Introduction

The theory of *motives* began in the early sixties when Grothendieck envisioned the existence of a “universal cohomology theory of algebraic varieties” acting as a gateway between algebraic geometry and the assortment of the classical Weil cohomology theories (de Rham, Betti, étale, crystalline). After the release of Manin’s foundational article [**Man68**] on the subject, Grothendieck’s ideas became popular and a powerful driving force in mathematics.

The theory of *noncommutative motives* is more recent. It began in the eighties when the Moscow school (Beilinson, Bondal, Kapranov, Manin, and others) started the study of algebraic varieties via their derived dg categories of coherent sheaves. It turns out that several invariants of algebraic varieties can be recovered from their derived dg categories. The idea of replacing algebraic varieties by arbitrary dg categories, which are morally speaking “noncommutative algebraic varieties”, later led Kontsevich [**Konb**] to envision the existence of a “universal invariant of noncommutative algebraic varieties”.

The purpose of this book is to give a rigorous overview of some of the main advances in the theory of noncommutative motives. It is based on a graduate course (18.917 - Noncommutative Motives) taught at MIT in the spring of 2014 and its intended audience consists of graduate students and mathematicians interested in noncommutative motives and their applications. We assume some familiarity with algebraic geometry and with homological/homotopical algebra. The contents of the book can be divided into three main parts:

- Part I: Differential graded categories – Chapter 1.
- Part II: Noncommutative pure motives – Chapters 2-7.
- Part III: Noncommutative mixed motives – Chapters 8-10.

A differential graded (=dg) category is a category enriched over complexes (morphism sets are complexes). An essential example to keep in mind is the derived dg category of an algebraic variety. Several invariants such as algebraic K-theory, cyclic homology (and all its variants), and topological Hochschild homology, can be defined directly on dg categories. In order to study all these invariants simultaneously, we introduce the notion of an *additive invariant* and of a *localizing invariant*.

A functor, defined on the category of (small) dg categories and with values in an additive category, is called an additive invariant if it inverts Morita equivalences and sends semi-orthogonal decompositions in the sense of Bondal-Orlov to direct sums. Chapter 2 is devoted to the study of this class of invariants and to the construction of the *universal* additive invariant. The theory of noncommutative pure motives can be roughly summarized as the study of the target additive category of this universal additive invariant. Making use of Kontsevich’s smooth proper dg categories, which are morally speaking the “noncommutative smooth proper algebraic varieties", we introduce in Chapter 4 several additive categories of noncommutative pure motives and relate them to their commutative counterparts. An important example is the category of noncommutative numerical motives. Among other properties, we prove that this category is abelian semi-simple. This result, combined with some noncommutative (standard) conjectures stated in Chapter 5, leads to a (conditional) theory of noncommutative motivic Galois groups and to the extension of the classical theory of intermediate Jacobians to "noncommutative algebraic varieties"; consult Chapters 6 and 7, respectively.

A functor, defined on the category of (small) dg categories and with values in a triangulated category, is called a localizing invariant if it inverts Morita equivalences, preserves filtered homotopy colimits, and sends Drinfeld's short exact sequences to distinguished triangles. The rigorous formalization of this notion requires the language of Grothendieck derivators, which can be found in Appendix A. Chapter 8 is devoted to the study of this class of invariants and to the construction of the *universal* localizing invariant. The theory of noncommutative mixed motives can be roughly summarized as the study of the target triangulated category of the universal localizing invariant. Making use once again of Kontsevich's smooth proper dg categories, we introduce in Chapter 9 several triangulated categories of noncommutative mixed motives and relate them with Voevodsky's triangulated category of geometric mixed motives. Finally, in Chapter 10 we briefly describe the (unconditional) theory of noncommutative motivic Hopf dg algebras, which is the mixed analogue of the theory of noncommutative motivic Galois groups.

Although recent, the theory of noncommutative motives already led to the (partial) solution of some open problems and conjectures in adjacent research areas. Some of these will be discussed throughout the book.

We refrain from giving a lengthy introduction to the contents of each chapter. The table of contents combined with the introduction of each chapter provides the corresponding information.

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