1 Introduction

The theory of minimal surfaces in three-dimensional Euclidean space has its roots in the calculus of variations developed by Euler and Lagrange in the 18-th century and in later investigations by Enneper, Scherk, Schwarz, Riemann and Weierstrass in the 19-th century. During the years, many great mathematicians have contributed to this theory. Besides the above mentioned names that belong to the nineteenth century, we find fundamental contributions by Bernstein, Courant, Douglas, Morrey, Morse, Radó and Shiffman in the first half of the last century. Paraphrasing Osserman, most of the activity in minimal surface theory in those days was focused almost exclusively on either Plateau's problem or PDE questions, and the only global result was the negative one of Bernstein's theorem[1].

Much of the modern global theory of complete minimal surfaces in three-dimensional Euclidean space has been influenced by the pioneering work of Osserman during the 1960's. Many of the global questions and conjectures that arose in this classical subject have only recently been addressed. These questions concern analytic and conformal properties, the geometry and asymptotic behavior, and the topology and classification of the images of certain injective minimal immersions $f\colon M \to \mathbb{R}^3$ which are complete in the induced Riemannian metric; we call the image of such a complete, injective, minimal immersion a *complete, embedded minimal surface* in $\mathbb{R}^3$.

[1] This celebrated result by Bernstein [**7**] asserts that the only minimal graphs over the entire plane are graphs given by affine functions, which means that the graphs are planes.

The following classification results solve two of these long standing conjectures[2].

Theorem 1.0.1. *A complete, embedded, simply connected minimal surface in $\mathbb{R}^3$ is a plane or a helicoid.*

Theorem 1.0.2. *Up to scaling and rigid motion, any connected, properly embedded, minimal planar domain in $\mathbb{R}^3$ is a plane, a helicoid, a catenoid or one of the Riemann minimal examples[3]. In particular, for every such surface there exists a foliation of $\mathbb{R}^3$ by parallel planes, each of which intersects the surface transversely in a connected curve which is a circle or a line.*

The proofs of the above theorems depend on a series of new results and theory that have been developed over the last two decades. The purpose of this monograph is two fold. The first is to explain these new global results in a manner accessible to the general mathematical public, and the second is to explain how these results transcend their application to the proofs of Theorems 1.0.1 and 1.0.2 and enhance dramatically our current understanding of the theory, giving rise to new theorems and conjectures, some of which were considered to be almost unapproachable dreams just 15 years ago. The interested reader can also find more detailed history and further results in the following surveys, reports and popular science articles [**3, 13, 32, 38, 47, 48, 77, 78, 80, 82, 87, 125, 132, 137, 184, 201, 212**]. We refer the reader to the excellent graduate texts on minimal surfaces by Dierkes et al [**50, 51**], Lawson [**111**] and Osserman [**190**], and especially see Nitsche's book [**186**] for a fascinating account of the history and foundations of the subject.

Before proceeding, we make a few general comments on the proof of Theorem 1.0.1 which we feel can suggest to the reader a visual idea of what is going on. The most natural motivation for understanding this theorem, Theorem 1.0.2 and other results presented in this survey is to try to answer the following question: *What are the possible shapes of surfaces which satisfy a variational principle and have a given topology?* For instance, if the variational equation expresses the critical points of the area functional, and if the requested topology is the simplest one of a disk, then Theorem 1.0.1 says that the possible shapes for complete examples are the trivial one given by a plane and (after a rotation) an infinite double spiral staircase, which is a visual description of a vertical helicoid. A more precise description of the double spiral staircase nature of a vertical helicoid is that this surface is the union of two infinite-sheeted multi-valued graphs (see Definition 4.1.1 for the notion of a multi-valued graph), which are glued along a vertical axis.

[2] Several authors pointed out to us that Osserman seems to be the first to ask the question about whether the plane and the helicoid were the only embedded, simply connected, complete minimal surfaces. He described this question as potentially the most beautiful extension and explanation of Bernstein's Theorem.

[3] The *Riemann minimal examples* referred to here were discovered by Riemann around 1860. See Chapter 2.5 for further discussion of these surfaces and images of them.

Crucial in the proof of Theorem 1.0.1 are local results of Colding and Minicozzi which describe the structure of compact, embedded minimal disks, as essentially being modeled by one of the above two examples, i.e., either they are graphs or pairs of finitely sheeted multi-valued graphs glued along an "axis". A last key ingredient in the proof of Theorem 1.0.1 is a result on the global aspects of limits of these shapes, which is given in the Lamination Theorem for Disks by Colding and Minicozzi, see Theorem 4.1.3 below.

For the reader's convenience, we include a guide on the next page of how chapters depend on each other, see below for a more detailed explanation of their contents.

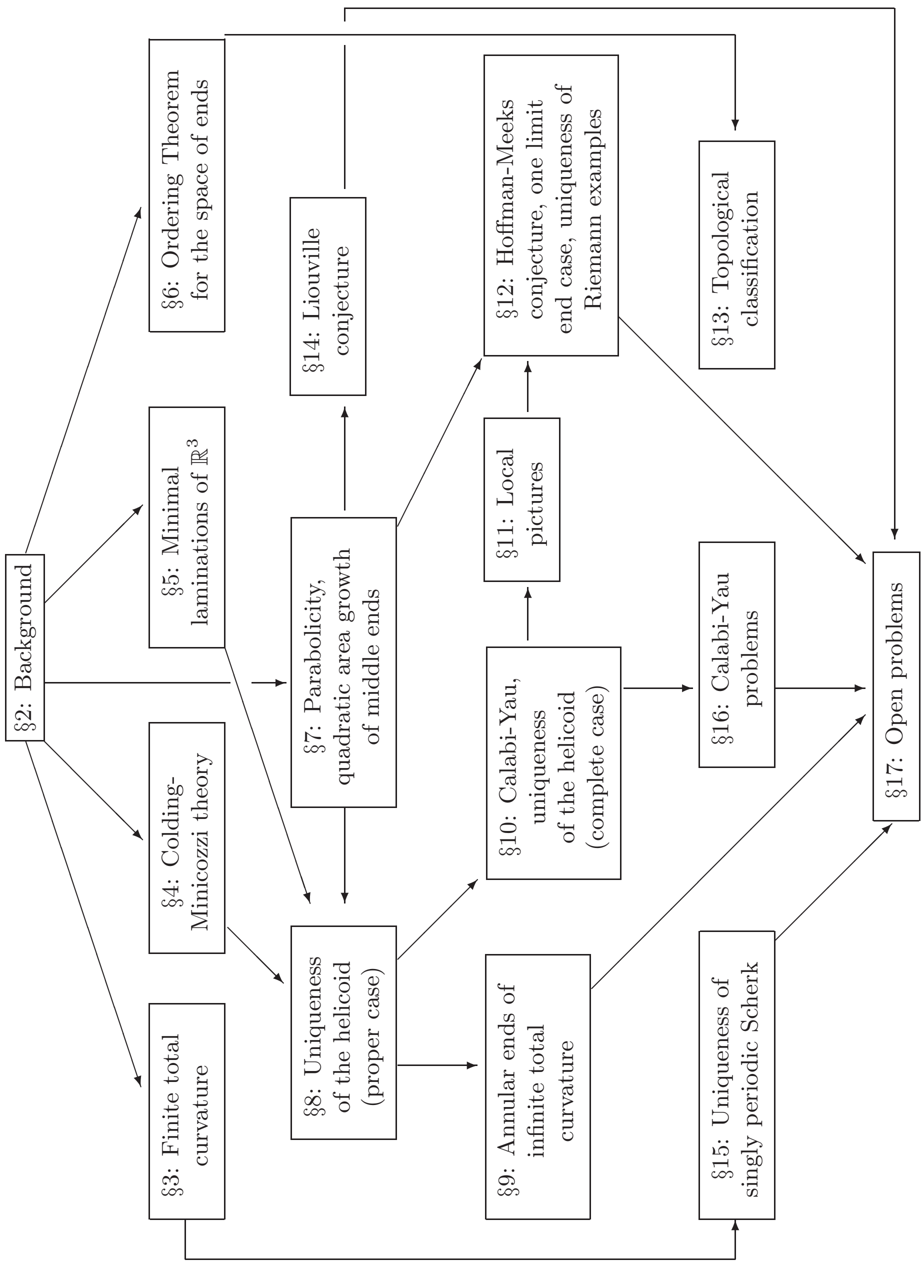
Our survey is organized as follows. We present the main definitions and background material in the introductory Chapter 2. In that chapter we also briefly describe geometrically, as well as analytically, many of the important classical examples of properly embedded minimal surfaces in $\mathbb{R}^3$. As in many other areas in mathematics, understanding key examples in this subject is crucial in obtaining a feeling for the subject, making important theoretical advances and asking the right questions. We believe that before going further, the unacquainted reader to this subject will benefit by taking a few minutes to view and identify the computer graphics images of these surfaces, which appear near the end of Chapter 2.5, and to read the brief historical and descriptive comments related to the individual images.

In Chapter 3, we describe the best understood family of complete embedded minimal surfaces: those with finite topology and more than one end. Recall that a compact, orientable surface is homeomorphic to a connected sum of tori, and the number of these tori is called the genus of the surface. An orientable surface M of finite topology is one which is homeomorphic to a surface of genus $g \in \mathbb{N} \cup \{0\}$ with finitely many points removed, called the *ends* of M. These punctures can be viewed as different ways to escape to infinity on M, and also can be identified with punctured disk neighborhoods of these points; these punctured disk neighborhoods are clearly annuli, hence they are called *annular ends.* The crucial result for minimal surfaces with finite topology and more that one end is Collin's Theorem, valid under the additional assumption of properness (a surface in $\mathbb{R}^3$ is *proper* if each closed ball in $\mathbb{R}^3$ contains a compact portion of the surface with respect to its intrinsic topology). Note that properness implies completeness.

Theorem 1.0.3 (Collin [**41**]). *If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with more than one end, then each annular end of M is asymptotic to the end of a plane or a catenoid. In particular, if M has finite topology and more than one end, then M has finite total Gaussian curvature*[4].

Collin's Theorem reduces the analysis of properly embedded minimal surfaces of finite topology and more than one end in $\mathbb{R}^3$ to complex function theory on compact Riemann surfaces. This reduction then leads to classification results and to interesting topological obstructions, which we include

[4]See equation (2.4) for the definition of total curvature.


§2: Background
§3: Finite total curvature
§4: Colding-Minicozzi theory
§5: Minimal laminations of $\mathbb{R}^3$
§6: Ordering Theorem for the space of ends
§7: Parabolicity, quadratic area growth of middle ends
§8: Uniqueness of the helicoid (proper case)
§9: Annular ends of infinite total curvature
§10: Calabi-Yau, uniqueness of the helicoid (complete case)
§11: Local pictures
§12: Hoffman-Meeks conjecture, one limit end case, uniqueness of Riemann examples
§13: Topological classification
§14: Liouville conjecture
§15: Uniqueness of singly periodic Scherk
§16: Calabi-Yau problems
§17: Open problems

in Chapter 3 as well. At the end of Chapter 3, we discuss several different methods for constructing properly embedded minimal surfaces of finite topology and for describing their moduli spaces.

In Chapter 4, we present an overview of some results concerning the geometry, compactness and regularity of limits of what are called *locally simply connected* sequences of minimal surfaces. These results are central in the proofs of Theorems 1.0.1 and 1.0.2 and are taken from a series of papers and surveys by Colding and Minicozzi [**26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40**]. We place special emphasis on the case of disks, explaining with details the strategy of proof of the two main results of this theory: the limit lamination theorem for disks (Theorem 4.1.3) and the 1-sided curvature estimate (Theorem 4.1.5). Since the purpose of this monograph is not to give a exhaustive treatment of Colding-Minicozzi theory, we do not include complete proofs of the steps in the above strategy, but we still hope the reader will find it useful to have a clear idea of the flow of arguments needed in the proofs of these crucial results, which run along not less than five highly demanding papers [**29, 33, 34, 35, 36**].

In Chapters 4 and 5, we define and develop the notion of *minimal lamination,* which is a natural limit object for a locally simply connected sequence of embedded minimal surfaces. The reader not familiar with the subject of minimal laminations should think about the closure of an embedded, noncompact geodesic γ on a complete Riemannian surface, a topic which has been widely covered in the literature (see e.g., Bonahon [**10**]). The closure of such a geodesic γ is a geodesic lamination $\mathcal{L}$ of the surface. When γ has no accumulation points, then it is proper and it is the unique leaf of $\mathcal{L}$. Otherwise, there pass complete, embedded, pairwise disjoint geodesics through the accumulation points, and these geodesics together with γ form the leaves of the geodesic lamination $\mathcal{L}$. The similar result is true for a complete, embedded minimal surface of locally bounded curvature (i.e., whose Gaussian curvature is bounded in compact extrinsic balls) in a Riemannian three-manifold [**158**]. We include in Chapter 5 a discussion of the *Structure Theorem for Minimal Laminations of* $\mathbb{R}^3$ (Meeks-Rosenberg [**158**] and Meeks-Pérez-Ros [**148**]).

In Chapter 6 we explain the *Ordering Theorem* of Frohman and Meeks [**65**] for the linear ordering of the ends of any properly embedded minimal surface with more than one end; this is a fundamental result for our purposes of classifying embedded, minimal planar domains in Theorem 1.0.2. We note that the results in this chapter play an important role in proving Theorems 1.0.1 and 1.0.2. For instance, they are useful when demonstrating that every properly embedded, simply connected minimal surface in $\mathbb{R}^3$ is conformally $\mathbb{C}$; this conformal property is a crucial ingredient in the proof of the uniqueness of the helicoid. Also, a key point in the proof of Theorem 1.0.2 is the fact that a properly embedded minimal surface in $\mathbb{R}^3$ cannot have more than two limit ends (see Definition 2.7.2 for the concept of limit end), which is a consequence of the results in Chapter 7.

Chapter 7 is devoted primarily to conformal questions on minimal surfaces. Roughly speaking, this means to study the conformal type of a given minimal surface (rather than its Riemannian geometry) considered as a Riemann surface, i.e., an orientable surface in which one can find an atlas by charts with holomorphic change of coordinates. To do this, we first define the notion of *universal superharmonic function* for domains in $\mathbb{R}^3$ and give examples. Next we explain how to use these functions to understand the conformal structure of properly immersed minimal surfaces in $\mathbb{R}^3$. We then follow the work of Collin, Kusner, Meeks and Rosenberg in [**42**] to analyze the asymptotic geometry and conformal structure of properly embedded minimal surfaces with an infinite number of ends.

In Chapter 8, we apply the results in the previous chapters to explain the main steps in the proof of Theorem 1.0.1 after replacing completeness by the stronger hypothesis of properness. This theorem together with Theorem 1.0.3 above, Theorem 1.0.5 below and with results by Bernstein and Breiner [**5**] or by Meeks and Pérez [**135**] lead to a complete understanding of the asymptotic geometry of any annular end of a complete, embedded minimal surface with finite topology in $\mathbb{R}^3$; namely, the annular end must be asymptotic to an end of a plane, catenoid or helicoid.

A discussion of the proof of Theorem 1.0.4 in the case of positive genus and a more general classification result of complete embedded minimal annular ends with compact boundary and infinite total curvature in $\mathbb{R}^3$ can be found in Chapter 9.

Theorem 1.0.4. *Every properly embedded, non-planar minimal surface in $\mathbb{R}^3$ with finite genus and one end has the conformal structure of a compact Riemann surface Σ minus one point, can be analytically represented by meromorphic data on Σ and is asymptotic to a helicoid. Furthermore, when the genus of Σ is zero, the only possible examples are helicoids.*

In Chapter 10, we complete our sketch of the proof of Theorem 1.0.1 by allowing the surface to be complete rather than proper. The problem of understanding the relation between the intrinsic notion of completeness and the extrinsic one of properness is known as the *embedded Calabi-Yau problem* in minimal surface theory; see [**14, 23, 177, 244, 245**] for the original Calabi-Yau problem in the complete immersed setting. Along these lines, we also describe the powerful *Minimal Lamination Closure Theorem* (Meeks-Rosenberg [**159**], Theorem 10.1.2 below), the $C^{1,1}$*-Regularity Theorem* for the singular set of convergence of a Colding-Minicozzi limit minimal lamination (Meeks [**131**] and Meeks-Weber [**162**]), the *Finite Genus Closure Theorem* (Meeks-Pérez-Ros [**143**]) and the *Lamination Metric Theorem* (Meeks [**133**]). The Minimal Lamination Closure Theorem (Theorem 10.1.2) is a refinement of the results and techniques used by Colding and Minicozzi [**39**] to prove the following deep result (see Chapter 10 where we deduce Theorem 1.0.5 from Theorem 10.1.2).

Theorem 1.0.5 (Colding, Minicozzi [**39**]). *A complete, embedded minimal surface of finite topology in* $\mathbb{R}^3$ *is properly embedded.*

With Theorem 1.0.5 in hand, we note that the hypothesis of properness for M in the statements of Theorem 1.0.4, of Collin's Theorem 1.0.3 and of the results in Chapter 9, can be replaced by the weaker hypothesis of completeness for M. Hence, Theorem 1.0.1 follows from Theorems 1.0.4 and 1.0.5.

In Chapter 11, we examine how the new theoretical results in the previous chapters lead to deep global results in the classical theory and to a general understanding of the local geometry of any complete, embedded minimal surface M in a homogeneously regular three-manifold (see Definition 2.8.5 below for the concept of homogeneously regular three-manifold). This local description is given in two local picture theorems by Meeks, Pérez and Ros [**142, 143**], each of which describes the local extrinsic geometry of M near points of concentrated curvature or of concentrated topology. The first of these results is called the *Local Picture Theorem on the Scale of Curvature,* and the second is known as the *Local Picture Theorem on the Scale of Topology.* In order to understand the second local picture theorem, we define and develop in Chapter 10 the important notion of a *parking garage structure* on $\mathbb{R}^3$, which is one of the possible limiting local pictures in the topological setting. Also discussed here is the *Fundamental Singularity Conjecture* for minimal laminations and the *Local Removable Singularity Theorem* of Meeks, Pérez and Ros for certain possibly singular minimal laminations of a three-manifold [**143**]. Global applications of the Local Removable Singularity Theorem to the classical theory are also discussed here. The most important of these applications are the *Quadratic Curvature Decay Theorem* and the *Dynamics Theorem* for properly embedded minimal surfaces in $\mathbb{R}^3$ by Meeks, Pérez and Ros [**143**]. An important consequence of the Dynamics Theorem and the Local Picture Theorem on the Scale of Curvature is that every complete, embedded minimal surface in $\mathbb{R}^3$ with infinite topology has a surprising amount of internal dynamical periodicity.

In Chapter 12, we present some of the results of Meeks, Pérez and Ros [**140, 141, 145, 148, 149**] on the geometry of complete, embedded minimal surfaces of finite genus with possibly an infinite number of ends. We first explain in Theorem 1.0.6 the existence of a bound on the number of ends of a complete, embedded minimal surface with finite total curvature solely in terms of the genus. So far, this is the best result towards the solution of the so called *Hoffman-Meeks conjecture:*

> A connected surface of finite topology with genus g and r ends, $r > 2$, can be properly minimally embedded in $\mathbb{R}^3$ if and only if $r \leq g + 2$.

By Theorem 1.0.3 and the Jorge-Meeks formula displayed in equation (2.10), a complete, embedded minimal surface M with genus $g \in \mathbb{N} \cup \{0\}$ and a finite number $r \geq 2$ of ends, has total finite total curvature $-4\pi(g + r - 1)$

and so the next theorem also gives a lower bound estimate on the total curvature in terms of its genus g. Then, by a theorem of Tysk [**235**], the index of stability of complete of a complete embedded minimal surface in $\mathbb{R}^3$ can be estimated in terms of the total curvature of the surface; for the definition of index of stability, see Definition 2.8.2. Closely related to Tysk's theorem is the beautiful result of Fischer-Colbrie [**61**] that a complete, orientable minimal surface with compact (possibly empty) boundary in $\mathbb{R}^3$ has finite index of stability if and only if it has finite total curvature.

Theorem 1.0.6 (Meeks, Pérez, Ros [**140**])**.** *For every non-negative integer g, there exists an integer $e(g)$ such that if $M \subset \mathbb{R}^3$ is a complete, embedded minimal surface of finite topology with genus g, then the number of ends of M is at most $e(g)$. Furthermore, if M has more than one end, then its finite index of stability is bounded solely as a function of the finite genus of M.*

In Chapter 12, we also describe the essential ingredients of the proof of Theorem 1.0.2 on the classification of properly embedded minimal planar domains in $\mathbb{R}^3$. At the end of this chapter, we briefly explain a structure theorem by Colding and Minicozzi for non-simply connected embedded minimal surfaces of prescribed genus. This result deals with limits of sequences of compact, embedded minimal surfaces M_n with boundaries diverging to infinity and uniformly bounded genus, and roughly asserts that there are only three allowed shapes for the surfaces M_n: either they converge smoothly (after choosing a subsequence) to a properly embedded, non-flat minimal surface of bounded Gaussian curvature and restricted geometry, or the Gaussian curvatures of the M_n blow up in a neighborhood of some points in space; in this case, the M_n converge to a lamination $\mathcal{L}$ of $\mathbb{R}^3$ by planes away from a non-empty, closed singular set of convergence $S(\mathcal{L})$, and the limiting geometry of the M_n consists either of a parking garage structure on $\mathbb{R}^3$ with two oppositely handed columns (see Chapter 10.2 for this notion), or small necks form in M_n around every point in $S(\mathcal{L})$.

The next goal of Chapter 12 is to focus on the proof of Theorem 1.0.2. In this setting of infinitely many ends for a connected open surface M, the set $\mathcal{E}(M)$ of ends has a natural topological structure of a compact, totally disconnected metric space with infinitely many points, see Definition 2.7.1 and the paragraph below it. This set $\mathcal{E}(M)$ has accumulation points, which lead to the notion of *limit end.* The first step in the proof of Theorem 1.0.2 is to notice that the number of limit ends of a properly embedded minimal surface in $\mathbb{R}^3$ is at most 2; this result appears as Theorem 7.3.1. Then one rules out the existence of a properly embedded minimal planar domain with just one limit end: this is the purpose of Theorem 12.2.1. At that point, we are ready to finish the proof of Theorem 1.0.2. This proof breaks into two parts, the first of which is to demonstrate a quasi-periodicity property coming from curvature estimates for any surface satisfying the hypotheses of Theorem 1.0.2 (see [**148**]), and the second one is based on the Shiffman function and its relation to integrable systems theory through the Korteweg-de Vries (KdV) equation, see the sketch of the proof of Theorem 12.3.1 below for details.

In Chapter 13, we explain how to topologically classify all properly embedded minimal surfaces in $\mathbb{R}^3$, through Theorem 1.0.7 below by Frohman and Meeks [**67**]. This result depends on their Ordering Theorem for the ends of a properly embedded minimal surface $M \subset \mathbb{R}^3$, which is discussed in Chapter 6. The Ordering Theorem states that there is a natural linear ordering on the set $\mathcal{E}(M)$ of ends of M. The ends of M which are not extremal in this ordering are called *middle ends* and have a parity which is even or odd (see Theorem 7.3.1 for the definition of this parity).

Theorem 1.0.7 (Topological Classification Theorem, Frohman, Meeks)**.** *Two properly embedded minimal surfaces in* $\mathbb{R}^3$ *are ambiently isotopic*[5] *if and only if there exists a homeomorphism between the surfaces that preserves both the ordering of their ends and the parity of their middle ends.*

At the end of Chapter 13, we present an explicit cookbook-type recipe for constructing a smooth (non-minimal) surface $\widehat{M}$ that represents the ambient isotopy class of a properly embedded minimal surface M in $\mathbb{R}^3$ with prescribed topological invariants.

Most of the classical periodic, properly embedded minimal surfaces $M \subset \mathbb{R}^3$ have infinite genus and one end, and so, the topological classification of non-compact surfaces shows that any two such surfaces are homeomorphic. For example, all non-planar doubly and triply-periodic minimal surfaces have one end and infinite genus [**17**], as do many singly-periodic minimal examples such as any of the classical singly-periodic Scherk minimal surfaces $\mathcal{S}_\theta$, for any $\theta \in (0, \frac{\pi}{2}]$ (see Chapter 2.5 for a description of these surfaces and a picture for $S_{\frac{\pi}{2}}$). Since there are no middle ends for these surfaces and they are homeomorphic, Theorem 1.0.7 demonstrates that there exists an orientation preserving diffeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $f(\mathcal{S}_{\frac{\pi}{2}}) = M$, where M is an arbitrarily chosen, properly embedded minimal surface with infinite genus and one end. Also, given two homeomorphic, properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$, we can always choose a diffeomorphism between the surfaces that preserves the ordering of the ends. Thus, since the parity of an annular end is always odd, one obtains the following classical topological uniqueness theorem of Meeks and Yau as a corollary.

Corollary 1.0.8 (Meeks, Yau [**168**])**.** *Two homeomorphic, properly embedded minimal surfaces of finite topology in* $\mathbb{R}^3$ *are ambiently isotopic.*

One of the important open questions in classical minimal surface theory asks whether a positive harmonic function on a properly embedded minimal surface in $\mathbb{R}^3$ must be constant; this question was formulated as a conjecture by Meeks [**132**] and is called the *Liouville Conjecture* for properly embedded

[5] See Definition 13.0.1.

minimal surfaces. The Liouville Conjecture is known to hold for many classes of minimal surfaces, which include all properly embedded minimal surfaces of finite genus and all of the classical examples discussed in Chapter 2.5. In Chapter 14, we present some recent positive results on this conjecture and on the related questions of *recurrence and transience of Brownian motion* for properly embedded minimal surfaces.

In Chapter 15, we explain a classification result which is closely related to the procedure of *minimal surgery;* in certain cases, Kapouleas [**99, 100**] has been able to approximate two embedded, transversely intersecting, minimal surfaces $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ by a sequence of embedded minimal surfaces $\{M_n\}_n$ that converges to $\Sigma_1 \cup \Sigma_2$. He obtains each M_n by first sewing in necklaces of singly-periodic Scherk minimal surfaces along $\Sigma_1 \cap \Sigma_2$ and then applies the implicit function theorem to get a nearby embedded minimal surface (see Chapter 3.2 for a description of the desingularization procedure of Kapouleas). The *Scherk Uniqueness Conjecture* (Meeks [**132**]) states that the only connected, properly embedded minimal surfaces in $\mathbb{R}^3$ with quadratic area growth[6] constant 2π (equal to the quadratic area growth constant of two planes) are the catenoid and the family $\mathcal{S}_\theta$, $\theta \in (0, \frac{\pi}{2}]$, of singly-periodic Scherk minimal surfaces. The validity of this conjecture would essentially imply that the only way to desingularize two embedded, transversely intersecting, minimal surfaces is by way of the *Kapouleas minimal surgery construction.* In Chapter 15, we outline the proof of the following important partial result on this conjecture.

Theorem 1.0.9 (Meeks, Wolf [**165**])**.** *A connected, properly embedded minimal surface in $\mathbb{R}^3$ with infinite symmetry group and quadratic area growth constant 2π must be a catenoid or one of the singly-periodic Scherk minimal surfaces.*

In Chapter 16, we discuss what are usually referred to as the *Calabi-Yau problems* for complete minimal surfaces in $\mathbb{R}^3$. These problems arose from questions asked by Calabi [**15**] and by Yau (see page 212 in [**23**] and problem 91 in [**244**]) concerning the existence of complete, immersed minimal surfaces which are constrained to lie in a given region of $\mathbb{R}^3$, such as in a bounded domain. Various aspects of the Calabi-Yau problems constitute an active field of research with an interesting mix of positive and negative results. We include here a few recent fundamental advances on this problem.

The final chapter of this survey is devoted to a discussion of the main outstanding conjectures in the subject. Many of these problems are motivated by the recent advances in classical minimal surface theory reported on in previous chapters. Research mathematicians, not necessarily schooled in differential geometry, are likely to find some of these problems accessible to attack with methods familiar to them. We invite anyone with an inquisitive

[6] A properly immersed minimal surface M in $\mathbb{R}^3$ has *quadratic area growth* if $\lim_{R\to\infty} R^{-2}\text{Area}(M \cap \mathbb{B}(R)) := A_M < \infty$, where $\mathbb{B}(R) = \{x \in \mathbb{R}^3 \mid \|x\| < R\}$. We call the number A_M the *quadratic area growth constant* of M in this case.

mind and strong geometrical intuition to join in the game of trying to solve these intriguing open problems.

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Outstanding problems and conjectures

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In this last chapter, we present many of the fundamental conjectures in classical minimal surface theory. Hopefully, our presentation and discussion of these problems will speed up their solution and stimulate further interest in this beautiful subject. We have listed in the statement of each conjecture the principal researchers to whom the conjecture might be attributed. We consider all of these problems to be of fundamental importance and we note that they are not listed in order of significance, difficulty or presumed depth.

Some of these problems and others appear in [**129**] or in [**132**], along with further discussions. Also see the first author's 1978 book [**123**] for a long list of conjectures in the subject, some of whose solutions we have discussed in our survey presented here.

Conjecture 17.0.14 (Convex Curve Conjecture, Meeks)**.** *Two convex Jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.*

There are some partial results on the Convex Curve Conjecture, under the assumption of some symmetry on the curves or more generally, under some assumptions about the flux of the compact minimal surface along its boundary components (see Meeks and White [**163**], Ros [**208**] and Schoen [**220**]). Also, the results of [**163, 164**] indicate that the Convex Curve Conjecture probably holds in the more general case where the two convex planar curves do not necessarily lie in parallel planes, but rather lie on the boundary of their convex hull; in this case, the planar Jordan curves are called *extremal.* Results by Ekholm, White and Wienholtz [**54**] imply that every compact, orientable minimal surface that arises as a counterexample to the Convex Curve Conjecture is embedded. Based on work in [**54**], Tinaglia [**226**] proved that for a fixed pair of extremal, convex planar curves, there is a bound on the genus of such a minimal surface. More generally, Meeks [**123**] has conjectured that if $\Gamma = \{\alpha, \beta_1, \beta_2, .., \beta_n\} \subset \mathbb{R}^3$ is a finite collection of planar, convex, simple closed curves with α in one plane and such that $\beta_1, \beta_2, ..., \beta_n$ bound a pairwise disjoint collection of disks in a parallel plane, then any compact minimal surface with boundary Γ must have genus zero.

Conjecture 17.0.15 (4π-Conjecture, Meeks, Yau, Nitsche)**.** *If Γ is a simple closed curve in $\mathbb{R}^3$ with total curvature at most 4π, then Γ bounds a unique compact, orientable, branched minimal surface and this unique minimal surface is an embedded disk.*

As partial results to this conjecture, it is worth mentioning that Nitsche [**185**] proved that a regular analytic Jordan curve in $\mathbb{R}^3$ whose total curvature is at most 4π bounds a unique minimal disk; we have already mentioned in Theorem 2.9.2 above that Meeks and Yau [**167**] demonstrated the conjecture if Γ is a C^2-extremal curve (they even allowed the minimal surface spanned by Γ to be non-orientable). Concerning this weakening of Conjecture 17.0.15 by removing the orientability assumption on the minimal surface spanning Γ, we mention the following generalized conjecture due to Ekholm, White and Wienholtz [**54**]:

Besides the unique minimal disk given by Nitsche's Theorem [**185**]*, only one or two Möbius strips can occur; and if the total curvature of Γ is at most 3π, then there are no such Möbius strip examples.*

The classical Euclidean isoperimetric inequality states that the inequality $4\pi A \leq L^2$ holds for the area A of a compact subdomain of $\mathbb{R}^2$ with boundary length L, with equality if and only if Ω is a round disk. The same inequality is known to hold for compact minimal surfaces with boundary in $\mathbb{R}^3$ with at most two boundary components (Reid [**203**], Osserman and Schiffer [**191**], Li, Schoen and Yau [**114**], see also Osserman's survey paper [**189**]).

Conjecture 17.0.16 (Isoperimetric Inequality Conjecture)**.** *Every connected, compact minimal surface Ω with boundary in $\mathbb{R}^3$ satisfies*

$$4\pi A \leq L^2,$$

where A is the area of Ω and L is the length of its boundary. Furthermore, equality holds if and only if Ω is a planar round disk.

Passing to a different conjecture, Gulliver and Lawson [**72**] proved that if Σ is an orientable, stable minimal surface with compact boundary that is properly embedded in the punctured unit ball $\mathbb{B} - \{\vec{0}\}$ of $\mathbb{R}^3$, then its closure is a compact, embedded minimal surface. If Σ is not stable, then the corresponding result is not known. Nevertheless, Meeks, Pérez and Ros [**143, 149**] proved that every properly embedded minimal surface M in $\mathbb{B} - \{\vec{0}\}$ with $\partial M \subset \mathbb{S}^2$ extends across the origin provided that $K|R|^2$ is bounded on M, where K is the Gaussian curvature function of M and

$R^2 = x_1^2 + x_2^2 + x_3^2$ (Theorem 2.8.6 implies that if M is stable, then $K|R|^2$ is bounded). In fact, the boundedness of $|K|R^2$ is equivalent to the removability of the singularity of M at the origin, and this removable singularity result holds true if we replace $\mathbb{R}^3$ by an arbitrary Riemannian three-manifold (Theorem 11.0.6). Meeks, Pérez and Ros have conjectured that this removable singularity result holds true if we replace the origin by any closed set in $\mathbb{R}^3$ with zero 1-dimensional Hausdorff measure and the surface is assumed to be properly embedded in the complement of this set, see Conjecture 17.0.18 below. The following conjecture can be proven to hold for any minimal surface of finite topology (in fact, with finite genus).

Conjecture 17.0.17 (Isolated Singularities Conjecture, Gulliver, Lawson). *The closure of a properly embedded minimal surface with compact boundary in the punctured ball $\mathbb{B} - \{\vec{0}\}$ is a compact, embedded minimal surface.*

In Chapter 11, we saw how the Local Removable Singularity Theorem 11.0.6 is a cornerstone for the proof of the Quadratic Curvature Decay Theorem 11.0.11 and the Dynamics Theorem 11.0.13, which illustrates the usefulness of removable singularities results.

The most ambitious conjecture about removable singularities for minimal surfaces is the following one, which deals with laminations instead of with surfaces. Recall that Examples I, II and III in Chapter 4.2 indicate that one cannot expect the next conjecture to be true if we replace $\mathbb{R}^3$ by $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ or by a ball in $\mathbb{R}^3$.

Conjecture 17.0.18 (Fundamental Singularity Conjecture, Meeks, Pérez, Ros). *If $A \subset \mathbb{R}^3$ is a closed set with zero 1-dimensional Hausdorff measure and $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - A$, then $\mathcal{L}$ extends to a minimal lamination of $\mathbb{R}^3$.*

In Chapter 8, we mentioned that a key part of the proof of the uniqueness of the helicoid by Meeks and Rosenberg relies on a finiteness result for the number of components of a minimal graph over a proper domain of $\mathbb{R}^2$ with zero boundary values. More precisely, they proved that if $\mathcal{D}$ is a proper, possibly disconnected domain of $\mathbb{R}^2$ and $u\colon \mathcal{D} \to \mathbb{R}$ is a solution of the minimal surface equation (2.1) in $\mathcal{D}$ with zero boundary values and bounded gradient, then $\mathcal{D}$ has at most a finite number of components where u is nonzero. This technical property can be viewed as an important partial result in the direction of the solution of the following conjecture, made by Meeks a number of years earlier.

Conjecture 17.0.19 (Connected Graph Conjecture, Meeks). *A minimal graph in $\mathbb{R}^3$ with zero boundary values over a proper, possibly disconnected domain in $\mathbb{R}^2$ can have at most two non-planar components. Furthermore, if the graph also has sublinear growth, then such a graph with no planar components is connected.*

There are several partial results related to Conjecture 17.0.19. Consider a proper, possibly disconnected domain $\mathcal{D}$ in $\mathbb{R}^2$ and a solution $u\colon \mathcal{D} \to \mathbb{R}$ of the minimal surface equation with zero boundary values, such that u is non-zero on each component of $\mathcal{D}$. In 1981, by Mikljukov [**170**] proved that if each component of $\mathcal{D}$ is simply connected with a finite number of boundary components, then $\mathcal{D}$ has at most three components (in fact, with current tools, it can be shown that his method applies to the case that $\mathcal{D}$ has finitely generated first homology group [**171**]). Earlier, Nitsche observed [**184**] that no component of $\mathcal{D}$ can be contained in a proper wedge (angle less than π). Collin [**41**] proved that at most one component of $\mathcal{D}$ can lie in any given half-plane. Spruck [**225**] demonstrated that under the assumption of sublinear growth in a suitably strong sense, $\mathcal{D}$ has at most two components. Without any assumption of the growth of the minimal graph, Li and Wang [**113**] proved that the number of disjointly supported minimal graphs with zero boundary values over an open subset of $\mathbb{R}^n$ is at most $(n+1)2^{n+1}$. Later, Tkachev [**227**] improved this exponential bound by a polynomial one, and in the case $n = 2$, he obtained that the number of disjointly supported minimal graphs is at most three. Also, Weitsman [**239**] has some related results that suggest that if $\mathcal{D}$ has finitely generated first homology group and u has sublinear growth, then the number of components of $\mathcal{D}$ should be at most one. We refer the reader to the end of his paper [**238**], where he discusses several interesting unsolved problems concerning the growth of u defined on a proper domain contained in a half-plane.

In the discussion of the conjectures that follow, it is helpful to fix some notation for certain classes of complete embedded minimal surfaces in $\mathbb{R}^3$.

- Let $\mathcal{C}$ be the space of connected, $\mathcal{C}$omplete, embedded minimal surfaces.
- Let $\mathcal{P} \subset \mathcal{C}$ be the subspace of $\mathcal{P}$roperly embedded surfaces.
- Let $\mathcal{M} \subset \mathcal{P}$ be the subspace of surfaces with $\mathcal{M}$ore than one end.

Conjecture 17.0.20 (Finite Topology Conjecture I, Hoffman, Meeks)**.** *An orientable surface M of finite topology with genus g and r ends, $r \neq 0, 2$, occurs as a topological type of a surface in $\mathcal{C}$ if and only if $r \leq g + 2$.*

See [**80, 85, 230, 237**], the discussion in Chapter 3.2 (the method of Weber and Wolf) and Chapter 12 for partial existence results which seem to indicate that the existence implication in the Finite Topology Conjecture holds when $r > 2$. Recall that Theorem 1.0.6 insures that for each positive genus g, there exists an upper bound $e(g)$ on the number of ends of an $M \in \mathcal{M}$ with finite topology and genus g. Hence, the non-existence implication in Conjecture 17.0.20 will be proved if one can show that $e(g)$ can be taken as $g + 2$. Concerning the case $r = 2$, Theorems 1.0.3 and 3.1.1 imply that the only examples in $\mathcal{M}$ with finite topology and two ends are catenoids. Also, by Theorems 1.0.3 and 3.1.2, if M has finite topology, genus zero and at least two ends, then M is a catenoid.

On the other hand, one of the central results in this monograph (Theorem 1.0.1) characterizes the helicoid among complete, embedded, non-flat minimal surfaces in $\mathbb{R}^3$ with genus zero and one end. Concerning one-ended surfaces in $\mathcal{C}$ with finite positive genus, first note that all these surfaces are proper by Theorem 1.0.5. Furthermore, every example $M \in \mathcal{P}$ of finite positive genus and one end has a special analytic representation on a once punctured compact Riemann surface, as follows from the works of Bernstein and Breiner [**5**] and Meeks and Pérez [**135**], see Theorem 1.0.4. In fact, these authors showed that any such minimal surface has *finite type*[1] and is asymptotic to a helicoid. This finite type condition could be used to search computationally for possible examples of genus-g helicoids, $g \in \mathbb{N}$. Along these lines, we have already mentioned the rigorous proofs by Hoffman, Weber and Wolf [**89**] and by Hoffman and White [**92**] of existence of a genus-one helicoid, a mathematical fact that was earlier computationally indicated by Hoffman, Karcher and Wei [**82**]. For genera $g = 2, 3, 4, 5, 6$, there are numerical existence results [**8, 9, 218, 228**] as mentioned in Chapter 2.5, see also the last paragraph in Chapter 8. All these facts motivate the next conjecture, which appeared in print for the first time in the paper [**158**] by Meeks and Rosenberg, although several versions of it as questions were around a long time before appearing in [**158**]. A step in the proof of the next conjecture in the case of genus one is the recent result of Bernstein and Breiner [**6**] that states that every genus 1 helicoid has an axis of rotational symmetry.

Conjecture 17.0.21 (Finite Topology Conjecture II, Meeks, Rosenberg)**.** *For every non-negative integer g, there exists a unique non-planar $M \in \mathcal{C}$ with genus g and one end.*

The Finite Topology Conjectures I and II together propose the precise topological conditions under which a non-compact orientable surface of finite topology can be properly minimally embedded in $\mathbb{R}^3$. What about the case where the non-compact orientable surface M has infinite topology? In this case, either M has infinite genus or M has an infinite number of ends. By Theorem 7.3.1, such an M must have at most two limit ends. Theorem 12.2.1 states that such an M cannot have one limit end and finite genus. We claim that these restrictions are the only ones.

Conjecture 17.0.22 (Infinite Topology Conjecture, Meeks)**.** *A non-compact, orientable surface of infinite topology occurs as a topological type of a surface in $\mathcal{P}$ if and only if it has at most one or two limit ends, and when it has one limit end, then its limit end has infinite genus.*

Recently, Traizet [**229**] constructed a properly embedded minimal surface with infinite genus and one limit end, all whose simple ends are annuli and whose Gaussian curvature function is unbounded. In a closely related paper, Morabito and Traizet [**174**] constructed a properly embedded minimal surface with two limit ends, one of which has genus zero and the other with infinite genus, such that all of its middle ends are annuli. These results support the above conjecture.

[1] See Definition 9.2.2 for the concept of minimal surface of finite type.

We now discuss two conjectures related to the underlying conformal structure of a minimal surface.

Conjecture 17.0.23 (Liouville Conjecture, Meeks)**.** *If $M \in \mathcal{P}$ and $h\colon M \to \mathbb{R}$ is a positive harmonic function, then h is constant.*

The above conjecture is closely related to work in [**42, 150, 160**]. For example, from the discussion in Chapters 7 and 12, we know that if $M \in \mathcal{P}$ has finite genus or two limit ends, then M is recurrent, which implies M satisfies the Liouville Conjecture. From results in [**150**], Chapter 14, we know the conjecture holds for all of the classical examples listed in Chapter 2.5. We also remark that Neel [**180**] proved that if a surface $M \in \mathcal{P}$ has bounded Gaussian curvature, then M does not admit non-constant bounded harmonic functions. A related conjecture is the following one:

Conjecture 17.0.24 (Multiple-End Recurrency Conjecture, Meeks)**.** *If $M \in \mathcal{M}$, then M is recurrent.*

Assuming that one can prove the last conjecture, the proof of the Liouville Conjecture would reduce to the case where $M \in \mathcal{P}$ has infinite genus and one end. Note that in this setting, a surface could satisfy Conjecture 17.0.23 while at the same time being transient. For example, every doubly or triply-periodic minimal surface with finite topology quotient satisfies the Liouville Conjecture (Theorem 14.0.8), and these minimal surfaces are never recurrent (Theorem 14.0.7). On the other hand, every doubly or triply-periodic minimal surface has exactly one end (Callahan, Hoffman and Meeks [**17**]), which implies that the assumption in Conjecture 17.0.24 that $M \in \mathcal{M}$, not merely $M \in \mathcal{P}$, is a necessary one. It should be also noted that the previous two conjectures need the hypothesis of global embeddedness, since there exist properly immersed minimal surfaces with two embedded ends and which admit bounded non-constant harmonic functions [**42**].

An open proper subdomain Ω with compact boundary of a non-compact Riemannian manifold M' is called *massive* if there exists a bounded subharmonic function $v\colon M' \to [0,\infty)$ such that $v = 0$ in $M' - \Omega$ and $\sup_\Omega v > 0$. An end $e \in \mathcal{E}(M')$ of M' is called *massive* if every open, proper subdomain $\Omega \subset M'$ with compact boundary that represents e is massive, see Grigor'yan [**71**]. Theorem 5.1 in [**71**] implies that M' has a massive end if and only if M' is transient. If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface, then M can have at most one massive end (if M has two massive ends e_1, e_2, then arguments in [**65**] imply that there exist end representatives E_1, E_2 with compact boundary of e_1, e_2, and an end of a plane or of a catenoid that separates $E_1 - \mathbb{B}(R)$ from $E_2 - \mathbb{B}(R)$, where $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$ and $R > 0$ is sufficiently large; in this situation, the work by Collin, Kusner, Meeks and Rosenberg [**42**] insures that at least one of E_1, E_2 is a parabolic surface with boundary, which easily contradicts massiveness). We now state a basic conjecture, due to Meeks, Pérez and Ros [**141**], which concerns the relationship between the Liouville Conjecture and transience for a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end, as well as two unanswered structure conjectures for the ends of properly embedded minimal surfaces which arise from [**42**].

Conjecture 17.0.25. *Let $M \in \mathcal{M}$ with horizontal limit tangent plane at infinity. Then:*

1. *M has a massive end if and only if it admits a non-constant, positive harmonic function* **(Massive End Conjecture)***.*
2. *Any proper, one-ended representative E with compact boundary for a middle end of M has vertical flux* **(Middle End Flux Conjecture)***.*
3. *Suppose that there exists a half-catenoid $\mathcal{C}$ with negative logarithmic growth in $\mathbb{R}^3 - M$. Then, any proper subdomain of M that only represents ends that lie below $\mathcal{C}$ (in the sense of the Ordering Theorem* [**65**]*) has quadratic area growth* **(Quadratic Area Growth Conjecture)***.*

Conjecture 17.0.26 (Isometry Conjecture, Choi, Meeks, White)**.** *If $M \in \mathcal{C}$, then every intrinsic isometry of M extends to an ambient isometry of $\mathbb{R}^3$. More generally, if M is not a helicoid, then it is minimally rigid, in the sense that any isometric minimal immersion of M into $\mathbb{R}^3$ is congruent to M.*

The Isometry Conjecture is known to hold if $M \in \mathcal{P}$ and either $M \in \mathcal{M}$ (Choi, Meeks and White [**24**]), M is doubly-periodic (Meeks and Rosenberg [**153**]), M is periodic with finite topology quotient (Meeks [**127**] and Pérez [**193**]) or M has finite genus (this follows from Theorem 1.0.4).

It can be shown that one can reduce the validity of the Isometry Conjecture to checking that whenever $M \in \mathcal{P}$ has one end and infinite genus, then there exists a plane in $\mathbb{R}^3$ that intersects M in a set that contains a simple closed curve. If $M \in \mathcal{P}$ and there exists such a simple closed intersecting curve γ of M with a plane, then the flux of M along γ is not zero, and hence, none of the associate surfaces to M are well-defined (see Footnote 16 for the definition of associate surface). But Calabi [**15**] proved that the associate surfaces are the only isometric minimal immersions from M into $\mathbb{R}^3$, up to congruence.

A consequence of the Dynamics Theorem 11.0.13 and the Local Picture Theorem on the Scale of Topology (Theorem 11.0.9) is that the first statement in the Isometry Conjecture holds for surfaces in $\mathcal{C}$ if and only if it holds for those surfaces in $\mathcal{P}$ which have bounded curvature and are quasi-dilation-periodic. Meeks (unpublished) has shown that any example in $\mathcal{P}$ with bounded curvature and invariant under a translation satisfies the conjecture. Since non-zero flux ($\mathcal{F} \neq \{0\}$ with the notation of the next conjecture) implies uniqueness of an isometric minimal immersion, the One-Flux Conjecture below implies the Isometry Conjecture.

Conjecture 17.0.27 (One-Flux Conjecture, Meeks, Pérez, Ros)**.** *Let $M \in \mathcal{C}$ and let $\mathcal{F} = \{F(\gamma) = \int_\gamma \mathrm{Rot}_{90^\circ}(\gamma') \mid \gamma \in H_1(M, \mathbb{Z})\}$ be the abelian group of flux vectors of M. If $\mathcal{F}$ has rank at most 1, then M is a plane, a helicoid, a* *catenoid, a Riemann minimal example or a doubly-periodic Scherk minimal surface.*

Conjecture 17.0.28 (Scherk Uniqueness Conjecture, Meeks, Wolf). *If M is a connected, properly immersed minimal surface in $\mathbb{R}^3$ and* $\text{Area}(M \cap \mathbb{B}(R)) \leq 2\pi R^2$ *holds in extrinsic balls $\mathbb{B}(R)$ of radius R, then M is a plane, a catenoid or one of the singly-periodic Scherk minimal surfaces.*

By the Monotonicity Formula, any connected, properly immersed minimal surface in $\mathbb{R}^3$ with

$$\lim_{R\to\infty} R^{-2}\text{Area}(M \cap \mathbb{B}(R)) \leq 2\pi,$$

is actually embedded. A related conjecture on the uniqueness of the doubly-periodic Scherk minimal surfaces was solved by Lazard-Holly and Meeks [**112**]; they proved that if $M \in \mathcal{P}$ is doubly-periodic and its quotient surface has genus zero, then M is one of the doubly-periodic Scherk minimal surfaces. The basic approach used in [**112**] was adapted later on by Meeks and Wolf [**165**] to prove that Conjecture 17.0.28 holds under the assumption that the surface is singly-periodic; see this precise statement in Theorem 1.0.9 and a sketch of its proof in Chapter 15. We recall that Meeks and Wolf's proof uses that the Unique Limit Tangent Cone Conjecture below holds in their periodic setting; this approach suggests that a good way to solve the general Conjecture 17.0.28 is first to prove Conjecture 17.0.29 on the uniqueness of the limit tangent cone of M, from which it follows (unpublished work of Meeks and Ros) that M has two Alexandrov-type planes of symmetry. Once M is known to have these planes of symmetry, one can describe the Weierstrass representation of M, which Meeks and Wolf (unpublished) claim would be sufficient to complete the proof of the conjecture. As explained in Chapter 15, much of the interest in Conjecture 17.0.28 arises from the role that the singly-periodic Scherk minimal surfaces play in desingularizing two transversally intersecting minimal surfaces by means of the Kapouleas' method.

Conjecture 17.0.29 (Unique Limit Tangent Cone at Infinity Conjecture, Meeks). *If $M \in \mathcal{P}$ is not a plane and has quadratic area growth, then* $\lim_{t\to\infty} \frac{1}{t}M$ *exists and is a minimal, possibly non-smooth cone over a finite balanced configuration of geodesic arcs in the unit sphere, with common ends points and integer multiplicities. Furthermore, if M has area not greater than $2\pi R^2$ in balls of radius R, then the limit tangent cone of M is either the union of two planes or consists of a single plane with multiplicity two passing through the origin.*

By unpublished work of Meeks and Wolf, the above conjecture is closely tied to the validity of the next classical one.

Conjecture 17.0.30. *Let $f\colon M \to \mathbb{B} - \{\vec{0}\}$ be a proper immersion of a surface with compact boundary in the punctured unit ball, such that $f(\partial M) \subset \partial\mathbb{B}$ and whose mean curvature function is bounded. Then, $f(M)$ has a unique limit tangent cone at the origin under homothetic expansions.*

If $M \in \mathcal{C}$ has finite topology, then M has finite total curvature or is asymptotic to a helicoid by Theorems 1.0.3, 1.0.4 and 1.0.5. It follows that for any such surface M, there exists a constant $C_M > 0$ such that the injectivity radius function $I_M\colon M \to (0, \infty]$ satisfies

$$I_M(p) \geq C_M \|p\|, \quad p \in M.$$

Work of Meeks, Pérez and Ros in [**142, 143**] indicates that this linear growth property of the injectivity radius function should characterize the examples in $\mathcal{C}$ with finite topology, in a similar manner that the inequality $K_M(p)\|p\|^2 \leq C_M$ characterizes finite total curvature for a surface $M \in \mathcal{C}$ (Theorem 11.0.11, here K_M denotes the Gaussian curvature function of M).

Conjecture 17.0.31 (Injectivity Radius Growth Conjecture, Meeks, Pérez, Ros)**.** *A surface $M \in \mathcal{C}$ has finite topology if and only if its injectivity radius function grows at least linearly with respect to the extrinsic distance from the origin.*

A positive solution of Conjecture 17.0.31 would give rise to an interesting dynamics theorem on the scale of topology, similar to Theorem 11.0.13. The results in [**142, 143**] and the earlier described Theorems 4.1.3 and 4.1.11 also motivated several conjectures concerning the limits of locally simply connected sequences of minimal surfaces in $\mathbb{R}^3$, like the following one.

Conjecture 17.0.32 (Parking Garage Structure Conjecture, Meeks, Pérez, Ros)**.** *Suppose $M_n \subset \mathbb{B}(R_n)$ is a locally simply connected sequence of embedded minimal surfaces with $\partial M_n \subset \partial\mathbb{B}(R_n)$ and $R_n \to \infty$ as $n \to \infty$. Assume also that the sequence M_n does not have uniformly bounded curvature in $\mathbb{B}(1)$. Then:*

1. *After a rotation and choosing a subsequence, the M_n converge to a minimal parking garage structure on $\mathbb{R}^3$ consisting of the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes, with singular set of convergence being a locally finite collection $\mathcal{S}(\mathcal{L})$ of vertical lines which are the columns of the parking garage structure.*
2. *For any two points $p, q \in \mathbb{R}^3 - \mathcal{S}(\mathcal{L})$, the ratio of the vertical spacing between consecutive sheets of the double multi-valued graphs defined by M_n near p and q, converges to one as $n \to \infty$. Equivalently, the Gaussian curvature of the sequence M_n blows up at the same rate along all the columns as $n \to \infty$.*

We next deal with the question of when a surface $M \in \mathcal{C}$ has strictly negative Gaussian curvature. Suppose again that a surface $M \in \mathcal{C}$ has finite topology, and so, M either has finite total curvature or is a helicoid with handles. It is straightforward to check that such a surface has negative curvature if and only if it is a catenoid or a helicoid (note that if $g\colon M \to \mathbb{C} \cup \{\infty\}$ is the stereographically projected Gauss map of M, then after a suitable rotation of M in $\mathbb{R}^3$, the meromorphic differential $\frac{dg}{g}$ vanishes exactly at the zeros of the Gaussian curvature of M; from here one deduces easily that if M has finite topology and strictly negative Gaussian curvature, then the genus of M is zero). More generally, if we allow a surface $M \in \mathcal{C}$ to be invariant under a proper discontinuous group G of isometries of $\mathbb{R}^3$, with M/G having finite topology, then M/G is properly embedded in $\mathbb{R}^3/G$ by an elementary application of the Minimal Lamination Closure Theorem (see Proposition 1.3 in [**200**]). Hence, in this case M/G has finite total curvature by a result of Meeks and Rosenberg [**153, 156**]. Suppose additionally that M/G has negative curvature, and we will discuss which surfaces are possible. If the ends of M/G are helicoidal or planar, then a similar argument using $\frac{dg}{g}$ gives that M has genus zero, and so, it is a helicoid. If M/G is doubly-periodic, then M is a Scherk minimal surface, see [**144**]. In the case M/G is singly-periodic, then M must have Scherk-type ends but we still do not know if the surface must be a Scherk singly-periodic minimal surface. These considerations motivate the following conjecture.

Conjecture 17.0.33 (Negative Curvature Conjecture, Meeks, Pérez, Ros)**.** *If $M \in \mathcal{C}$ has negative curvature, then M is a catenoid, a helicoid or one of the singly or doubly-periodic Scherk minimal surfaces.*

Passing to a different question, some of the techniques developed by Meeks, Pérez and Ros in [**143**] and discussed in Chapter 11 provide a beginning theory for analyzing and possibly characterizing examples in $\mathcal{C}$ whose Gauss maps exclude two or more points on $\mathbb{S}^2$. A classical result of Fujimoto [**68**] establishes that the Gauss map of any orientable, complete, non-flat, minimally immersed surface in $\mathbb{R}^3$ cannot exclude more than 4 points, which improved the earlier result of Xavier [**242**] that the Gauss map of such a surface cannot miss more than 6 points. If one assumes that a surface $M \in \mathcal{C}$ is periodic with finite topology quotient, then Meeks, Pérez and Ros solved the first item in the next conjecture [**144**]. Also see Kawakami, Kobayashi and Miyaoka [**104**] for related results on this problem, including some partial results on the conjecture of Osserman that states that the Gauss map of an orientable, complete, non-flat, immersed minimal surface with finite total curvature in $\mathbb{R}^3$ cannot miss 3 points of $\mathbb{S}^2$.

Conjecture 17.0.34 (Four Point Conjecture, Meeks, Pérez, Ros)**.** *Suppose $M \in \mathcal{C}$. Then:*

1. *If the Gauss map of M omits 4 points on $\mathbb{S}^2$, then M is a singly or doubly-periodic Scherk minimal surface.*
2. *If the Gauss map of M omits exactly 3 points on $\mathbb{S}^2$, then M is a singly-periodic Karcher saddle tower whose flux polygon is a convex unitary*

hexagon[2] *(note that any three points in a great circle are omitted by one of these examples, see* [**200**] *for a description of these surfaces).*

3. *If the Gauss map of M omits exactly 2 points, then M is a catenoid, a helicoid, one of the Riemann minimal examples or one of the KMR doubly-periodic minimal tori described in Chapter* 2.5*. In particular, the pair of points missed by the Gauss map of M must be antipodal.*

The following three conjectures are related to the embedded Calabi-Yau problem.

Conjecture 17.0.35 (Finite Genus Properness Conjecture, Meeks, Pérez, Ros)**.** *If $M \in \mathcal{C}$ and M has finite genus, then $M \in \mathcal{P}$.*

In [**141**], Meeks, Pérez and Ros proved Conjecture 17.0.35 under the additional hypothesis that M has a countable number of ends (recall that this assumption is necessary for M to be proper in $\mathbb{R}^3$ by Theorem 7.3.1). An equivalent conjecture by Meeks, Pérez and Ros [**148**] states that if $M \in \mathcal{C}$ has finite genus, then M has bounded Gaussian curvature; note that Theorem 5.0.8 implies that if $M \in \mathcal{C}$ has locally bounded Gaussian curvature in $\mathbb{R}^3$ and finite genus, then M is properly embedded.

Conjecture 17.0.36 (Embedded Calabi-Yau Conjectures, Martín, Meeks, Nadirashvili, Pérez, Ros)**.**

1. *There exists an $M \in \mathcal{C}$ contained in a bounded domain in $\mathbb{R}^3$.*
2. *There exists an $M \in \mathcal{C}$ whose closure $\overline{M}$ has the structure of a minimal lamination of a slab, with M as a leaf and with two planes as limit leaves. In particular, $\mathcal{P} \neq \mathcal{C}$.*
3. *A necessary and sufficient condition for a connected, open topological surface M to admit a complete bounded minimal embedding in $\mathbb{R}^3$ is that every end of M has infinite genus.*
4. *A necessary and sufficient condition for a connected, open topological surface M to admit a proper minimal embedding in every smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that M is orientable and every end of M has infinite genus.*
5. *A necessary and sufficient condition for a connected, non-orientable open topological surface M to admit a proper minimal embedding in some bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that every end of M has infinite genus.*

The next conjecture deals with the embedded Calabi-Yau problem in a Riemannian three-manifold N. We remark that Conjecture 17.0.35 can be shown to follow from the next conjecture.

Conjecture 17.0.37 (Finite Genus Conjecture in three-manifolds, Meeks, Pérez, Ros)**.** *Suppose M is a connected, complete, embedded minimal surface with empty boundary and finite genus in a Riemannian three-manifold N.*

[2] Not necessarily regular.

Let $\overline{M} = M \cup L(M)$, where $L(M)$ is the set of limit points[3] *of M. Then, one of the following possibilities holds.*

1. *$\overline{M}$ has the structure of a minimal lamination of N.*
2. *$\overline{M}$ fails to have a minimal lamination structure, $L(M)$ is a non-empty minimal lamination of N consisting of stable leaves and M is properly embedded in $N - L(M)$.*

Our next problem is related to integral curves of the gradient vector fields of the harmonic coordinate functions on a properly embedded minimal surface in $\mathbb{R}^3$. Theorem 7.2.3 implies that for any properly immersed minimal surface M in $\mathbb{R}^3$ and for any $t \in \mathbb{R}$, the surface $M(t) = M \cap \{x_3 \leq t\}$ is parabolic. In [**132**], Meeks used the parabolicity of $M(t)$ to show that the scalar flux of ∇x_3 across $\partial M(t)$ does not depend on t (this result is called the *Algebraic Flux Lemma*). If M were recurrent, then it is known (Tsuji [**234**]) that the following stronger property holds: almost all integral curves of ∇x_3 begin at $x_3 = -\infty$ and end at $x_3 = \infty$. The next conjecture is a strengthening of this geometric flux property to arbitrary properly embedded minimal surfaces in $\mathbb{R}^3$.

Conjecture 17.0.38 (Geometric Flux Conjecture, Meeks, Rosenberg)**.** *Let $M \in \mathcal{P}$ and $h \colon M \to \mathbb{R}$ be a non-constant coordinate function on M. Consider the set I of integral curves of ∇h. Then, there exists a countable set $C \subset I$ such that for any integral curve $\alpha \in I - C$, the composition $h \circ \alpha \colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism*[4]*.*

One could weaken the hypothesis in the above conjecture that "except for a countable subset of I, h restricted to an element $\alpha \in I$ is a diffeomorphism with $\mathbb{R}$" to the hypothesis that "for almost-all elements of I, h restricted to an an element $\alpha \in I$ is a diffeomorphism with $\mathbb{R}$". We feel that the proof of the Geometric Flux Conjecture will have important theoretical consequences.

We have seen examples of how one can produce stable minimal surfaces by using barrier constructions (Chapter 2.9), and how these stable minimal surfaces act as guide posts which are useful for deciphering the structure of complete, embedded minimal surfaces (e.g., in the proofs of Theorems 2.9.1 and 6.0.2). Below, we have collected some outstanding problems that concern stable minimal surfaces. Regarding item *1* in the next conjecture, Ros (unpublished) proved that a complete, non-orientable minimal surface without boundary which is stable outside a compact set[5] must have finite total curvature. The validity of item *2* implies that the sublamination of limit leaves of the lamination $\mathcal{L}$ in Conjecture 17.0.18 extends to a lamination of $\mathbb{R}^3$ by planes. In reference to item *3,* we remark that complete, stable

[3] See the paragraph just before Theorem 10.1.2 for the definition of $L(M)$.

[4] After a choice of $p \in \alpha$, we are identifying α with the parameterized curve $\alpha \colon \mathbb{R} \to M$ such that $\alpha(0) = p$ and $\alpha'(t) = \nabla h(\alpha(t))$, $t \in \mathbb{R}$.

[5] See Footnote 19 for the definition of stability in the non-orientable case.

minimal surfaces with boundary are not in general parabolic (see pages 22 and 23 in our survey [**137**]). Concerning item *4*, Pérez [**196**] proved this conjecture under the additional assumptions that the surface is proper and has quadratic area growth.

Conjecture 17.0.39 (Stable Minimal Surface Conjectures)**.**

1. *A complete, non-orientable, stable minimal surface in* $\mathbb{R}^3$ *with compact boundary has finite total curvature* **(Ros)**.
2. *If* $A \subset \mathbb{R}^3$ *is a closed set with zero* 1*-dimensional Hausdorff measure and* $M \subset \mathbb{R}^3 - A$ *is a connected, stable, minimally immersed surface which is complete outside of* A*, then the closure of* M *is a plane* **(Meeks)**.
3. *If* $M \subset \mathbb{R}^3$ *is a complete, stable minimal surface with boundary, then* M *is* δ*-parabolic, i.e., given* $\delta > 0$*, the set* $M(\delta) = \{p \in M \mid \text{dist}_M(p, \partial M) \geq \delta\}$ *is parabolic* **(Meeks, Rosenberg)**.
4. *A complete, embedded, stable minimal surface in* $\mathbb{R}^3$ *with boundary a straight line is a half-plane, a half of the Enneper minimal surface or a half of the helicoid* **(Pérez, Ros, White)**.

Any end of a surface $M \in \mathcal{C}$ with finite total curvature is C^2-asymptotic to the end of a plane or catenoid (equation (2.11)). Our last conjecture can be viewed as a potential generalization of this result.

Conjecture 17.0.40 (Standard Middle End Conjecture, Meeks)**.** *If* $M \in \mathcal{M}$ *and* $E \subset M$ *is a one-ended representative for a middle end of* M*, then* E *is* C^0*-asymptotic to the end of a plane or catenoid. In particular, if* M *has two limit ends, then each middle end is* C^0*-asymptotic to a plane.*

For a non-annular, one-ended middle end representative E (i.e., E has infinite genus) in the above conjecture, $\lim_{t \to \infty} \frac{1}{t} E$ is a plane P passing through the origin with positive integer multiplicity at least two by Theorem 7.3.1. Also, if M has two limit ends and horizontal limit tangent plane at infinity, then for such a middle end representative E, every divergent sequence of horizontal translates of E has a subsequence which converges to a finite collection of horizontal planes. This limit collection might depend on the sequence; in this case it remains to prove there is only one plane in limit collections of this type.