Preface

The Riemann zeta function $\zeta(s)$ in the real variable s was introduced by L. Euler (1737) in connection with questions about the distribution of prime numbers. Later B. Riemann (1859) derived deeper results about the prime numbers by considering the zeta function in the complex variable. He revealed a dual correspondence between the primes and the complex zeros of $\zeta(s)$, which started a theory to be developed by the greatest minds in mathematics. Riemann was able to provide proofs of his most fundamental observations, except for one, which asserts that all the non-trivial zeros of $\zeta(s)$ are on the line $\operatorname{Re} s = \frac{1}{2}$. This is the famous Riemann Hypothesis – one of the most important unsolved problems in modern mathematics.

These lecture notes cover closely the material which I presented to graduate students at Rutgers in the fall of 2012. The theory of the Riemann zeta function has expanded in different directions over the past 150 years; however my goal was limited to showing only a few classical results on the distribution of the zeros. These results include the Riemann memoir (1859), the density theorem of F. Carlson (1920) about the zeros off the critical line, and the estimates of G. H. Hardy - J. E. Littlewood (1921) for the number of zeros on the critical line.

Then, in Part 2 of these lectures, I present in full detail the result of N. Levinson (1974), which asserts that more than one third of the zeros are critical (lie on the line $\operatorname{Re} s = \frac{1}{2}$). My approach had frequent detours so that students could learn different techniques with interesting features. For instance, I followed the stronger construction invented by J. B. Conrey (1983), because it reveals clearly the esssence of Levinson's ideas.

After establishing the principal inequality of the Levinson-Conrey method, it remains to evaluate asymptotically the second power-moment of a relevant Dirichlet polynomial, which is built out of derivatives of the zeta function and its mollifier. This task was carried out differently than by the traditional arguments and in greater generality than it was needed. The main term coming from the contribution of the diagonal terms fits with results in sieve theory and can be useful elsewhere.

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