CHAPTER 1

Introduction

1.1. The model of first-passage percolation and its history

First-passage percolation (FPP) was introduced by Hammersley and Welsh [**108**] in 1965 as a model of fluid flow through a random medium. It has been a stage of research for probabilists since its origin, but despite all efforts through the past decades, most of the predictions about its important statistics remain to be understood. Most of the beauty of the model lies in its simple definition (as a random metric space) and that several of its fascinating conjectures do not require much effort to be stated. During these 50 years, FPP has attracted the attention of theoretical physicists, biologists, and computer scientists, and also gave birth to some now fundamental mathematical tools, like the sub-additive ergodic theorem. Here, we will focus on the model defined on the lattice $\mathbb{Z}^d$ with independent and identically distributed (i.i.d.) edge-weights; some variants will be discussed in Chapter 7.

The model is defined as follows. We place a non-negative random variable τ_e, called the passage time of the edge e, at each nearest-neighbor edge in $\mathbb{Z}^d$. The collection (τ_e) is assumed to be i.i.d. with common distribution function F and corresponding probability measure ν. The random variable τ_e is interpreted as the time or the cost needed to traverse the edge e.

A path Γ is a finite or infinite sequence of edges $e(1), e(2), \ldots$, in $\mathbb{Z}^d$ such that for each $n \geq 1$, $e(n)$ and $e(n+1)$ share at least one endpoint. For any finite path Γ we define the passage time of Γ to be

$$T(\Gamma) = \sum_{e \in \Gamma} \tau_e.$$

Given two points $x, y \in \mathbb{R}^d$ one then sets

$$T(x,y) = \inf_{\Gamma} T(\Gamma), \tag{1.1}$$

where the infimum is over all finite paths Γ that contain both x' and y', and x' is the unique vertex in $\mathbb{Z}^d$ such that $x \in x' + [0,1)^d$ (similarly for y'). The random variable $T(x,y)$ will be called the passage time between points x and y. In the original interpretation of the model, $T(x,y)$ represents the time that a fluid with source at x takes to reach the location y.

For each $t \geq 0$ let

$$B(t) = \{y \in \mathbb{R}^d \ : \ T(0,y) \leq t\}.$$

In the case that $F(0) = 0$, the pair $(\mathbb{Z}^d, T(\cdot,\cdot))$ is almost surely (a.s.) a metric space and $B(t) \cap \mathbb{Z}^d$ is the (random) ball of radius t centered at the origin. (See

Figure 1.) The ultimate goal of first-passage percolation is to understand the large-scale properties of this metric. Some of the main questions include the following. We write $|\cdot|$ for the ℓ^2 norm in $\mathbb{R}^d$.

(1) What is the typical distance between two points that are far from each other in the lattice? Or in other words, what can we say about $T(x,y)$ as $|x-y| \to \infty$? Does it converge, when possibly rescaled and recentered? If so, what is the rate of convergence?
(2) What does a ball of large radius look like? Is there a scaling limit and fluctuation theory for the set $B(t)$?
(3) What is the geometry of geodesics (time-minimizing paths) between two distant points? How different are they from straight lines?
(4) What role does the distribution of the passage times play in describing the metric?

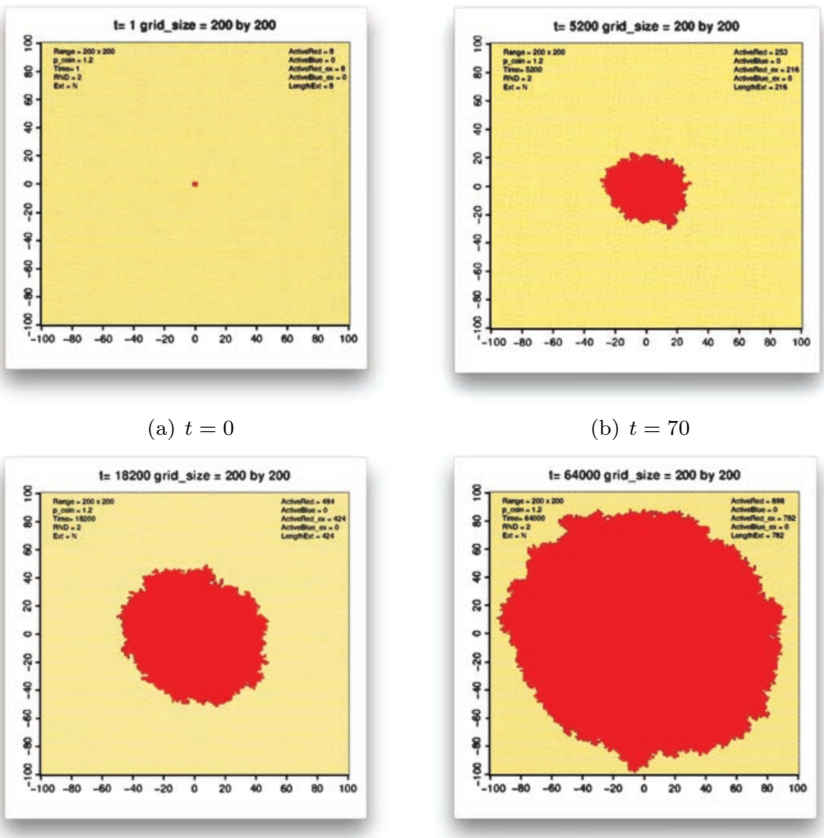
In this book, we will discuss progress on these and related questions. The purpose is twofold. First, we hope that this book will serve as a quick guide for readers who are not necessarily experts in the field. We will try to provide not only the main results, but also the main techniques and a large collection of open problems. Second, the field has had a burst of activity in the past five years and the most complete survey is more than a decade old. We hope that this book will fill this gap, or at least share some of the beautiful mathematical ideas and constructions that arise through FPP and which have enchanted many throughout these years.

Let us return to questions 1 to 4. The original paper of Hammersley and Welsh [**108**] considered question 1 for a class of passage times in $\mathbb{Z}^2$. If we write e_1 for the first coordinate vector, they showed that $T(0, ne_1)$ grows linearly in n. Their result was extended in the famous work of Kingman [**44**, **128**, **129**]. It was also the building block for the classical "shape theorem" of Richardson [**155**], improved by Cox and Durrett [**62**] and Kesten [**125**], that gives the analogue of the law of large numbers for the random ball $B(t)$. The shape theorem roughly says that $B(t)$ grows linearly in t and, when properly normalized, it converges to a deterministic subset $\mathcal{B}$ of $\mathbb{R}^d$, called the limit shape. The set $\mathcal{B} = \mathcal{B}_\nu$ is not universal and depends on the distribution ν of the passage times. Chapter 2 is devoted to explaining the shape theorem and certain properties of the limit shape $\mathcal{B}$.

In Chapter 3, we discuss the variance and the order of fluctuations of the passage time T. In two dimensions, it is expected that under certain assumptions on ν the fluctuations are governed by the predictions of physicists, including Kardar, Parisi, and Zhang [**76**, **120**, **121**, **132**]. In higher dimensions, the picture is less clear and some of the predictions disagree. After stating what is conjectured, we focus our attention on presenting proofs of more recent results, including sublinear variance of $T(0,x)$ in x valid under minimal assumptions on the passage time.

Chapter 4 is devoted to the study of geodesics. We discuss the existence and properties of finite geodesics between any two points, then move to the study of geodesic rays. We present results on coalescence, directional properties of geodesic rays, and a proof sketch of the absence of geodesic lines (or bigeodesics) in the upper half-plane. We also present the important connection between geodesic lines and ground states of the two-dimensional Ising ferromagnet.

In Chapter 5 we describe the role of Busemann functions in the model. We explain a beautiful argument by Hoffman for the existence of two or more geodesic



(a) $t = 0$

(b) $t = 70$

(c) $t = 135$

(d) $t = 250$

FIGURE 1. Simulation of the ball $B(t)$ at $t = 0, t = 70, t = 135$ and $t = 250$. The passage times τ_e have exponential distribution with mean one. Simulations by Si Tang.

rays. We then focus on Busemann-type limits and their relation to limiting geodesic graphs. Chapter 6 introduces the vast relation between FPP, growth processes, and infection models. We focus on questions of coexistence of multiple species and the limiting interface.

Chapter 7 is our attempt to show the reader what this book is not about. In the literature, there are thousands of pages of related (and equally fascinating) questions and models, similar to or inspired by FPP. We collect a few of these examples and point out the appropriate references. In particular, we briefly discuss FPP with non-independent weights, FPP on different graphs, the maximum flow problem, and exactly solvable models for last-passage percolation. Chapter 8 recalls open questions spread throughout this manuscript, put in one place for easy reference.

This book is intended to serve as an introduction to the field for researchers, as a reference, and also as a textbook for a graduate course. No previous knowledge of percolation theory is assumed. A first-year graduate course in probability theory should suffice.

1.2. Acknowledgments

We thank the American Institute of Mathematics and its staff for helping us organize a workshop on this subject. A.A. thanks the hospitality of the mathematics department at Indiana University, and its visiting professor program, during which part of this book was completed. He also thanks Elizabeth Housworth, who introduced him to the Japanese eraser and chalk dust cleaner. M.D. thanks the mathematics department at Northwestern University for hospitality. A. A. and J. H. thank the School of Mathematics at Georgia Tech for hospitality. We thank Phil Sosoe for several fruitful discussions on this topic, especially on concentration estimates. We thank Si Tang for making the simulations used in Figure 1 available to us, Sven Erick Alm and Maria Deijfen for simulations and Figures 2, 4, and 5, and Xuan Wang for discussions on nonrandom fluctuations. We also thank Daniel Ahlberg, Rick Durrett, Chris Hoffman, Wai-Kit Lam, Timo Seppäläinen, Vladas Sidoravicius, Phil Sosoe, Si Tang, Chen Xu, three anonymous reviewers, and countless others for spotting typos and giving comments. Special thanks to Vladas Sidoravicius who encouraged us to write a book on first-passage percolation. Last, we are indebted to our families for love and support.