \mathcal{A}[\epsilon]\log\epsilon$ *with constant term given by*

$$\begin{aligned}\int_0^{\infty,\mathrm{Had}} x^{-k}\, \phi(x)\, dx &= -\frac{1}{(k-1)!}\int_0^\infty \log x\, \phi^{(k)}(x)\, dx\\ &+ \mathrm{Res}_{a=-k}F_a(\phi)\sum_{j=1}^{k-1}\frac{1}{j}\\ &= \tilde{F}_{-k}(\phi),\end{aligned}$$

where the sum is set to zero if $k=1$.

PROOF. (1) If $\mathrm{Re}(a) > -1$, we have

$$\begin{aligned}\int_\epsilon^\infty x^a\, \phi(x)\, dx &= \int_0^\infty x^a\, \phi(x)\, dx - \int_0^\epsilon x^a\, \phi(x)\, dx\\ &= \int_0^\infty x^a\, \phi(x)\, dx - \epsilon^{a+1}\int_0^1 x^a\, \phi(\epsilon\, x)\, dx.\end{aligned}$$

The map $\epsilon \mapsto \int_0^1 x^a\, \phi(\epsilon\, x)\, dx$ lies in $\mathcal{A}[\epsilon]$ so that the map $\epsilon \mapsto \int_\epsilon^\infty x^a\, \phi(x)\, dx$ lies in $\mathbb{C}\oplus\mathcal{A}[\epsilon]\epsilon^{a+1}$ and

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty x^a\, \phi(x)\, dx = \int_0^\infty x^a\, \phi(x) dx.$$

(2) If $a \notin -\mathbb{N}$, we proceed by induction on k with $\mathrm{Re}(a) \in]-(k+1), -k]$ using integration by parts to show that $\int_\epsilon^\infty x^a\, \phi(x)\, dx$ lies in $\mathbb{C}\oplus\mathcal{A}[\epsilon]\epsilon^{a+1}$. The step $k=0$ holds by the previous item. One integration by parts provides the induction step. Indeed, if the map $\epsilon \mapsto \int_\epsilon^\infty x^{a+1}\, \phi(x)\, dx$ lies in $\mathbb{C}\oplus\mathcal{A}[\epsilon]\epsilon^{a+2}$, then the map $\epsilon \mapsto \int_\epsilon^\infty x^a \phi(x)\, dx$ lies in $\mathbb{C}\oplus\mathcal{A}[\epsilon]\epsilon^{a+1}$ since

$$\begin{aligned}\int_\epsilon^\infty x^a\, \phi(x)\, dx &= -\int_\epsilon^\infty \frac{x^{a+1}}{a+1}\,\phi'(x)\, dx + \left[\frac{x^{a+1}}{a+1}\phi(x)\right]_\epsilon^\infty\\ &= -\frac{1}{a+1}\int_\epsilon^\infty x^{a+1}\,\phi'(x)\, dx - \frac{\epsilon^{a+1}}{a+1}\phi(\epsilon).\end{aligned}$$

For $\mathrm{Re}(a) \in]-(k+1), -k]$ and $a \notin -\mathbb{N}$, using a Taylor expansion of $\phi^{(l)}$ at 0, we have

$$\begin{aligned}\int_\epsilon^\infty x^a\,\phi(x)\,dx &= \frac{(-1)^k}{(a+1)\cdots(a+k)}\int_\epsilon^\infty x^{a+k}\,\phi^{(k)}(x)\,dx\\ &\quad + (-1)^k\frac{\epsilon^{a+k}}{(a+1)\cdots(a+k)}\phi^{(k-1)}(\epsilon)+\cdots\\ &\quad + (-1)^j\frac{\epsilon^{a+j}}{(a+1)\cdots(a+j)}\phi^{(j-1)}(\epsilon)+\cdots-\frac{\epsilon^{a+1}}{a+1}\phi(\epsilon)\\ &= \frac{(-1)^k}{(a+1)\cdots(a+k)}\int_\epsilon^\infty x^{a+k}\,\phi^{(k)}(x)\,dx\\ &\quad + \sum_{j=1}^k(-1)^j\sum_{i_j=0}^{N_j}\frac{\epsilon^{a+j+i_j}}{(a+1)\cdots(a+j)\,i_j!}\phi^{(j-1+i_j)}(0)\\ &\quad + \text{remainder terms.}\end{aligned}$$

The constant term reads

$$\int_0^{\infty,\mathrm{Had}} x^a\,\phi(x)\,dx = \frac{(-1)^k}{(a+1)\cdots(a+k)}\int_0^\infty x^{a+k}\,\phi^{(k)}(x)\,dx$$

since the remaining terms do not contribute to the constant term.

(3) If $a=-1$ then,

$$\begin{aligned}\int_\epsilon^\infty x^{-1}\,\phi(x)\,dx &= -\int_\epsilon^\infty \log x\,\phi'(x)\,dx + [\log x\,\phi(x)]_\epsilon^\infty\\ &= -\int_\epsilon^\infty \log x\,\phi'(x)\,dx - \log\epsilon\,\phi(\epsilon),\end{aligned}$$

which lies in $\mathcal{A}[\epsilon]\bigoplus\mathcal{A}[\epsilon]\log\epsilon$ as a consequence of Lemma 1.19 and has finite part at zero given by

$$\int_0^{\infty,\mathrm{Had}} x^{-1}\,\phi(x)\,dx = -\int_0^\infty \log x\,\phi'(x)\,dx.$$

(4) If $a=-k$ for some integer $k>1$, then by induction on k we show that the map $\epsilon\mapsto\int_\epsilon^\infty x^{-k}\,\phi(x)\,dx$ lies in $\mathcal{A}[\epsilon]\epsilon^{-k+1}\bigoplus\mathcal{A}[\epsilon]\log\epsilon$. The previous step gives the statement for $k=1$. Using integration by parts, we easily prove the induction step,

$$\begin{aligned}\int_\epsilon^\infty x^{-k}\,\phi(x)\,dx &= -\int_\epsilon^\infty \frac{x^{-k+1}}{-k+1}\,\phi'(x)\,dx + \left[\frac{x^{-k+1}}{-k+1}\,\phi(x)\right]_\epsilon^\infty\\ &= \frac{1}{k-1}\int_\epsilon^\infty x^{-k+1}\,\phi'(x)\,dx + \frac{\epsilon^{-k+1}}{k-1}\,\phi(\epsilon).\end{aligned}$$

Iterating the integration by parts procedure gives

$$\begin{aligned}\int_\epsilon^\infty x^{-k}\,\phi(x)\,dx &= \frac{1}{(k-1)!}\int_\epsilon^\infty x^{-1}\,\phi^{(k-1)}(x)\,dx + \frac{\epsilon^{-k+1}}{k-1}\,\phi(\epsilon) + \cdots \\ &+ \frac{\epsilon^{-k+j}}{(k-1)\cdots(k-j)}\,\phi^{(j-1)}(\epsilon) + \cdots + \frac{\epsilon^{-1}}{(k-1)!}\phi^{(k-2)}(\epsilon) \\ &= -\frac{1}{(k-1)!}\int_\epsilon^\infty \log x\,\phi^{(k)}(x)\,dx - \frac{\log\epsilon}{(k-1)!}\phi^{(k-1)}(\epsilon) \\ &+ \frac{\epsilon^{-k+1}}{k-1}\,\phi(\epsilon) + \cdots + \frac{\epsilon^{-k+j}}{(k-1)\cdots(k-j)}\,\phi^{(j-1)}(\epsilon) \\ &+ \cdots + \frac{\epsilon^{-1}}{(k-1)!}\phi^{(k-2)}(\epsilon). \end{aligned} \tag{1.16}$$

A Taylor expansion around $\epsilon = 0$ of ϕ and its derivatives then gives rise to the finite part

$$\begin{aligned}&\int_0^{\infty,\mathrm{Had}} x^{-k}\,\phi(x)\,dx \\ &= -\frac{1}{(k-1)!}\int_0^\infty \log x\,\phi^{(k)}(x)\,dx + \frac{\phi^{(k-1)}(0)}{(k-1)!(k-1)} + \cdots \\ &+ \frac{\phi^{(k-1)}(0)}{(k-1)\cdots(k-j)\,(k-j)!} + \cdots + \frac{\phi^{(k-1)}(0)}{(k-1)!} \\ &= -\frac{1}{(k-1)!}\int_0^\infty \log x\,\phi^{(k)}(x)\,dx + \frac{\phi^{(k-1)}(0)}{(k-1)!}\sum_{j=1}^{k-1}\frac{1}{k-j}. \end{aligned}$$

□

EXERCISE 1.22. *Show that for any positive real number* $\lambda = e^\mu$ *we have*

$$\mathrm{fp}_{\epsilon=0}\int_{\lambda\epsilon}^\infty x^a\,\phi(x)\,dx = \int_0^{\infty,\mathrm{Had}} x^a\,\phi(x)\,dx - \delta_{a+k}\,\mu\frac{\phi^{(k-1)}(0)}{(k-1)!}.$$

Compare with Exercise 1.17.

Applying Proposition 1.21 to $\phi(x) = e^{-x}$ leads to the following **Hadamard extension** of the Gamma function (which we also call **cut-off Gamma function**):

$$\Gamma^{\mathrm{Had}}(b) := \mathrm{fp}_{\epsilon=0}\left(\int_\epsilon^\infty x^{b-1}\,e^{-x}\,dx\right).$$

EXERCISE 1.23. (1) *Show that if* $\mathrm{Re}(b) > 0$*, the map* $\epsilon \mapsto \int_\epsilon^\infty x^{b-1}\,e^{-x}\,dx$ *lies in* $\mathcal{A}[\epsilon]$*. The corresponding asymptotic expansion has constant term given by*

$$\Gamma^{\mathrm{Had}}(b) = \lim_{\epsilon\to 0}\left(\int_\epsilon^\infty x^{b-1}\,e^{-x}\,dx\right) = \Gamma(b).$$

(2) *Show that for* $b \notin \mathbb{Z}_{\leq 0}$*, the map* $\epsilon \mapsto \int_\epsilon^\infty x^{b-1}\,e^{-x}\,dx$ *lies in* $\mathbb{C} \oplus \mathcal{A}[\epsilon]\,\epsilon^b$*. The corresponding asymptotic expansion has constant term given by*

$$\Gamma^{\mathrm{Had}}(b) = \frac{\Gamma(b+k)}{b(b+1)\cdots(b+k-1)}.$$

(3) *Show that if* $b = -k \in -\mathbb{N} \cup \{0\}$, *the map* $\epsilon \mapsto \int_\epsilon^\infty x^{-k-1}\, e^{-x}\, dx$ *lies in* $\mathcal{A}[\epsilon]\epsilon^{-k+1} \bigoplus \mathcal{A}[\epsilon] \log \epsilon$. *The corresponding asymptotic expansion has constant term given by*

$$\begin{aligned} \Gamma^{\mathrm{Had}}(0) &= -\gamma \quad \text{if} \quad k = 0, \\ \Gamma^{\mathrm{Had}}(-k) &= \frac{(-1)^k}{k!}\left[\sum_{j=1}^{k} \frac{1}{j} - \gamma\right], \end{aligned}$$

where the sum over j *is set to zero when* $k = 1$.

1.5. Discrepancies

The following theorem sums up the results of the two previous paragraphs.

THEOREM 1.24. *The Riesz and Hadamard regularisation methods yield the same extended distribution* $\tilde{F}_a$ *for any complex value* a *which, when applied to a Schwartz function* ϕ, *reads*

$$\tilde{F}_a(\phi) := \int_0^{\infty,\mathrm{Riesz}} x^a\, \phi(x)\, dx = \int_0^{\infty,\mathrm{Had}} x^a\, \phi(x)\, dx. \tag{1.17}$$

These coincide with the ordinary integral $\int_0^\infty x^a\, \phi(x)\, dx$ *whenever* $\mathrm{Re}(a) > -1$.

(1) *If* $a \notin -\mathbb{N}$, *then*

$$\tilde{F}_a(\phi) = \frac{(-1)^k}{(a+1)\cdots(a+k)} F_{a+k}(\phi^{(k)}),$$

where k *is any integer such that* $\mathrm{Re}(a+k) > -1$.

(2) *Furthermore, for a positive integer* k

$$\tilde{F}_{-k}(\phi) = \frac{\phi^{(k-1)}(0)}{(k-1)!}\sum_{j=1}^{k-1}\frac{1}{j} - \frac{1}{(k-1)!}\int_0^\infty \log x\, \phi^{(k)}(x)dx,$$

setting the sum over j *equal to zero if* $k = 1$.

This applied to the Schwartz function $\phi(x) = e^{-x}$ confirms the results of Exercises 1.23 and 1.18, which show that Riesz and Hadamard finite part regularisations lead to the same extended Gamma function $\tilde{\Gamma}(-k)$ at nonpositive integers:

$$\begin{aligned} \tilde{\Gamma}(0) := \Gamma^{\mathrm{Had}}(0) = \Gamma^{\mathrm{Riesz}}(0) &= -\gamma \quad \text{if} \quad k = 0, \\ \tilde{\Gamma}(-k) := \Gamma^{\mathrm{Had}}(-k) = \Gamma^{\mathrm{Riesz}}(-k) &= \frac{(-1)^k}{k!}\left[\sum_{j=1}^{k}\frac{1}{j} - \gamma\right] \quad \text{if} \quad k > 0. \end{aligned}$$

Extending a homogeneous distribution $F_a \to \tilde{F}_a$ to negative integers unfortunately has a cost for we lose various properties along the way.

1.5.1. Loss of homogeneity. Recall from (1.1) that F_a is positively homogeneous for $\mathrm{Re}(a) > -1$:

$$F_a(\phi_t) = t^a\, F_a(\phi) \qquad \forall t > 0.$$

As a result, since $\phi_t^{(k)} = t^{-k-1}\phi^{(k)}(t^{-1}\cdot)$, for any $a \notin -\mathbb{N}$ we have

$$\begin{aligned}\tilde{F}_a(\phi_t) &:= \frac{(-1)^k}{(a+1)\cdots(a+k)} F_{a+k}\left(\phi_t^{(k)}\right) \\ &= t^a \tilde{F}_a(\phi),\end{aligned}$$

i.e., $\tilde{F}_a$ is still a homogeneous distribution. However, for $a = -k$ with $k \in \mathbb{N}$ we have

$$\begin{aligned}\tilde{F}_a(\phi_t) &= \frac{1}{(k-1)!}\left(t^{-k}\sum_{j=1}^{k}\frac{\phi^{(k-1)}(0)}{j} - t^{-k-1}\int_0^\infty \log(x)\,\phi^{(k)}(t^{-1}x)\,dx\right) \quad \text{for} \\ &= \frac{1}{(k-1)!}\left(t^{-k}\sum_{j=1}^{k}\frac{\phi^{(k-1)}(0)}{j} - t^{-k}\int_0^\infty \log(tx)\,\phi^{(k)}(x)\,dx\right) \\ &= t^{-k}\left[\tilde{F}_a(\phi) + \frac{\phi^{(k-1)}(0)}{(k-1)!}\log t\right] \\ &= t^{-k}\left[\tilde{F}_a(\phi) + \mathrm{Res}_{a=-k}\tilde{F}_a(\phi)\,\log t\right],\end{aligned}$$

so that the extended distribution is no longer homogeneous. A discrepancy arises with the loss of homogeneity of the extended homogeneous distribution at negative integers. Consequently, $\tilde{F}_a$ is homogeneous whenever $z \mapsto \tilde{F}_{a+z}(\phi)$ is holomorphic at zero.

1.5.2. The extended Gamma function: obstruction to the functional equation. The extended Gamma function $\tilde{\Gamma}$ obeys the following property for $\mathrm{Re}(b) > 0$:

$$\tilde{\Gamma}(b+1) = b\,\tilde{\Gamma}(b), \tag{1.18}$$

but a discrepancy arises since property (1.18) breaks down at nonpositive integers.

EXERCISE 1.25. *For any complex value b, show that*

$$\tilde{\Gamma}(b+1) = b\,\tilde{\Gamma}(b) + \mathrm{Res}_{z=0}\tilde{\Gamma}(b+z).$$

Consequently, $\tilde{\Gamma}$ obeys the functional equation $\tilde{\Gamma}(b+1) = b\,\tilde{\Gamma}(b)$ outside the poles, but at a pole $-k$ in $\mathbb{Z}_{\leq 0}$ we have

$$\tilde{\Gamma}(-k+1) = -k\,\tilde{\Gamma}(-k) + \frac{(-1)^k}{k!}.$$

In both cases investigated in sections 1.5.1 and 1.5.2, the presence of a residue is responsible for an obstruction to the expected property.