CHAPTER 1

Notation and basic tools

1.1. General notation

1.1.1. Sets of numbers. The set of natural numbers (that is, positive integers) is denoted by $\mathbb{N}$. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote respectively the set of all integers, rational numbers, real numbers and complex numbers. The nonnegative integers and nonnegative real numbers are denoted respectively by $\mathbb{Z}^+$ and $\mathbb{R}^+$.

1.1.2. Interval of integers. The notation $[\![n, m]\!]$ denotes an interval of integers, that is, $[\![n, m]\!] := \{k \in \mathbb{Z} \mid n \leq k \leq m\}$.

We shall often deal with sets $X_1, \ldots, X_n$ that are cyclically permuted. The notation $X_{i+1 \bmod n}$ means X_{i+1} if $i \in [\![1, n-1]\!]$ and X_1 if $i = n$. More generally, if the set of indices $\mathcal{I}$ under consideration is $[\![1, n]\!]$ (resp. $[\![0, n-1]\!]$), then $i \bmod n$ denotes the integer $j \in \mathcal{I}$ such that $j \equiv i \bmod n$.

1.1.3. Cardinality of a set. If E is a finite set, $\#E$ denotes its cardinality, that is, the number of elements in E.

A set is *countable* if it can be written as $\{x_n \mid n \in \mathbb{N}\}$. A finite set is countable.

1.1.4. Notation of topology. The definitions of the topological notions used in this book are recalled in the appendix. Here we only give some notation.

Let X be a metric space and let Y be a subset of X. Then $\overline{Y}$, $\mathrm{Int}\,(Y)$, $\mathrm{Bd}(Y)$ denote respectively the closure, the interior and the boundary of Y.

Remark 1.1. When talking about topological notions (neighborhood, interior, etc.), we always refer to the induced topology on the ambient space X. For instance, in Example 1.2 below, $[0, 1/2)$ is an open set since the ambient space is $[0, 1]$.

The distance on a metric space X is denoted by d. If $x \in X$ and $r > 0$, the open ball of center x and radius r is $B(x, r) := \{y \in X \mid d(x, y) < r\}$, and the closed ball of center x and radius r is $\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}$.

The diameter of a set $Y \subset X$ is $\mathrm{diam}(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$. If Y is compact, then the supremum is reached.

1.1.5. Restriction of a map. Let $f \colon X \to X'$ be a map and $Y \subset X$. The restriction of f to Y, denoted by $f|_Y$, is the map $f|_Y \colon \begin{array}{ccc} Y & \to & X' \\ x & \mapsto & f(x) \end{array}$.

1.2. Topological dynamical systems, orbits, ω-limit sets

Our purpose is to study dynamical systems on intervals. However we prefer to give the notation in a broader context because most of the definitions have a meaning for any dynamical system, and a few properties will not be specific to the interval case.

1.2.1. Topological dynamical systems, invariant set. A *topological dynamical system* (X, f) is given by a continuous map $f: X \to X$, where X is a nonempty compact metric space. The evolution of the system is given by the successive iterations of the map. If $n \in \mathbb{N}$, the n-th iterate of f is denoted by f^n, that is,

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

By convention, f^0 is the identity map on X. We can think of n as time: starting from an initial position x at time 0, the point $f^n(x)$ represents the new position at time n.

EXAMPLE 1.2. Let $f: [0,1] \to [0,1]$ be the map defined by $f(x) = 3x(1-x)$. The successive iterates of x can be plotted on the graph of f, as illustrated in Figure 1.1; the diagonal $y = x$ is utilized to re-use the result of an iteration.

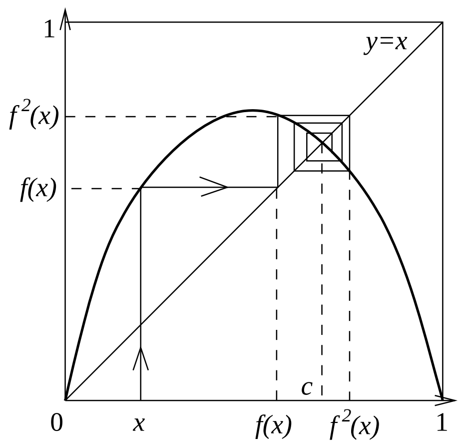

FIGURE 1.1. The first iterates of x plotted on the graph of f.

Let (X, f) be a topological dynamical system. An *invariant* (or f-*invariant*) set is a nonempty closed set $Y \subset X$ such that $f(Y) \subset Y$; it is *strongly invariant* if in addition $f(Y) = Y$. If Y is an invariant set, let $f|_Y$ denote the map f restricted to Y and *arriving in* Y, that is, $f|_Y: Y \to Y$. With this slight abuse of notation, $(Y, f|_Y)$ is a topological dynamical system, called a *subsystem* of (X, f), and we shall speak of the properties of $f|_Y$ (e.g., "$f|_Y$ is transitive").

1.2.2. Trajectory, orbit, periodic point. In the literature, the words *trajectory* and *orbit* often have the same meaning. However we prefer to follow the terminology of Block-Coppel [**41**] because it is convenient to make a distinction between two notions. In this book, when (X, f) is a topological dynamical system and x is a point in X, the *trajectory* of x is the infinite sequence $(f^n(x))_{n \geq 0}$ (there may be repetitions in the sequence) and the *orbit* of x is the set $\mathcal{O}_f(x) := \{f^n(x) \mid n \geq 0\}$. Similarly, if E is a subset of X, then $\mathcal{O}_f(E) := \bigcup_{n \geq 0} f^n(E)$.

A point x is *periodic* (for the map f) if there exists a positive integer n such that $f^n(x) = x$. The *period* of x is the least positive integer p such that $f^p(x) = x$.

It is easy to see that, if x is periodic of period p and $n \in \mathbb{N}$, then $f^n(x) = x$ if and only if n is a multiple of p; moreover $\mathcal{O}_f(x)$ is a finite set of p distinct points: $\mathcal{O}_f(x) = \{x, f(x), \ldots, f^{p-1}(x)\}$. If x is a periodic point of period p, then its orbit is called a *periodic orbit of period* p. Each point of a periodic orbit is a periodic point with the same period and the same orbit. If $f(x) = x$, then x is called a *fixed point*. Let

$$P_n(f) := \{x \in X \mid f^n(x) = x\};$$

this is the set of periodic points whose periods divide n.

A point x is *eventually periodic* if there exists an integer $n \geq 0$ such that $f^n(x)$ is periodic.

1.2.3. Omega-limit set. Let (X, f) be a topological dynamical system. The *ω-limit set* of a point $x \in X$, denoted by $\omega(x, f)$, is the set of all limit points of the trajectory of x, that is,

$$\omega(x, f) := \bigcap_{n \geq 0} \overline{\{f^k(x) \mid k \geq n\}}.$$

The *ω-limit set of the map* f is

$$\omega(f) := \bigcup_{x \in X} \omega(x, f).$$

LEMMA 1.3. *Let (X, f) be a topological dynamical system, $x \in X$ and $n \geq 1$. Then*

i) *$\omega(x, f)$ is a closed set, and it is strongly invariant,*
ii) *$\omega(f^n(x), f) = \omega(x, f)$,*
iii) *$\forall i \geq 0$, $\omega(f^i(x), f^n) = f^i(\omega(x, f^n))$,*
iv) *$\omega(x, f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n)$,*
v) *if $\omega(x, f)$ is infinite, then $\omega(f^i(x), f^n)$ is infinite for all $i \geq 0$,*
vi) *$f(\omega(f)) = \omega(f)$,*
vii) *$\omega(f^n) = \omega(f)$.*

PROOF. Assertions (i) to (iv) can be easily deduced from the definition. Assertion (vi) follows from (i), assertions (v) and (vii) follow from (iii)-(iv). $\square$

LEMMA 1.4. *Let (X, f) be a topological dynamical system and $x \in X$. If $\omega(x, f)$ is finite, then it is a periodic orbit.*

PROOF. Let F be a nonempty subset of $\omega(x, f)$ different from $\omega(x, f)$. We set $F' := \omega(x, f) \setminus F$. Both F, F' are finite and nonempty. Let U, U' be two open sets such that $F \subset U$, $F' \subset U'$, $\overline{U} \cap F' = \emptyset$ and $\overline{U'} \cap F = \emptyset$. Thus, for every large enough integer n, the point $f^n(x)$ belongs to $U \cup U'$. Moreover, there are infinitely many integers n such that $f^n(x) \in U$ and infinitely many n such that $f^n(x) \in U'$. Therefore, there exists an increasing sequence $(n_i)_{i \geq 0}$ such that, $\forall i \geq 0$, $f^{n_i}(x) \in U$ and $f^{n_i+1}(x) \in U'$. By compactness, the sequence $(f^{n_i}(x))_{i \geq 0}$ has a limit point $y \in \overline{U} \cap \omega(x, f) = F$. Since f is continuous, $f(y)$ is a limit point of $(f^{n_i+1}(x))_{i \geq 0}$, and hence $f(y) \in \omega(x, f) \cap \overline{U'} = F'$. Thus $f(F) \cap F' \neq \emptyset$, and so $\omega(x, f)$ contains no invariant subset except itself. This implies that f acts as a cyclic permutation on $\omega(x, f)$, that is, $\omega(x, f)$ is a periodic orbit. $\square$

1.2.4. Semi-conjugacy, conjugacy. Let (X, f) and (Y, g) be two topological dynamical systems. The system (Y, g) is said to be *(topologically) semi-conjugate* to (X, f) if there exists a continuous onto map $\varphi\colon X \to Y$ such that $\varphi \circ f = g \circ \varphi$. If in addition the map φ is a homeomorphism, (Y, g) is *(topologically) conjugate* to (X, f); conjugacy is an equivalence relation. Two conjugate dynamical systems share the same dynamical properties as long as topology is concerned (differential properties may not be preserved if φ is only assumed to be continuous).

1.3. Intervals, interval maps

1.3.1. Intervals, endpoints, length, nondegenerate interval, inequalities between subsets of $\mathbb{R}$. The (real) intervals are exactly the connected sets of $\mathbb{R}$. An interval J is either the empty set or one of the following forms:

- $J = [a, b]$ with $a, b \in \mathbb{R}$, $a \leq b$ (if $a = b$, then $J = \{a\}$),
- $J = (a, b)$ with $a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{+\infty\}$, $a < b$,
- $J = [a, b)$ with $a \in \mathbb{R}, b \in \mathbb{R} \cup \{+\infty\}$, $a < b$,
- $J = (a, b]$ with $a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R}$, $a < b$.

Suppose that J is nonempty and bounded (i.e., when $a, b \in \mathbb{R}$). The *endpoints* of J are a and b; let ∂J denote the set $\{a, b\}$. The *length* of J, denoted by $|J|$, is equal to $b - a$.

An interval is *degenerate* if it is either empty or reduced to a single point, and it is *nondegenerate* otherwise.

If $a, b \in \mathbb{R}$, let $\langle a, b\rangle$ denote the smallest interval containing $\{a, b\}$, that is, $\langle a, b\rangle = [a, b]$ if $a \leq b$ and $\langle a, b\rangle = [b, a]$ if $b \leq a$.

If X and Y are two nonempty subsets of $\mathbb{R}$, the notation $X < Y$ means that, $\forall x \in X, \forall y \in Y$, $x < y$ (in this case X and Y are disjoint) and $X \leq Y$ means that, $\forall x \in X, \forall y \in Y$, $x \leq y$ (X and Y may have a common point, equal to $\max X = \min Y$). We may also say that X is on the left of Y.

Lemma 1.5. *Every open set $U \subset \mathbb{R}$ can be written as the union of countably many disjoint open intervals.*

Proof. The connected components of U are disjoint nonempty open intervals, and every nondegenerate interval contains a rational number, which implies that the connected components of U are countable. □

1.3.2. Interval maps, monotonicity, critical points, piecewise monotone and piecewise linear maps. We say that $f\colon I \to I$ is an *interval map* if I is a nondegenerate compact interval and f is a continuous map.

When dealing with an interval map $f\colon I \to I$, we shall always refer to the ambient space. The topology is the induced topology on I; points and sets are implicitly points in I and subsets of I, and intervals are subintervals of I (and hence are bounded intervals).

Remark 1.6. The fixed points of an interval map f can be easily seen on the graph of f. Indeed, x is a fixed point if and only if (x, x) is in the intersection of the graph $y = f(x)$ with the diagonal $y = x$. E.g., in Example 1.2, the map has two fixed points, 0 and c. Similarly, the points of $P_n(f)$ correspond to the intersection of the graph of f^n with $y = x$.

Let $f\colon I \to \mathbb{R}$ be a continuous map, where I is an interval, and let J be a nondegenerate subinterval of I.

- The map f is increasing (resp. decreasing) on J if for all points $x, y \in J$, $x < y \Rightarrow f(x) < f(y)$ (resp. $f(x) > f(y)$).
- The map f is nondecreasing (resp. nonincreasing) on J if for all $x, y \in J$, $x < y \Rightarrow f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$).
- The map f is *monotone* (resp. *strictly monotone*) on J if f is either nondecreasing or nonincreasing (resp. either increasing or decreasing) on J.

A *critical point of* f is a point $x \in I$ such that there exists no neighborhood of x on which f is strictly monotone. Notice that if f is differentiable, the set of critical points is included in the set of zeros of f'.

The map f is *piecewise monotone* if the interval I can be divided into finitely many subintervals on each of which f is monotone, that is, there exist points $a_0 = \min I < a_1 < \dots a_{n-1} < a_n = \max I$ such that f is monotone on $[a_i, a_{i+1}]$ for all $i \in [\![0, n-1]\!]$. The set of critical points of f is included in $\{a_1, \dots, a_{n-1}\}$. Conversely, if the set of critical points of f is finite, then f is piecewise monotone.

REMARK 1.7. The critical points are also called *turning points*, especially when the map f is piecewise monotone.

Let $f\colon I \to \mathbb{R}$ be a continuous map, where $I := [a, b]$, $a < b$. The map f is *linear* if there exist $\alpha, \beta \in \mathbb{R}$ such that $f(x) = \alpha x + \beta$ for all $x \in [a, b]$. The *slope* of f is $\mathsf{slope}(f) := \alpha$. One has $\mathsf{slope}(f) = \frac{f(b)-f(a)}{b-a}$ and $|\mathsf{slope}(f)| = \frac{|f(I)|}{|I|}$.

f is *piecewise linear* if there exist $a_0 = \min I < a_1 < \dots < a_{n-1} < a_n = \max I$ such that f is linear on $[a_i, a_{i+1}]$ for all $i \in [\![0, n-1]\!]$. In particular, a piecewise linear map is piecewise monotone.

Most of our examples will be piecewise linear.

1.3.3. Rescaling. If two interval maps f and g are conjugate by an increasing linear homeomorphism, they have the same graph up to the action of a homothety or a translation. We call this action a *rescaling*. If g is conjugate to f by a decreasing linear homeomorphism, the graph of g is obtained from the one of f by a half-turn rotation and a rescaling. Not only are the maps f and g conjugate, but they have exactly the same properties (when the conjugacy is decreasing, it just reverses the order when order is involved in a property).

REMARK 1.8. When dealing with interval maps, one may assume that the interval is $[0, 1]$. Indeed, if $f\colon [a, b] \to [a, b]$ is an interval map, let $\varphi\colon [0, 1] \to [a, b]$ be the linear homeomorphism defined by $\varphi(x) := a + (b-a)x$ and let $g := \varphi^{-1} \circ f \circ \varphi$. The maps $f\colon [a, b] \to [a, b]$ and $g\colon [0, 1] \to [0, 1]$ are conjugate, and g is a mere rescaling of f.

1.3.4. Periodic intervals. Let $f\colon I \to I$ be an interval map. If $J_1, \dots, J_p$ are disjoint nondegenerate closed subintervals of I such that $f(J_i) = J_{i+1 \bmod p}$ for all $i \in [\![1, p]\!]$, then $(J_1, \dots, J_p)$ (as well as the set $C := J_1 \cup \dots \cup J_p$) is called a *cycle of intervals of period* p. Moreover, J_1 is called a *periodic interval of period* p.

1.3.5. Intermediate value theorem. The intermediate value theorem is fundamental and we shall use it constantly. For convenience, we give several equivalent statements.

THEOREM 1.9 (Intermediate value theorem). *Let $f\colon I \to \mathbb{R}$ be a continuous map, where I is a nonempty interval.*

- *Let J be a nonempty subinterval of I. Then $f(J)$ is also a nonempty interval.*
- *Let $x_1, x_2 \in I$ with $x_1 \leq x_2$. Then $f([x_1, x_2]) \supset \langle f(x_1), f(x_2)\rangle$. In particular, for every c between $f(x_1)$ and $f(x_2)$, there exists $x \in [x_1, x_2]$ such that $f(x) = c$.*

PROOF. The first assertion follows from the fact that the image of a connected set by a continuous map is connected (Theorem 8.74 in Appendix) and the image of a nonempty set is nonempty. The second assertion is a straightforward consequence of the first one with $J = [x_1, x_2]$. □

Definition of graph maps. A *topological graph* is a compact connected metric space G containing a finite subset V such that $G \setminus V$ has finitely many connected components and every connected component of $G \setminus V$ is homeomorphic to $(0, 1)$. A topological graph is *nondegenerate* if it contains more than one point. A *subgraph* of G is a closed connected subset of G; a subgraph is a topological graph. A *tree* is a topological graph containing no subset homeomorphic to a circle. A *branching point* is a point having no neighborhood homeomorphic to a real interval. An *endpoint* is a point having a neighborhood homeomorphic to the half-closed interval $[0, 1)$. The sets of branching points and endpoints are finite (they are included in V). If H is a subgraph of G, the set of endpoints of H is denoted by ∂H. A subset of G is called an *interval* (resp. a *circle*) if it is homeomorphic to an interval of the real line (resp. a circle of positive radius).

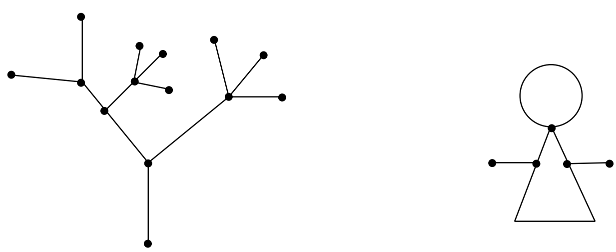

FIGURE 1.2. A tree (on the left) and a topological graph (on the right). The branching points and the endpoints are indicated by big dots.

A *graph* (resp. *tree*) *map* is a continuous map $f\colon X \to X$, where X is a nondegenerate topological graph (resp. tree). If $G_1, \ldots, G_p$ are disjoint nondegenerate subgraphs of X such that $f(G_i) = G_{i+1 \bmod p}$ for all $i \in [\![1, p]\!]$, then $(G_1, \ldots, G_p)$ is called a *cycle of graphs of period p*.

DEFINITION 1.10. Let $f\colon G \to G$ be a graph map. If $I \subset G$ is either a nondegenerate interval or a circle, the map $f|_I$ is said to be *monotone* if it is locally monotone at every point $x \in I$, that is, there exists an open neighborhood U of x with respect to the topology of I such that:

- U contains $K(x)$, where $K(x) \subset I$ is the largest subinterval of I containing x on which f is constant,
- U and $f(U)$ are homeomorphic to intervals,

- $f|_U: U \to f(U)$, seen as a map between intervals, is monotone (more precisely, there exist intervals $J, J' \subset \mathbb{R}$ and homeomorphisms $h: U \to J$, $h': f(U) \to J'$ such that $h' \circ f|_U \circ h^{-1}: J \to J'$ is monotone).

Notice that, when G is a tree, the fact that $f|_I$ is monotone implies that $f(I)$ is necessarily an interval, whereas in general $f(I)$ may not be an interval (in particular, $f(I)$ may wrap around circles).

1.4. Chains of intervals and periodic points

The next lemma is a basic tool to prove the existence of fixed points. Below, Lemma 1.13 states the existence of periodic points when some intervals are nested under the action of f.

LEMMA 1.11. *Let $f: [a,b] \to \mathbb{R}$ be a continuous map. If $f([a,b]) \subset [a,b]$ or $f([a,b]) \supset [a,b]$, then f has a fixed point.*

PROOF. Let $g(x) := f(x) - x$. If $f([a,b]) \subset [a,b]$, then

$$g(a) = f(a) - a \geq a - a = 0 \quad \text{and} \quad g(b) = f(b) - b \leq b - b = 0.$$

By the intermediate value theorem applied to g, there exists $c \in [a,b]$ with $g(c) = 0$. If $f([a,b]) \supset [a,b]$, there exist $x, y \in [a,b]$ such that $f(x) \leq a$ and $f(y) \geq b$. We then have

$$g(x) = f(x) - x \leq a - x \leq 0 \quad \text{and} \quad g(y) = f(y) - y \geq b - y \geq 0.$$

Thus there exists $c \in \langle x, y \rangle$ with $g(c) = 0$ by the intermediate value theorem. In both cases, c is a fixed point of f. □

DEFINITION 1.12 (Covering, chain of intervals). Let f be an interval map.

- Let J, K be two nonempty closed intervals. Then J is said to *cover* K (for f) if $K \subset f(J)$. This is denoted by $J \underset{f}{\to} K$, or simply $J \to K$ if there is no ambiguity. If k is a positive integer, J *covers* K k *times* if J contains k closed subintervals with disjoint interiors such that each one covers K.
- Let $J_0, \dots, J_n$ be a nonempty closed interval such that J_{i-1} covers J_i for all $i \in [\![1,n]\!]$. Then $(J_0, J_1, \dots, J_n)$ is called a *chain of intervals (for f)*. This is denoted by $J_0 \to J_1 \to \dots \to J_n$.

LEMMA 1.13. *Let f be an interval map and $n \geq 1$.*

i) *Let $J_0, \dots, J_n$ be nonempty intervals such that $J_i \subset f(J_{i-1})$ for all i in $[\![1,n]\!]$. Then there exists an interval $K \subset J_0$ such that $f^n(K) = J_n$, $f^n(\partial K) = \partial J_n$ and $f^i(K) \subset J_i$ for all $i \in [\![0,n]\!]$. If in addition $J_0, \dots, J_n$ are closed (and so $(J_0, \dots, J_n)$ is a chain of intervals), then K can be chosen to be closed.*

ii) *Let $(J_0, \dots, J_n)$ be a chain of intervals such that $J_0 \subset J_n$. Then there exists $x \in J_0$ such that $f^n(x) = x$ and $f^i(x) \in J_i$ for all $i \in [\![0,n-1]\!]$.*

iii) *Suppose that, for every $i \in [\![1,p]\!]$, $(J_0^i, \dots, J_n^i)$ is a chain of intervals and, for every pair (i,j) of distinct indices in $[\![1,p]\!]$, there exists $k \in [\![0,n]\!]$ such that J_k^i and J_k^j have disjoint interiors. Then there exist closed intervals $K_1, \dots, K_p$ with pairwise disjoint interiors such that*

$$\forall i \in [\![1,p]\!],\ f^n(K_i) = J_n^i,\ f^n(\partial K_i) = \partial J_n^i$$

$$\text{and} \quad \forall k \in [\![0,n]\!],\ \forall i \in [\![1,p]\!],\ f^k(K_i) \subset J_k^i.$$

PROOF. We first prove by induction on n the following:

FACT 1. Let $J_0, \ldots, J_n$ be nonempty intervals such that $J_i \subset f(J_{i-1})$ for all $i \in [\![1, n]\!]$. Then there exist intervals $K_n \subset K_{n-1} \subset \cdots \subset K_1 \subset J_0$ such that, for all $k \in [\![1, n]\!]$ and all $i \in [\![0, k]\!]$,

$$f^i(K_k) \subset J_i, f^k(K_k) = J_k, f^k(\partial K_k) = \partial J_k \quad \text{and} \quad f^k(\mathrm{Int}\,(K_k)) = \mathrm{Int}\,(J_k).$$

Moreover, if $J_0, \ldots, J_n$ are closed, then $K_1, \ldots, K_n$ can be chosen to be closed too.

• Case $n = 1$. We write $\overline{J_1} = [a, b]$. There exist $x, y \in \overline{J_0}$ such that $f(x) = a$ and $f(y) = b$. If a (resp. b) belongs to $f(J_0)$, we choose x (resp. y) in J_0. If a (resp. b) does not belong to $f(J_0)$, then it does not belong to J_1 either, and x (resp. y) is necessarily an endpoint of J_0. With no loss of generality, we may suppose that $x \le y$ (the other case being symmetric). We define

$$y' := \min\{z \ge x \mid f(z) = b\}, \quad x' := \max\{z \le y' \mid f(z) = a\}$$

and $K_1' := [x', y']$. Then $f(K_1') = \overline{J_1}$, $f(\{x', y'\}) = \{a, b\}$ and no other point in K_1' is mapped to a or b by f. If J_1 is closed, then $K_1 := K_1'$ is suitable. Otherwise, it is easy to check that K_1 can be chosen among $(x', y'), [x', y'), (x', y']$ in such a way that $f(K_1) = J_1$ and $K_1 \subset J_0$.

• Suppose that Fact 1 holds for n and consider nonempty intervals $J_0, \ldots, J_n, J_{n+1}$ such that $J_i \subset f(J_{i-1})$ for all $i \in [\![1, n+1]\!]$. Let $K_1, \ldots, K_n$ be the intervals given by Fact 1 applied to $J_0, \ldots, J_n$. Since $f^{n+1}(K_n) = f(J_n) \supset J_{n+1}$, we can apply the case $n = 1$ for the map $g := f^{n+1}$ and the two intervals K_n, J_{n+1}. We deduce that there exists an interval $K_{n+1} \subset K_n$, which is closed if $J_0, \ldots, J_{n+1}$ are closed, and such that

$$f^{n+1}(K_{n+1}) = J_{n+1},$$
$$f^{n+1}(\partial K_{n+1}) = \partial J_{n+1} \quad \text{and} \quad f^{n+1}(\mathrm{Int}\,(K_{n+1})) = \mathrm{Int}\,(J_{n+1}).$$

Moreover, $f^i(K_{n+1}) \subset J_i$ for all $i \in [\![0, n]\!]$ because $K_{n+1} \subset K_n$. This ends the proof of Fact 1, which trivially implies (i).

Let $(J_0, \ldots, J_n)$ be a chain of intervals such that $J_0 \supset J_n$. Fact 1 implies that there exists a closed interval $K_n \subset J_0$ such that $f^n(K_n) = J_n$ and $f^i(K_n) \subset J_i$ for all $i \in [\![0, n]\!]$. Thus $f^n(K_n) \supset K_n$ and it is sufficient to apply Lemma 1.11 to $g := f^n|_{K_n}$ in order to find a point $x \in K_n$ such that $f^n(x) = x$. For all $i \in [\![0, n-1]\!]$, $f^i(x)$ obviously belongs to J_i. This proves (ii).

Let $(J_0^i, \ldots, J_n^i)_{1 \le i \le p}$ be chains of intervals satisfying the assumptions of (iii). For every $i \in [\![1, p]\!]$, let $(K_1^i, \ldots, K_n^i)$ be the closed intervals given by Fact 1 for $(J_0^i, \ldots, J_n^i)$, and set $K_i := K_n^i$. We fix $i \ne j$ in $[\![1, p]\!]$. By assumption, there exists $k \in [\![0, n]\!]$ such that J_k^i and J_k^j have disjoint interiors. If $k = 0$, then K_i and K_j have trivially disjoint interiors because they are respectively included in J_0^i and J_0^j. From now on, we assume that $k \ge 1$. Suppose that $K_i^k \cap K_j^k \ne \emptyset$. The set $f^k(K_i^k \cap K_j^k)$ is included in $J_k^i \cap J_k^j$ and, by assumption, J_k^i and J_k^j have disjoint interiors. Therefore the intervals J_k^i and J_k^j have a common endpoint, say b, and $f^k(K_i^k \cap K_j^k) = \{b\}$. By definition of K_i^k, there is a unique point z in K_i^k such that $f^k(z) = b$, and the same holds for K_j^k. Hence $K_i^k \cap K_j^k$ contains at most one point. Since $K_i \subset K_i^k$ and $K_j \subset K_j^k$, the intervals K_i and K_j have disjoint interiors. This concludes the proof of (iii). □

Definitions for graph maps. The notion of covering extends to graph maps provided Definition 1.12 is phrased differently. A modification is needed for two reasons:

- one may want to consider circles as "intervals" whose endpoints are equal,
- for a graph map f, it may occur that a compact interval I satisfies $f(I) \supset I$ but contains no fixed point, as illustrated in Figure 1.3.

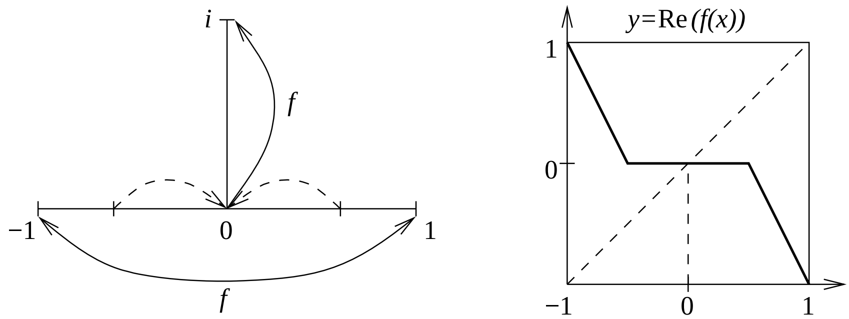

FIGURE 1.3. Let $f: X \to X$ be a continuous map, where X is the tree $[-1,1]\cup i[0,1] \subset \mathbb{C}$ (on the left), and f is such that $f(-1) = 1$, $f(1) = -1$, $f(0) = i$ and f is one-to-one on $[-1,0]$ and $[0,1]$ (the definition of f on $i[0,1]$ does not matter). Set $I := [-1,1]$. It is clear that $f(I) \supset I$. Nevertheless f has no fixed point in I. On the right is represented the real part of $f|_I$; the constant interval corresponds to the points $x \in I$ such that $f(x) \in i[0,1]$.

DEFINITION 1.14. Let $f: G \to G$ be a graph map and let J, K be two non-degenerate intervals in G. Then J is said to *cover* K if there exists a subinterval $J' \subset J$ such that $f(J') = K$ and $f(\partial J') = \partial K$. If $J_0, J_1, \dots, J_n$ are intervals in X such that J_{i-1} covers J_i for all $i \in [\![1, n]\!]$, then $(\overline{J_0}, \dots, \overline{J_n})$ is a *chain of intervals* (this is a slight abuse of notation since, if $\overline{J_i}$ is a circle, it is necessary to remember the endpoint of J_i).

Using this definition, Lemma 1.13(ii)-(iii) remains valid for graph maps. In particular, if $(\overline{J_0}, \dots, \overline{J_{n-1}}, \overline{J_0})$ is a chain of intervals for a graph map f, then there exists a point $x \in \overline{J_0}$ such that $f^n(x) = x$ and $f^i(x) \in \overline{J_i}$ for all $i \in [\![1, n-1]\!]$.

A variant, called *positive covering*, has been introduced in [**16**]. Positive covering does not imply covering, but implies the same conclusions concerning periodic points. We do not state the definition because it will not be needed in this book. See [**16, 17**] for the details.

1.5. Directed graphs

A (finite) directed graph G is a pair (V, A) where V, A are finite sets and there exist two maps $i, f: A \to V$. The elements of V are the *vertices* of G and the elements of A are the *arrows* of G. An arrow $a \in A$ goes from its *initial vertex* $u = i(a)$ to its *final vertex* $v = f(a)$. The arrow a is also denoted by $u \xrightarrow{a} v$. A directed graph is often given by a picture, as in Example 1.15. If $V = \{v_1, \dots, v_p\}$, the *adjacency matrix* of G is the matrix $M = (m_{ij})_{1\le i,j\le p}$, where m_{ij} is equal to the number of arrows from v_i to v_j. Conversely, if $M = (m_{ij})_{1 \le i,j \le p}$ is a matrix such that $m_{ij} \in \mathbb{Z}^+$ for all $i, j \in [\![1, p]\!]$, one can build a directed graph whose adjacency matrix is M: it has p vertices $\{v_1, \ldots, v_p\}$ and there are m_{ij} arrows from v_i to v_j for all $i, j \in [\![1, p]\!]$.

A directed graph is *simple* if, for every pair of vertices (u, v), there is at most one arrow from u to v. In this case, an arrow $u \xrightarrow{a} v$ is simply denoted by $u \to v$ since there is no ambiguity. A directed graph is simple if and only if all the coefficients of its adjacency matrix belong to $\{0, 1\}$.

There are several, equivalent norms for matrices. We shall use the following one: if $M = (m_{ij})_{1 \le i,j \le p}$, we set $\|M\| := \sum_{1 \le i,j \le p} |m_{ij}|$.

EXAMPLE 1.15. Figure 1.4 represents a directed graph with three vertices v_1, v_2, v_3. Its adjacency matrix is $\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

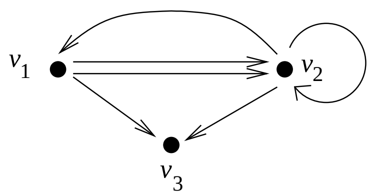

FIGURE 1.4. An example of a directed graph.

Let G be a directed graph. A *path of length n from u_0 to u_n* is a sequence

$$u_0 \xrightarrow{a_1} u_1 \xrightarrow{a_2} u_2 \xrightarrow{a_3} \cdots u_{n-1} \xrightarrow{a_n} u_n,$$

where $u_0, \ldots, u_n$ are vertices of G and $u_i \xrightarrow{a_i} u_{i+1}$ is an arrow in G for all i in $[\![0, n-1]\!]$. Such a path is called a *cycle* if $u_0 = u_n$.

If $\mathcal{A} := A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} A_n$ and $\mathcal{B} := B_0 \xrightarrow{b_1} B_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} B_m$ are two paths such that $A_n = B_0$, the *concatenation* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{AB}$, is the path

$$A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} A_n \xrightarrow{b_1} B_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} B_m.$$

If $\mathcal{A}$, $\mathcal{B}$ are of respective lengths n, m, then $\mathcal{AB}$ is of length $n + m$.

A cycle is *primitive* if it is not the repetition of a shorter cycle, that is, it cannot be written $\underbrace{\mathcal{A}\mathcal{A} \cdots \mathcal{A}}_{n \text{ times}}$ where $\mathcal{A}$ is a cycle and $n \ge 2$.

A straightforward computation leads to the following result.

PROPOSITION 1.16. *Let G be a directed graph and let $\{v_1, \ldots, v_p\}$ denote its set of vertices. Let M be its adjacency matrix. For every $n \in \mathbb{N}$, let $M^n = (m^n_{ij})_{1 \le i,j \le p}$. Then, $\forall n \ge 1$, $\forall i, j \in [\![1, p]\!]$, the number of paths of length n from v_i to v_j is equal to m^n_{ij}. As a consequence, the number of paths of length n in G is equal to*

$$\|M^n\| = \sum_{1 \le i,j \le p} m^n_{ij}.$$