Preface

This book is based on a semester lecture course Analysis on Graphs that I taught a number of years ago at the Department of Mathematics of the University of Bielefeld. The purpose of the book is to provide an introduction to the subject of the discrete Laplace operator on locally finite graphs. It should be accessible to undergraduate and graduate students with enough background in linear algebra, analysis and elementary probability theory.

The book starts with elementary material at the level of first semester mathematics students, and concludes with the results proved in the mathematical literature in 1990s. However, the book covers only some selected topics about the discrete Laplacian and is complementary to many existing books on similar subjects.

Let us briefly describe the contents of the book.

In Chapter 1 we give the definition and prove some basic properties of the discrete Laplace operator such as solvability of the Dirichlet problem and the existence of the associated random walk (= a reversible Markov chain).

In Chapter 2 we are concerned with the eigenvalues of the Laplace operator on finite graphs and their relation to the rate of convergence to the equilibrium of the corresponding random walk.

Chapter 3 contains some estimates of the eigenvalues on finite graphs, in particular, Cheeger's inequality, as well as the relation of eigenvalues to the expansion rate of subsets of graphs [**37**], [**39**].

In Chapter 4 we deal with the Laplace operator on infinite graphs and its restriction to finite domains – the Dirichlet Laplacian. The central topic is the relation between the eigenvalues of the Dirichlet Laplacian and the isoperimetric properties of the graph, which is based on a version of Cheeger's inequality. In Section 4.5 we prove a beautiful theorem of Coulhon and Saloff-Coste [**51**] about isoperimetric inequalities on Cayley graphs.

Chapter 5 is devoted to *heat kernel* estimates on infinite graphs, where the heat kernel is the density of the transition probability of the random walk with respect to the underlying measure (for example, the degree measure). In the case of a simple random walk in $\mathbb{Z}$ we obtain the estimates directly from the definition by means of Stirling's formula. In Section 5.3 we prove a universal Gaussian upper bound of the heat kernel that is due to Carne [**32**] and Varopoulos [**136**] . In Section 5.4 we prove the on-diagonal upper bound of the heat kernel of [**47**], [**81**] assuming that the graph satisfies a Faber-Krahn inequality.

In Sections 5.5-5.6 we prove some lower bounds of the heat kernel of [**49**], [**113**]. In Section 5.7 we use the heat kernel techniques to prove a universal upper bound for escape rate of the random walk on graphs with the polynomial volume growth. This can be regarded as a far-reaching generalization of the Hardy-Littlewood $\sqrt{n \log n}$-estimate for the escape rate of a simple random walk in $\mathbb{Z}^m$, which was obtained in

1914 (ten years before Khinchin's law of the iterated logarithm). For graphs with polynomial volume growth the $\sqrt{n \log n}$-estimate is sharp as was shown in [**15**].

In Chapter 6 we are concerned with the problem of deciding whether the random walk is recurrent or transient. Here we give a number of analytic conditions for recurrence and transience. In particular, the heat kernel bounds from the previous chapter lead immediately to the celebrated theorem of Polya: the simple random walk in $\mathbb{Z}^m$ is recurrent if and only if $m \leq 2$ (Section 6.1). A far-reaching generalization of Polya's theorem is the Varopoulos criterion [**135**] for recurrence on Cayley graphs that is presented in Section 6.2. In the remaining part of this chapter, we prove for general graphs Nash-Williams' [**115**] and volume tests for recurrence (the latter being a discrete version of a theorem of Cheng-Yau [**33**] about parabolicity of Riemannian manifolds), as well as an isoperimetric test for transience [**72**].

Chapter 7 contains exercises that were actually used for homework in the aforementioned lecture course. Solutions to all exercises are available on my home page.

Some remarks are due concerning the bibliography. Initially I planned to limit myself to a minimal bibliography list containing only necessary references from the text. However, it was suggested by an anonymous referee that the bibliography should contain also references to the sources in adjacent areas thus providing a broader coverage of topics of analysis on graphs. Furthermore, the referee had kindly offered a long list of such references, which greatly facilitated my work on the bibliography list. Hence, here is a list of sources for further reading.

- Classical (combinatorial) graph theory: [**31**], [**38**], [**59**], [**122**], [**123**].
- Various aspects of analysis on graphs: [**29**], [**45**], [**46**], [**52**], [**53**], [**64**], [**65**], [**66**], [**129**], [**130**].
- Spectral theory on graphs: [**6**], [**19**], [**21**], [**22**], [**24**], [**27**], [**28**], [**30**], [**35**], [**39**], [**40**], [**42**], [**43**], [**44**], [**45**], [**52**], [**53**], [**67**], [**70**], [**90**], [**92**], [**93**], [**105**], [**106**], [**114**], [**125**], [**132**], [**133**], [**134**].
- Potential-theoretic aspects of graphs: [**63**], [**96**], [**97**], [**127**], [**137**], [**139**].
- Analysis on Cayley and Schreier graphs: [**16**], [**17**], [**49**], [**51**], [**54**], [**65**], [**66**], [**71**], [**83**], [**84**], [**85**], [**112**], [**118**], [**121**], [**119**].
- Random processes on graphs: [**9**], [**10**], [**15**], [**86**], [**120**], [**124**], [**131**], [**139**], [**140**].
- Heat kernels on graphs: [**10**], [**11**], [**12**], [**13**], [**14**], [**47**], [**48**], [**50**], [**55**], [**56**], [**68**], [**69**], [**81**], [**82**], [**87**], [**88**], [**100**], [**101**], [**102**], [**116**], [**126**]
- Curvature on graphs: [**20**], [**23**], [**41**], [**94**], [**95**], [**107**], [**108**], [**109**].
- Homology theory on graphs: [**4**], [**7**], [**60**], [**75**], [**76**], [**77**], [**78**], [**79**], [**80**].
- Analysis on metric/quantum graphs: [**26**], [**27**], [**64**], [**67**], [**98**].
- Analysis on fractals and ultra-metric spaces: [**3**], [**8**], [**25**], [**64**], [**73**], [**99**], [**128**].

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