Preface

Generally speaking, a *dynamical system* is a space in which the points (which can be viewed as configurations) move along with time according to a given rule, usually not depending on time. Time can be either continuous (the motion of planets, fluid mechanics, etc.) or discrete (the number of bees each year, etc.). In the discrete case, the system is determined by a map $f\colon X \to X$, where X is the space, and the evolution is given by successive iterations of the transformation: starting from the point x at time 0, the point $f(x)$ represents the new position at time 1 and $f^n(x) = f \circ f \circ \cdots \circ f(x)$ (f iterated n times) is the position at time n.

A dynamical system ruled by a deterministic law can nevertheless be unpredictable. In particular, in the early 1960s, Lorenz underlined this phenomenon after realizing by chance that in his meteorological model, two very close initial values may lead to totally different results [**114, 115, 116**]; he discovered the so-called "butterfly effect". This kind of behavior has also been exhibited in other dynamical systems. One of the first to be studied, among the simplest, is given by the map $f(x) = rx(1-x)$ acting on the interval $[0,1]$ and models the evolution of a population. If the parameter r is small enough, then all the trajectories converge to a fixed point – the population stabilizes. However, May showed that for larger values of r, the dynamics may become very complicated [**120**].

This book focuses on dynamical systems given by the iteration of a continuous map on an interval. These systems were broadly studied because they are simple but nevertheless exhibit complex behaviors. They also allow numerical simulations using a computer or a mere pocket calculator, which enabled the discovery of some chaotic phenomena. Moreover, the "most interesting" part of some higher-dimensional systems can be of lower dimension, which allows, in some cases, to boil down to systems in dimension one. This idea was used for instance to reduce the study of Lorenz flows in dimension 3 to a class of maps on the interval. However, continuous interval maps have many properties that are not generally found in other spaces. As a consequence, the study of one-dimensional dynamics is very rich but not representative of all systems.

In the 1960s, Sharkovsky began to study the structure of systems given by a continuous map on an interval, in particular the co-existence of periodic points of various periods, which is ruled by *Sharkovsky's order* [**150**]. Non-Russian-speaking scientists were hardly aware of this striking result until a new proof of this theorem was given in English by Štefan in 1976 in a preprint [**162**] (published one year later in [**163**]). In 1975, in the paper "Period three implies chaos" [**111**], Li and Yorke proved that a continuous interval map with a periodic point of period 3 has periodic points of all periods – which is actually a part of Sharkovsky's Theorem eleven years earlier; they also proved that, for such a map f, there exists an uncountable set such that, if x, y are two distinct points in this set, then $f^n(x)$ and $f^n(y)$ are arbitrarily close for some n and are further than some fixed positive distance for other integers n tending to infinity; the term "chaos" was introduced in mathematics in this paper of Li and Yorke, where it was used in reference to this behavior.

Afterwards, various definitions of chaos were proposed. They do not coincide in general and none of them can be considered as the unique "good" definition of chaos. One may ask, "What is chaos then?" It relies generally on the idea of unpredictability or instability, i.e., knowing the trajectory of one point is not enough to know what happens elsewhere. The map $f\colon X \to X$ is said to be *sensitive to initial conditions* if near every point x there exists a point y arbitrarily close to x such that the distance between $f^n(x)$ and $f^n(y)$ is greater than a given $\delta > 0$ for some n. *Chaos in the sense of Li-Yorke* (see above) asks for more instability, but only on a subset. For Devaney, chaos is seen as a mixing of unpredictability and regular behavior: a system is *chaotic in the sense of Devaney* if it is transitive, sensitive to initial conditions and has a dense set of periodic points [**74**]. Others put as a part of their definition that the entropy should be positive, which means that the number of different trajectories of length n, up to some approximation, grows exponentially fast.

In order to obtain something uniform, the system is often assumed to be transitive. Roughly speaking, this means that it cannot be decomposed into two parts with nonempty interiors that do not interact under the action of the transformation. This "basic" assumption actually has strong consequences for systems on one-dimensional spaces. For a continuous interval map, it implies most of the other notions linked to chaos: sensitivity to initial conditions, dense set of periodic points, positive entropy, chaos in the sense of Li-Yorke, etc. This leads us to search for (partial) converses: for instance, if the interval map f is sensitive to initial conditions, then, for some integer n, the map f^n is transitive on a subinterval.

The study of periodic points has taken an important place in the works on interval maps. For these systems, chaotic properties not only imply existence of periodic points, but the possible periods also provide some information about the system. For instance, for a transitive interval map, there exist periodic points of all even periods, and an interval map has positive entropy if and only if there exists a periodic point whose period is not a power of 2. This kind of relationship is very typical of one-dimensional systems.

The aim of this book is not to collect all the results about continuous interval maps but to survey the relations between the various kinds of chaos and related notions for these systems. The papers on this topic are numerous but very scattered in the literature, sometimes little known or difficult to find, sometimes originally published in Russian (or Ukrainian, or Chinese), and sometimes without proof. Furthermore some results were found twice independently, which was often due to a lack of communication and to language barriers, leading research to develop separately in English and Russian literature. This has complicated our task when attributing authorship; we want to apologize for possible errors or omissions when indicating who first proved the various results.

We adopt a topological point of view; i.e., we do not speak about invariant measures or ergodic properties. Moreover, we are interested in the set of continuous interval maps, not in particular families such as piecewise monotone, C^∞ or unimodal maps. We give complete proofs of the results concerning interval maps.

Many results for interval maps have been generalized to other one-dimensional systems. We briefly describe them in paragraphs called "Remarks on graph maps" at the end of the concerned sections. We indicate some main ideas and give the references. This subject is still in evolution, and the most recent works and references may be missing.

This book is addressed to both graduate students and researchers. We have tried to keep to the elementary level. The prerequisites are basic courses of real analysis and topology, and some linear algebra.

Contents of the book

In the **first chapter**, we define some elementary notions and introduce some notation. Throughout this book, a continuous map $f\colon I \to I$ on a nondegenerate compact interval I will be called an *interval map*. We also provide some basic results about ω-limit sets and tools to find periodic points.

In **Chapter 2**, we study the links between transitivity, topological mixing and sensitivity to initial conditions. We first prove that a transitive interval map has a dense set of periodic points. Then we show that transitivity is very close to the notion of topological mixing in the sense that for a transitive interval map $f\colon I \to I$, either f is topologically mixing or the interval I is divided into two subintervals J, K which are swapped under the action of f and such that both $f^2|_J$ and $f^2|_K$ are topologically mixing. Furthermore, the notions of topological mixing, topological weak mixing and total transitivity are proved to be equivalent for interval maps.

Next we show that a transitive interval map is sensitive to initial conditions and, conversely, if the map is sensitive, then there exists a subinterval J such that $f^n|_J$ is transitive for some positive integer n.

Chapter 3 is devoted to periodic points. First we prove that topological mixing is equivalent to the specification property, which roughly means that any collection of pieces of orbits can be approximated by the orbit of a periodic point.

Next we show that, if the set of periodic points is dense for the interval map f, then there exists a nondegenerate subinterval J such that either $f|_J$ or $f^2|_J$ is transitive provided that f^2 is not equal to the identity map.

Then we present Sharkovsky's Theorem, which says that there is a total order on $\mathbb{N}$ – called *Sharkovsky's order* – such that, if an interval map has a periodic point of period n, then it also has periodic points of period m for all integers m greater than n with respect to this order. The *type* of a map f is the minimal integer n for Sharkovsky's order such that f has a periodic point of period n; if there is no such integer n, then the set of periods is exactly $\{2^n \mid n \geq 0\}$ and the type is 2^∞. We build an interval map of type n for every $n \in \mathbb{N} \cup \{2^\infty\}$.

Next, we study the relation between the type of a map and the existence of horseshoes. Finally, we compute the type of transitive and topologically mixing interval maps.

In **Chapter 4**, we are concerned with topological entropy. A *horseshoe* for the interval map f is a family of two or more closed subintervals $J_1, \ldots, J_p$ with disjoint interiors such that $f(J_i) \supset J_1 \cup \cdots \cup J_p$ for all $1 \leq i \leq p$. We show that the existence of a horseshoe implies that the topological entropy is positive. Reciprocally, Misiurewicz's Theorem states that, if the entropy of the interval map f is positive, then f^n has a horseshoe for some positive integer n.

Next we show that an interval map has a homoclinic point if and only if it has positive topological entropy. For an interval map f, x is a *homoclinic point* if there exists a periodic point z different from x such that x is in the unstable manifold of z and z is a limit point of $(f^{np}(x))_{n\geq 0}$, where p is the period of z.

We then give some upper and lower bounds on the entropy, focusing on lower bounds for transitive and topologically mixing maps and lower bounds depending on the periods of periodic points (or, in other words, on the type of the map for Sharkovsky's order). In particular, an interval map has positive topological entropy if and only if it has a periodic point whose period is not a power of 2. The sharpness of these bounds is illustrated by some examples.

To conclude this chapter, we show that a topologically mixing interval map has a uniformly positive entropy; that is, every cover by two open nondense sets has positive topological entropy. Actually, this property is equivalent to topological mixing for interval maps.

Chapter 5 is devoted to chaos in the sense of Li-Yorke. Two points x, y form a *Li-Yorke pair of modulus* δ for the map f if

$$\limsup_{n\to+\infty} |f^n(x) - f^n(y)| \geq \delta \quad \text{and} \quad \liminf_{n\to+\infty} |f^n(x) - f^n(y)| = 0.$$

A *δ-scrambled set* is a set S such that every pair of distinct points in S is a Li-Yorke pair of modulus δ; the set S is *scrambled* if for every $x, y \in S$, $x \neq y$, (x, y) is a Li-Yorke pair (of modulus δ for some $\delta > 0$ depending on x, y). The map is *chaotic in the sense of Li-Yorke* if it has an uncountable scrambled set. We prove that an interval map of positive topological entropy admits a δ-scrambled Cantor set for some $\delta > 0$ and is thus chaotic in the sense of Li-Yorke. We also show that a topologically mixing map has a dense δ-scrambled set which is a countable union of Cantor sets.

Next, we study an equivalent condition for zero entropy interval maps to be chaotic in the sense of Li-Yorke, which implies the existence of a δ-scrambled Cantor set as in the positive entropy case. A zero entropy interval map that is chaotic in the sense of Li-Yorke is necessarily of type 2^∞ for Sharkovsky's order, but the converse is not true; we build two maps of type 2^∞ having an infinite ω-limit set, one being chaotic in the sense of Li-Yorke and the other not.

Then we state that the existence of one Li-Yorke pair for an interval map is enough to imply chaos in the sense of Li-Yorke.

Finally, we show that an interval map is chaotic in the sense of Li-Yorke if and only if it has positive topological sequence entropy.

In **Chapter 6**, we study some notions related to Li-Yorke pairs.

Generic chaos and *dense chaos* are somehow two-dimensional notions. A topological system $f\colon X \to X$ is generically (resp. densely) chaotic if the set of Li-Yorke pairs is residual (resp. dense) in $X \times X$. A transitive interval map is generically chaotic; conversely, a generically chaotic interval map has exactly one or two transitive subintervals. Dense chaos is strictly weaker than generic chaos: a densely chaotic interval map may have no transitive subinterval, as illustrated by an example. We show that, if f is a densely chaotic interval map, then f^2 has a horseshoe, which implies that f has a periodic point of period 6 and the topological entropy of f is at least $\frac{\log 2}{2}$.

Distributional chaos is based on a refinement of the conditions defining Li-Yorke pairs. We show that, for interval maps, distributional chaos is equivalent to positive topological entropy.

In **Chapter 7**, we focus on the existence of some kinds of chaotic subsystems and we relate them to the previous notions.

A system is said to be *chaotic in the sense of Devaney* if it is transitive, sensitive to initial conditions and has a dense set of periodic points. For an interval map, the existence of an invariant closed subset that is chaotic in the sense of Devaney is equivalent to positive topological entropy. We also show that an interval map has an invariant uncountable closed subset X on which f^n is topologically mixing for some $n \geq 1$ if and only if f has positive topological entropy.

Finally, we study the existence of an invariant closed subset on which the map is transitive and sensitive to initial conditions. We show that this property is implied by positive topological entropy and implies chaos in the sense of Li-Yorke. However these notions are distinct: there exist zero entropy interval maps with a transitive

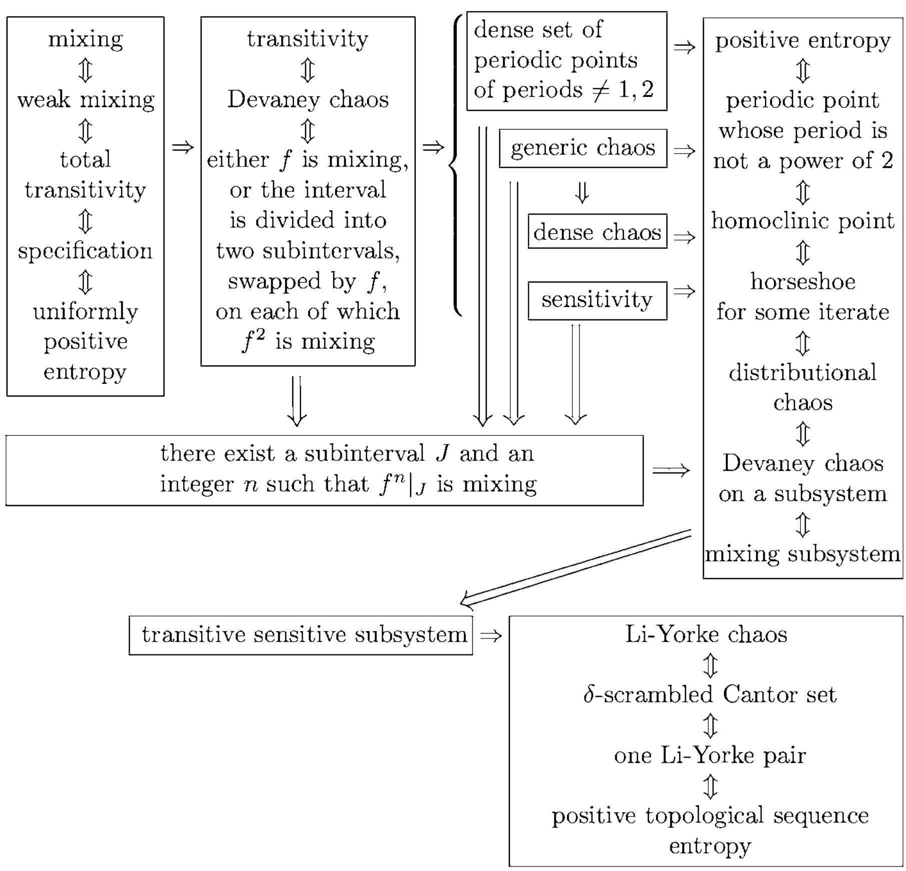

FIGURE 0.1. Diagram summarizing the relations between the main notions related to chaos for an interval map f.

sensitive subsystem and interval maps with no transitive sensitive subsystem that are chaotic in the sense of Li-Yorke.

The **last chapter** is an appendix that recalls succinctly some background in topology.

The relations between the main notions studied in this book are summarized by the diagram in Figure 0.1 on page xi.

I thank all the people who have contributed to improve this book by remarks or translations: Jozef Bobok, Victor Jiménez López, Sergiy Kolyada, Jian Li, Michał Misiurewicz, T. K. Subrahmonian Moothathu, Ľubomar Snoha, Émilie Tyberghein, Zheng Wei, and Dawei Yang. I particularly thank Roman Hric who helped me to fill a gap in a proof.

I want to thank CNRS (Centre National de la Recherche Scientifique) for giving me time (two sabbatical semesters) to write this book, which allowed me to finish this long-standing project.

Sylvie Ruette