CHAPTER 1

Panorama of Arithmetic Functions

Throughout these notes we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ the sets of integers, rationals, real and complex numbers, respectively. The positive integers are called *natural numbers*. The set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

of natural numbers contains the subset of *prime numbers*

$$\mathcal{P} = \{2, 3, 5, 7, 11, \dots\}.$$

We will often denote a prime number by the letter p.

A function $f : \mathbb{N} \to \mathbb{C}$ is called an *arithmetic function*. Sometimes an arithmetic function is extended to all $\mathbb{Z}$. If f has the property

$$f(mn) = f(m) + f(n) \tag{1.1}$$

for all m, n relatively prime, then f is called an *additive function*. Moreover, if (1.1) holds for all m, n, then f is called *completely additive*; for example, $f(n) = \log n$ is completely additive. If f has the property

$$f(mn) = f(m)f(n) \tag{1.2}$$

for all m, n relatively prime, then f is called a *multiplicative function*. Moreover, if (1.2) holds for all m, n, then f is called *completely multiplicative*; for example, $f(n) = n^{-s}$ for a fixed $s \in \mathbb{C}$, is completely multiplicative.

If $f : \mathbb{N} \to \mathbb{C}$ has at most a polynomial growth, then we can associate with f the *Dirichlet series*

$$D_f(s) = \sum_1^\infty f(n)n^{-s}$$

which converges absolutely for $s = \sigma + it$ with σ sufficiently large. The product of Dirichlet series is a Dirichlet series

$$D_f(s)D_g(s) = D_h(s),$$

with $h = f * g$ defined by

$$h(l) = \sum_{mn=l} f(m)g(n) = \sum_{d|l} f(d)g(l/d), \tag{1.3}$$

which is called the *Dirichlet convolution*.

The constant function $f(n) = 1$ for all $n \in \mathbb{N}$ has the Dirichlet series

$$\zeta(s) = \sum_1^\infty n^{-s} \tag{1.4}$$

which is called the *Riemann zeta-function*. Actually $\zeta(s)$ was first introduced by L. Euler who studied the distribution of prime numbers using the infinite product formula

$$\begin{aligned}\zeta(s) &= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.\end{aligned} \tag{1.5}$$

The series (1.4) and the product (1.5) converge absolutely in the half-plane $s = \sigma + it$, $\sigma > 1$. Since $\zeta(s)$ for $s > 1$ is well approximated by the integral

$$\int_1^\infty y^{-s}\,\mathrm{d}y = \frac{1}{s-1}$$

as $s \to 1$, it follows from (1.5) that

$$\sum_p \frac{1}{p^s} \sim \log \frac{1}{s-1}, \qquad \text{as } s > 1,\ s \to 1. \tag{1.6}$$

By the Euler product for $\zeta(s)$ it follows that $1/\zeta(s)$ also has a Dirichlet series expansion

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_1^\infty \frac{\mu(m)}{m^s} \tag{1.7}$$

where $\mu(m)$ is the multiplicative function defined at prime powers by

$$\mu(p) = -1, \qquad \mu(p^\alpha) = 0, \quad \text{if } \alpha \geqslant 2. \tag{1.8}$$

This is a fascinating function (introduced by A. F. Möbius in 1832) which plays a fundamental role in the theory of prime numbers.

Translating the obvious formula $\zeta(s) \cdot \zeta(s)^{-1} = 1$ into the language of Dirichlet convolution we obtain the δ-function

$$\delta(n) = \sum_{m|n} \mu(m) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases} \tag{1.9}$$

Clearly the two relations

$$g = 1 * f, \qquad f = \mu * g \tag{1.10}$$

are equivalent. This equivalence is called the *Möbius inversion*; more explicitly,

$$g(n) = \sum_{d|n} f(d) \quad \Longleftrightarrow \quad f(n) = \sum_{d|n} \mu(d) g(n/d). \tag{1.11}$$

If f, g are multiplicative, then so are $f \cdot g$, $f * g$. If g is multiplicative, then

$$\sum_{d|n} \mu(d) g(d) = \prod_{p|n} \big(1 - g(p)\big). \tag{1.12}$$

In various contexts one can view the left side of (1.12) as an "exclusion-inclusion" procedure of events occurring at divisors d of n with densities $g(d)$. Then the right side of (1.12) represents the probability that none of the events associated with prime divisors of n takes place.

Now we are going to give some basic examples of arithmetic functions. We begin by exploiting Dirichlet convolutions. The first one is the *divisor function*

$$\tau = 1 * 1, \qquad \tau(n) = \sum_{d|n} 1, \qquad \zeta^2(s) = \sum_1^\infty \tau(n) n^{-s}.$$

This is multiplicative with $\tau(p^\alpha) = \alpha + 1$. We have

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_1^\infty \tau(n)^2 n^{-s}$$

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_1^\infty \tau(n^2) n^{-s}$$

$$\zeta^3(s) = \sum_1^\infty \Big(\sum_{d^2 m = n} \tau(m^2) \Big) n^{-s}.$$

Note that

$$\sum_{d^2 m = n} \tau(m^2) = \sum_{klm = n} 1 = \tau_3(n), \tag{1.13}$$

say. Next we get

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_1^\infty 2^{\omega(n)} n^{-s}$$

where $\omega(n)$ denotes the number of distinct prime divisors of n, so $2^{\omega(n)}$ is the number of squarefree divisors of n. The characteristic function of squarefree numbers is

$$|\mu(n)| = \mu^2(n) = \sum_{d^2 | n} \mu(d),$$

its Dirichlet series is

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_1^\infty |\mu(n)| n^{-s} = \prod_p \Big(1 + \frac{1}{p^s}\Big).$$

Inverting this we get the generating series for the *Liouville function* $\lambda(n)$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_1^\infty \lambda(n) n^{-s}.$$

Note that

$$\lambda(n) = (-1)^{\Omega(n)}$$

where $\Omega(n)$ is the total number of prime divisors of n (counted with the multiplicity).

The *Euler φ-function* is defined by $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$; it is the number of reduced residue classes modulo n. This function satisfies

$$\varphi(n) = n \prod_p \Big(1 - \frac{1}{p}\Big) = n \sum_{d|n} \frac{\mu(d)}{d}.$$

Hence

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_1^\infty \varphi(n) n^{-s}.$$

A different class of arithmetic functions (very important for the study of prime numbers) is obtained by differentiating relevant Dirichlet series

$$-D_f'(s) = \sum_1^\infty f(n)(\log n)n^{-s}.$$

In particular

$$-\zeta'(s) = \sum_1^\infty (\log n)n^{-s}.$$

Since $L(n) = \log n$ is additive, we have the formula

$$L \cdot (f * g) = (L \cdot f) * g + f * (L \cdot g)$$

which says that the multiplication by L is a derivation in the Dirichlet ring of arithmetic functions.

By the Euler product we have

$$\log \zeta(s) = \sum_{l=1}^\infty \sum_p l^{-1} p^{-ls}. \tag{1.14}$$

Hence differentiating we get

$$-\frac{\zeta'}{\zeta}(s) = \sum_1^\infty \Lambda(n) n^{-s} \tag{1.15}$$

with $\Lambda(n)$ (popularly named *von Mangoldt function*) given by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^\alpha,\ \alpha \geqslant 1 \\ 0, & \text{otherwise.} \end{cases} \tag{1.16}$$

From the left side of (1.15) we get

$$\Lambda = \mu * L, \qquad \Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d. \tag{1.17}$$

Hence, by Möbius inversion we get

$$L = 1 * \Lambda, \qquad \log n = \sum_{d|n} \Lambda(d). \tag{1.18}$$

Similarly we define the von Mangoldt functions Λ_k of any degree $k \geqslant 0$ by

$$\Lambda_k = \mu * L^k, \qquad \Lambda_k(n) = \sum_{d|n} \mu(d) \left(\log \frac{n}{d}\right)^k. \tag{1.19}$$

We have

$$L^k = 1 * \Lambda_k, \qquad (\log n)^k = \sum_{d|n} \Lambda_k(d). \tag{1.20}$$

Note that $\Lambda_0 = \delta$, $\Lambda_1 = \Lambda$, and we have the recurrence formula

$$\Lambda_{k+1} = L \cdot \Lambda_k + \Lambda * \Lambda_k. \tag{1.21}$$

This follows by writing

$$\Lambda_{k+1}(n) = \sum_{d|n} \mu(d) \left(\log \frac{n}{d}\right)^k (\log n - \log d)$$

and using (1.18).

From (1.21) we derive by induction in k that $\Lambda_k(n) \geqslant 0$ and $\Lambda_k(n)$ is supported on positive integers having at most k distinct prime divisors. Moreover we get by (1.20) that

$$0 \leqslant \Lambda_k(n) \leqslant (\log n)^k. \tag{1.22}$$

Exercise. Prove the formula

$$\Lambda_k(mn) = \sum_{0\leqslant j\leqslant k} \binom{k}{j} \Lambda_j(m)\Lambda_{k-j}(n). \tag{1.23}$$

Exercise. Prove the formula

$$\sum_p \frac{1}{p^s} = \sum_1^\infty \frac{\mu(n)}{n} \log \zeta(ns), \qquad \text{if } s > 1. \tag{1.24}$$

CHAPTER 5

The Riemann Memoir

In 1859, B. Riemann [**Rie59**] wrote a short paper (8 pages) called "On the number of primes less than a given magnitude" (in German) in which he expressed fundamental properties of $\zeta(s)$ in the complex variable $s = \sigma + it$. We state these in the modern forms.

A. The function $\zeta(s)$ defined in $s > 1$ by

$$\zeta(s) = \sum n^{-s}$$

has analytic continuation to the whole complex plane, and it is holomorphic except for a simple pole at $s = 1$ with residue 1.

B. The functional equation holds

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

where $\Gamma(s)$ denotes the gamma function of Euler, see Appendix B

C. The zeta function has simple real zeros at $s = -2, -4, -6, \ldots$, which are called *trivial zeros*, and infinitely many non-trivial zeros of the form

$$\rho = \beta + i\gamma, \qquad 0 \leqslant \beta \leqslant 1,\ \gamma \in \mathbb{R}.$$

The number $N(T)$ of non-trivial zeros of height $0 < \gamma < T$ satisfies

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T), \qquad T \geqslant 2.$$

D. The product formula holds

$$s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = e^{-Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$

with

$$B = 1 + \frac{\gamma}{2} - \log 2\sqrt{\pi}.$$

E. The prime numbers formula holds for $x > 1$,

$$\psi^{\flat}(x) = \sum_{n\leqslant x}{}^{\flat}\Lambda(n) = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right).$$

We shall explain the convergence issues in Corollary 10.3.

F. The Riemann Hypothesis. Every non-trivial zero of $\zeta(s)$ is on the line $\operatorname{Re} s = 1/2$, i.e.,

$$\rho = \frac{1}{2} + i\gamma.$$