CHAPTER 1

Introduction

In this introductory chapter, after a brief historical outlook and motivation (Section 1.1), we present Atiyah-Segal picture of quantum field theory (Section 1.2). In Section 1.3 we present the heuristic idea of the path integral as giving a solution of Atiyah-Segal's axioms. We then proceed to explain the logic of making mathematical sense of the path integral using the stationary phase formula, producing the answer as an asymptotic series in $\hbar$ with coefficients given by contributions of Feynman diagrams. Finally, we explain the role of the BV construction as resolving the issue of degenerate critical points of the action, obstructing the stationary phase evaluation of the integral. In Section 1.4 we lay out the plan of the subsequent exposition.

1.1. Prologue

Batalin-Vilkovisky formalism ("BV formalism") was developed in the early 1980's [**9**, **10**] as a tool in mathematical physics designed to define the perturbative path integral for gauge theories. Gauge theories—models of field theory possessing a local symmetry ("gauge symmetry")—have been at the center of attention in mathematical physics for a long while (the first appearance of gauge symmetry dates back to Maxwell's classical electromagnetism). E.g., Yang-Mills theory is a prime example of a gauge theory, and its special case is the Standard Model of strong and electroweak interactions of elementary particles.

BV formalism was developed on the basis of the so-called BRST formalism due to Becchi-Rouet-Stora [**11**] and, independently, Tyutin [**93**]; and BRST formalism, in turn, was a development of Faddeev-Popov construction of Feynman diagrams for Yang-Mills theory [**38**]. All three of these constructions (Faddeev-Popov, BRST, BV) are methods of gauge-fixing of progressive generality, i.e., they deal with the following issue. One tries to define the path integral describing the quantities of interest in the quantum field theory (partition function, correlation functions, S-matrix, etc.), in the perturbation theory framework, by formally applying the stationary phase formula for oscillatory integrals (see Sections 3.2, 3.7). This results in a sum over critical points of the action functional of contributions, which are asymptotic series in the Planck constant and/or the coupling constants with coefficients given by Feynman diagrams (whose values are certain configuration space integrals in a local field theory). However, for the stationary phase formula to be applicable, the critical points of the action (stationary phase points of the oscillatory integrand) must be isolated. In gauge theories, the critical points are not isolated and the stationary phase formula cannot be applied directly. All the constructions of gauge-fixing (Faddeev-Popov, BRST, BV) amount to replacing the integral with the one which does have isolated critical points and thus can be evaluated by the stationary phase formula. All these constructions involve an arbitrary choice of "gauge" (e.g. a section of the gauge orbits in Faddeev-Popov case and a Lagrangian submanifold in the BV-extended space of fields in the BV case) and the integral replacing the original one is, in fact, an integral over a supermanifold, and its value is independent of the choice of the gauge (and equals the original integral when both sides make measure-theoretic sense).

The necessary appearance of integration over supermanifolds in gauge theories can be traced to the appearance of "ghost fields" (Faddeev-Popov ghosts)—odd (anti-commuting or "fermionic") fields, which were first spotted by Feynman [**42**] as necessary in Feynman diagrams for Yang-Mills theory, and were understood by Faddeev and Popov as coming from an integral formula for the Jacobian associated, roughly speaking, to the "angle" at which the chosen section intersects the gauge orbits. Thus, ghosts effectively restore the gauge-fixing independence of the answers in perturbative gauge theory. In BRST formalism, ghosts, instead of being merely auxiliary fields arising from an integral formula for the determinant, were understood on a different conceptual level, as generators of the Chevalley-Eilenberg algebra associated to the Lie algebra of gauge transformations. In BV formalism, in addition to the ghosts, one adjoins the "anti-fields"—the generators of the Koszul-Tate resolution of the critical locus, see [**41, 45, 90**]. Thus, in gauge theories one encounters odd fields arising from homological resolutions of either the gauge symmetry or the critical locus.[1]

Faddeev-Popov construction, BRST and BV formalisms (Sections 4.1, 4.3, 4.8) each introduced a new point of view on what is a gauge theory. E.g., BRST formalism introduced the structures of homological algebra to gauge theory and identified the space of fields as a differential graded manifold (or equivalently[2] a Q-manifold,—a supermanifold equipped with an odd vector field Q squaring to zero)—an object combining the features of a manifold and of a cochain complex—with the action being a distinguished cocycle, see Section 4.2.3. In BV formalism, ideas of symplectic geometry are added to the recipe (see Section 4.4): the space of fields turns out to be an odd-symplectic Q-manifold, with the action being the Hamiltonian function generating the cohomological vector field Q; the geometry of this setup was elucidated by A. S. Schwarz in [**87**]. Apart from conceptual advancements, each next formalism is more general as a method of gauge-fixing: while Faddeev-Popov construction deals with gauge symmetry given by a group action (for instance, Yang-Mills theory), BRST can, e.g., deal with the symmetry given by a Lie algebroid. BV is the most general known method of gauge-fixing and can deal, in particular, with gauge symmetry given by a non-integrable distribution on classical fields (for example, the Poisson sigma model responsible for Kontsevich's celebrated deformation quantization construction [**17, 62**], see Sections 4.9.3, 5.3) and with "reducible" gauge symmetry (e.g. the BF theory in dimension $\geqslant 4$). In particular, BV formalism turned out to be particularly well suited to dealing with the complicated gauge symmetry structure of a special type of gauge theories—the topological field theories.

[1] An entirely different mechanism through which odd fields can arise in a field theory is from having particles of half-integer spin in the theory. Then the corresponding field takes values in sections of a spinor bundle over the spacetime manifold. In this exposition we focus on bosonic theories, i.e., without fermions among the physical fields.

[2] In fact, there is a technical difference: a dg manifold is a Q-manifold endowed with a compatible $\mathbb{Z}$-grading. Thus, Q-manifolds are somewhat more general. For the purposes of this text, we always have a $\mathbb{Z}$-grading available and the distinction between the two notions is irrelevant.

Topological quantum field theory (TQFT) originated in the idea, put forward by Albert S. Schwarz [**85**] (see also [**86**] on the history of the development of TQFT), to construct topological invariants[3] of manifolds by studying a field theory on a manifold, defined classically by a diffeomorphism-invariant action functional.[4] This paved the way for an exciting and unexpected application of the machinery of quantum field theory, developed originally for the needs of physics, to mathematics. The field has truly blossomed starting from the late 1980's, following the work of Witten [**99**] on Chern-Simons theory on 3-manifolds and its application for constructing topological invariants of 3-manifolds and knots inside them.

The mathematical effort to summarize the data of topological quantum field theory resulted in Atiyah's axiomatic description of TQFT [**3**] which essentially postulates the expected gluing/cutting property of the path integral without referring to the path integral explicitly (Section 1.2).[5] Atiyah's axioms are a modification of Segal's axioms of 2-dimensional conformal field theory [**88**]. In Atiyah-Segal's picture, a quantum field theory can be seen as a functor from the "spacetime category" (the category of cobordisms endowed with appropriate geometric structure) to a target category—typically, the category of vector spaces (and often, Hilbert spaces). This functorial picture is a natural (but certainly not straightforward) extension to the setting of quantum field theory of the operator formalism in quantum mechanics which describes the evolution of the system in time t as a unitary evolution operator between the in-Hilbert space (states at time t_0) and the out-Hilbert space (states at time $t_1 = t_0 + t$).

An important extension of Atiyah's axiomatics of TQFT, allowing gluing and cutting with corners (as opposed to just along closed codimension 1 submanifolds) was suggested by Baez-Dolan [**6**]. Such TQFTs are called *extended* (into higher codimension), and *fully extended* if the top codimension strata—points—are allowed. The program of fully extended TQFTs was developed further and a general classification result for them was proven by J. Lurie [**67**].

Several constructions proposed in the early 1990's give mathematical examples of TQFT's in the sense of Atiyah's axioms, which are not constructed starting with a classical action functional, e.g. the Dijkgraaf-Witten theory [**35**] (Section 1.2.3), Turaev-Viro model [**92**], Reshetikhin-Turaev construction [**80**]. The latter yields answers compatible with Witten's quantum Chern-Simons theory, though a direct

[3] "Topological invariants" normally stands for "invariants under diffeomorphism" in this context. Note that in the dimension up to 3, one does not have to make a distinction with invariance under homeomorphism, thanks to Hauptvermutung (equivalence of topological, piecewise-linear and smooth categories) valid in low dimensions.

[4] One can find precursors of topological quantum field theory predating Schwarz's seminal paper: in particular, the work of Ponzano-Regge [**78**] suggesting a state-sum model for the 3-dimensional quantum gravity and the solution of 2-dimensional Yang-Mills theory (which becomes a topological theory in the limit when areas of the surfaces are taken to zero) by A. A. Migdal [**70**].

[5] The path integral itself has the status of a heuristic object, which admits a mathematical definition in nice cases, e.g., as the perturbative path integral, given by the stationary phase formula as a formal power series with terms corresponding to Feynman diagrams. However, one still needs to check, case by case, that the expected properties of the path integral—like the behavior under the change of variables or the Fubini theorem—are compatible with its mathematical definition. Alternatively, one can try to define the path integral (again, in nice cases) as a limit of finite-dimensional integrals (see e.g. [**43**])—then Fubini theorem becomes straightforward, but change of variables is not, and neither is the existence of the continuum limit (and if it does exist, it depends on the details of the finite-dimensional replacements).

comparison to the path integral quantization of the Chern-Simons action functional is still lacking.

On the other hand, the study of the perturbative Chern-Simons theory by Axelrod-Singer [**4, 5**] (Section 5.2) has led to the understanding that the Feynman diagrams, giving the coefficients of the expansion of the partition function in $\hbar$ (or in the inverse level), are given by the integrals of certain differential forms over the Fulton-MacPherson-Axelrod-Singer compactification of the configuration spaces of n points (with n the number of vertices in the Feynman graph) on the underlying 3-manifold. Kontsevich's deformation quantization formula [**62**] (Section 5.3) exhibits a similar structure: it is a formal power series with coefficients given by integrals over compactified configuration spaces of points in a disk with two vertices on the boundary. The reason for this is that "secretly" Kontsevich's formula is the calculation of a correlator in a particular 2-dimensional topological field theory (in Schwarz's sense, i.e., defined classically by a diffeomorphism-invariant action functional)—the Poisson sigma model—on a disk, with two observables placed on the boundary circle, by means of a perturbative expansion of the path integral, see [**17**]. So, technically the perturbative invariants of 3-manifolds coming from Chern-Simons theory and Kontsevich's deformation quantization formula look very similar—they are both computations of the perturbative path integral for two topological field theories. Moreover, proving the crucial statements about these results—independence of the perturbative Chern-Simons invariants of 3-manifolds on the metric (which determines the form integrated over the configuration space) and the associativity of Kontsevich's star product—both rely on a Stokes' theorem argument for the integrals over compactified configuration spaces (see Sections 5.2.2, 5.3.2).

The reason why perturbative answers in the two topological field theories—Chern-Simons and Poisson sigma model—are so similar, is explained by the fact that they both arise as examples of the AKSZ construction [**2**] of topological sigma models in Batalin-Vilkovisky formalism (Section 4.9). Essentially, the key observation is that Chern-Simons and Poisson sigma model fit in a class of theories where classical fields, ghosts (and possibly higher ghosts) and anti-fields can be assembled into "superfields" which are nonhomogeneous differential forms on the spacetime manifold with values in some target supermanifold. Assuming that the target is a Q-manifold with a compatible symplectic structure of appropriate degree, one can construct a BV action S on such superfields, with the kinetic term defined by the de Rham operator on the source and the interaction term defined by the target data. This action defines a consistent gauge theory in the BV sense, i.e. it satisfies the BV master equation $\{S, S\} = 0$, and is diffeomorphism-invariant. Moreover, the Stokes' theorem arguments for the configuration space integrals arising as coefficients in the perturbative answers, have an origin in BV formalism,—the so-called "BV-Stokes' theorem."[6]

[6]BV-Stokes' theorem (see Sections 4.4.4, 4.7) has the status of a mathematical theorem for finite-dimensional BV integrals, whereas for the path integrals it becomes a heuristic principle that needs to be checked directly on the level of Feynman diagrams. —This is the point where Stokes' theorem for configuration space integrals provides a mathematically rigorous counterpart for BV-Stokes' theorem in the setting of perturbative path integral, see Sections 5.2, 5.3.

Finally, we would like to mention some of the connections and applications of the BV formalism that are outside of the scope of this exposition.

- Zwiebach's work on closed string field theory and the BV structure naturally appearing there [**105**].
- Applications of BV to renormalization, see [**31**, **46**, **104**].
- Chas-Sullivan string topology—the BV algebra structure (see Section 4.5.1 for the definition) on the homology $H_\bullet(LM)$ of the free loop space [**29**] of an oriented manifold.
- By a result of Getzler [**49**], BV algebra structure naturally appears on the BRST cohomology of the space of local observables in any two-dimensional conformal field theory with a BRST-exact stress-energy tensor (i.e. a "topological conformal field theory" or TCFT). The reason for this is that the genus-zero content (operator product expansions) of a TCFT is described by an algebra over framed E_2 operad; the cohomology of the latter is the operad of BV algebras.[7]
- Calaque, Pantev, Toën, Vaquié and Vezzosi [**15**, **76**] have recently developed a promising realization of the AKSZ construction in the language of derived symplectic geometry, which should allow for a better grip on the global phenomena in AKSZ theories.
- Costello and Gwilliam [**32**] have developed a "factorization algebra" formulation of locality in quantum gauge theories, based on BV formalism and developing on the idea of operator product expansions.
- A refinement of BV formalism for field theories on manifolds with boundary was developed by Cattaneo, Reshetikhin and the author in [**21**, **24**], see also [**25**] for a brief survey. In this setting, perturbative partition functions are defined for manifolds with boundary and satisfy the Atiyah-Segal gluing axiom. Thus, the cutting-pasting axiomatic picture of quantum field theory becomes explicitly realized in terms of Feynman diagrams (written in terms of configuration space integrals).

1.2. Atiyah-Segal picture of quantum field theory

The idea of locality, in the interpretation of Atiyah-Segal, is as follows: a quantum field theory (QFT) assigns certain values—"partition functions"—to manifolds. It can be evaluated on manifolds and satisfies a gluing/cutting property. So, a manifold can be cut into simple pieces (simple topologically and/or small geometrically), then the QFT can be evaluated on those pieces and subsequently assembled into the value of the QFT on the entire manifold.

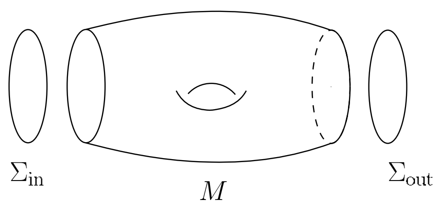
---

[7] In fact, this construction yields a BV algebra with "wrong degrees" (as opposed to the algebraic structure on functions on the space of BV fields)—a BV Laplacian and the anti-bracket of degree -1 (while the BRST differential has the usual degree $+1$). This has to do with the fact that Getzler's BV algebra is associated to the 2-dimensional geometry of the surface. The shift by -2 w.r.t. the grading of the operations in the BV algebra structure on functions of fields can be identified with the codimension of punctures on a surface. For degree shifts in the BV structures on strata of the spacetime manifold, cf. [**21**]. For an explicit example of Getzler's BV structure, see [**66**].

1.2.1. Atiyah's axioms of topological quantum field theory. In the following definition, due to Atiyah [**3**], we assume that all manifolds are smooth and oriented.[8]

An n-dimensional topological quantum field theory (TQFT) is the following association.

- To a **closed** $(n-1)$**-dimensional manifold** Σ, the TQFT associates a vector space $\mathcal{H}_\Sigma$ over $\mathbb{C}$—the "space of states."
- To an n**-manifold** M **with boundary** split into in- and out-components, $\partial M = \bar{\Sigma}_{\text{in}} \sqcup \Sigma_{\text{out}}$ (bar refers to reversing the orientation on the in-boundary), the TQFT associates a $\mathbb{C}$-linear map $Z_M : \mathcal{H}_{\Sigma_{\text{in}}} \to \mathcal{H}_{\Sigma_{\text{out}}}$—the "partition function."[9]



We call such M a *cobordism* between Σ_{in} and Σ_{out}, and we denote

$$\Sigma_{\text{in}} \xrightarrow{M} \Sigma_{\text{out}}$$

- **Diffeomorphisms of closed** $(n-1)$**-manifolds** act on spaces of states: to $\phi : \Sigma \to \Sigma'$ a diffeomorphism, the TQFT associates an isomorphism $\rho(\phi) : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$ (in the way compatible with composition of diffeomorphisms).[10] For ϕ orientation-preserving, $\rho(\phi)$ is $\mathbb{C}$-linear; for ϕ orientation-reversing, $\rho(\phi)$ is $\mathbb{C}$-anti-linear.

This association should satisfy the following axioms:

- **Multiplicativity**: disjoint unions are mapped to tensor products. Explicitly,

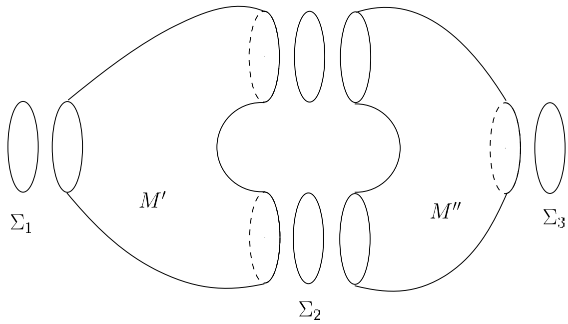
$$\mathcal{H}_{\Sigma \sqcup \Sigma'} = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\Sigma'}, \qquad Z_{M \sqcup M'} = Z_M \otimes Z_{M'}$$

- **Gluing**: given two cobordisms $\Sigma_1 \xrightarrow{M'} \Sigma_2$ and $\Sigma_2 \xrightarrow{M''} \Sigma_3$, with out-boundary of the first one coinciding with the in-boundary of the second one,

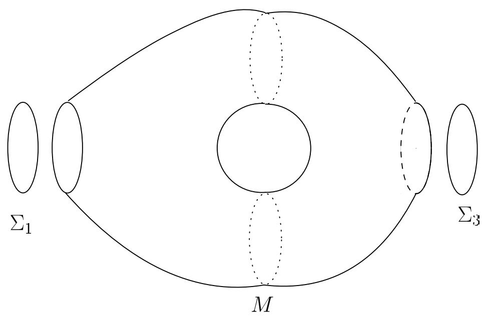
[8] An orientation is a natural structure for field theories defined by an action functional, which requires integration and thus naturally uses an orientation. However, there do exist examples of TQFTs that do not need an orientation. For instance, the Dijkgraaf-Witten model [**35**] which, in its simplest form, counts the number of G-coverings on a manifold, weighted with the inverse size of the stabilizer, for G a finite group, see Section 1.2.3.

[9] Another possible name for Z_M is the "evolution operator".

[10] In fact, Atyiah's axioms—see axiom (1.2.2) below—force $\rho(\phi)$ to coincide with the partition function of a mapping cylinder $\Sigma \xrightarrow{[0,1]\times\Sigma} \Sigma'$ associated to ϕ.



we can *glue* (or "concatenate") them over Σ_2 into a new cobordism $M := M' \cup_{\Sigma_2} M''$, going as $\Sigma_1 \xrightarrow{M} \Sigma_3$.



Then the partition function for M is the *composition* of partition functions for M' and M'' as linear maps:

$$Z_M = Z_{M''} \circ Z_{M'} : \quad \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_3} \tag{1.2.1}$$

- **Normalization**:
  - For $\varnothing$ the empty $(n-1)$-manifold,
  $$\mathcal{H}_\varnothing = \mathbb{C}$$
  - For Σ a closed $(n-1)$-manifold, the partition function for the *cylinder* $\Sigma \xrightarrow{\Sigma\times[0,1]} \Sigma$ is the identity on $\mathcal{H}_\Sigma$.
- **Behavior under diffeomorphisms**: for $\phi : M \to M'$ a diffeomorphism between two cobordisms, denote $\phi|_{\text{in}}$, $\phi|_{\text{out}}$ the restrictions of ϕ to the in- and out-boundary. We have a commutative diagram

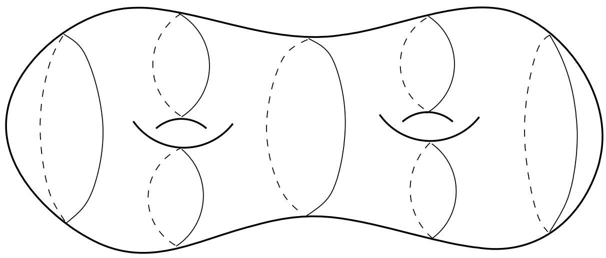
$$\begin{tikzcd} \mathcal{H}_{\Sigma_{\text{in}}} \arrow[r, "Z_M"] \arrow[d, "\rho(\phi|_{\text{in}})"'] & \mathcal{H}_{\Sigma_{\text{out}}} \arrow[d, "\rho(\phi|_{\text{out}})"] \\ \mathcal{H}_{\Sigma'_{\text{in}}} \arrow[r, "Z_{M'}"'] & \mathcal{H}_{\Sigma'_{\text{out}}} \end{tikzcd} \tag{1.2.2}$$

Remark 1.2.1. Atiyah's TQFT is a functor of symmetric monoidal categories, $\mathrm{Cob}_n \to \mathrm{Vect}_{\mathbb{C}}$, where the structure is as follows:

	Cob_n	$\mathrm{Vect}_{\mathbb{C}}$
objects	closed $(n-1)$-manifolds	vector spaces over $\mathbb{C}$
morphisms	cobordisms $\Sigma_{\mathrm{in}} \xrightarrow{M} \Sigma_{\mathrm{out}}$	linear maps
composition	gluing	composition of maps
identity morphism	cylinder $\Sigma \xrightarrow{\Sigma\times[0,1]} \Sigma$	identity map $\mathrm{id} : V \to V$
monoidal product	disjoint union $\sqcup$	tensor product $\otimes$
monoidal unit	$\varnothing$	$\mathbb{C}$

Remark 1.2.2. A *closed* n-manifold M can be viewed as a cobordism from $\varnothing$ to $\varnothing$, thus $Z_M : \mathbb{C} \to \mathbb{C}$ is a multiplication by some number $z \in \mathbb{C}$. By abuse of notations, we denote $Z_M := z \in \mathbb{C}$. Thus, with this convention, the partition function for a closed n-manifold is a complex number, invariant under diffeomorphisms and compatible with gluing-cutting. E.g., for $n = 2$, we can cut any closed surface into disks and pairs of pants

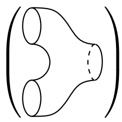
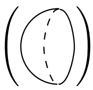
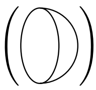
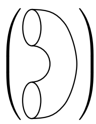
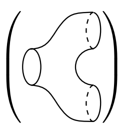
(1.2.3)



Thus, Z for any surface can be calculated from the gluing axiom, provided that Z is known for a disk and for a pair of pants.

Considering TQFTs modulo equivalence relation given by natural transformations of functors $\mathrm{Cob}_n \to \mathrm{Vect}_{\mathbb{C}}$, one has the following classification results in low dimensions $n = 1, 2$.

- For $n = 1$, the TQFT modulo equivalence is determined by a natural number N—the dimension of the space of states of a point $\mathcal{H}_{\mathrm{pt}} = \mathbb{C}^N$. This determines all spaces of states and partition functions. E.g., for a circle one gets $Z_{S^1} = N$.
- For $n = 2$, the classification result was obtained by Dijkgraaf [**34**]: 2-dimensional TQFTs modulo equivalence are in one-to-one correspondence with Frobenius algebras $(V, m, \mathbf{1}, \epsilon)$ modulo isomorphisms. Here V is a complex vector space, $m : V \otimes V \to V$ is an associative commutative product, $\mathbf{1} \in V$ is the unit for the product and $\epsilon : V \to \mathbb{C}$ is the counit or trace such that the bilinear pairing $(x, y) = \epsilon \circ m(x, y) : V \otimes V \to \mathbb{C}$ is symmetric and non-degenerate. The pairing induces an isomorphism $V \simeq V^*$ and a coproduct $\Delta : V \to V \otimes V$ (from dualizing the product). The correspondence between TQFTs and Frobenius algebras is as follows: space $V = \mathcal{H}_{S^1}$ is the space of states on a circle. Structure maps of the Frobenius algebra structure on V arise as partition functions of pairs of pants, disks and cylinders by the following rules.

TQFT partition function	Frobenius algebra structure map
Z 	multiplication $m : V \otimes V \to V$
Z 	unit $\mathbf{1} \in V$ (or rather $\mathbf{1}\cdot : \mathbb{C} \to V$)
Z 	counit $\epsilon : V \to \mathbb{C}$
Z 	pairing $(,) : V \otimes V \to \mathbb{C}$
Z 	coproduct $\Delta : V \to V \otimes V$

Thus one reads a Frobenius algebra off a TQFT. Also, given a Frobenius algebra, one can construct the associated TQFT by these rules. Its value on a closed surface of genus g is:

$$Z = \epsilon \circ (m \circ \Delta)^g \mathbf{1} \qquad \in \mathbb{C}$$

which arises from cutting the surface into disks and pairs of pants as in (1.2.3).

Example 1.2.3. For G a finite group, the center of the group ring $\mathbb{C}[G]$ has the natural structure of a Frobenius algebra. The associated 2-dimensional TQFT is the 2-dimensional Dijkgraaf-Witten model (without a group cocycle twist), counting G-coverings on a surface, which we discuss in Section 1.2.3 below.

Remark 1.2.4. The target category of a TQFT does not necessarily have to be the category $\mathrm{Vect}_{\mathbb{C}}$. E.g. the category $\mathrm{dgVect}_{\mathbb{C}}$ of chain complexes, with morphisms given by chain maps, arises naturally as the target category in the context of BRST or BV quantization.

Remark 1.2.5. A classification result for Atiyah's TQFTs in dimension $n \geqslant 3$ is beyond reach, since one cannot construct e.g. all 3-manifolds from some small list of building blocks by gluing along closed codimension 1 manifolds. Motivated by this, a far-reaching refinement of Atiyah's axiomatics was developed in [**6**, **67**], where gluing and cutting with corners is permitted, which allows one to disassemble manifolds into simple pieces. In this setting one speaks about "fully extended"[11] TQFTs and they are monoidal functors from the category of n-cobordisms extended to an (∞, n)-category to an appropriate target (∞, n)-category. In particular, a

[11] "Extended" here means "extended into higher codimension," i.e., gluing/cutting along higher-codimension strata is allowed. "Fully extended" pertains to the case where strata of all codimensions are allowed.

fundamental classification result (the "cobordism hypothesis") was proven by Lurie [**67**], which states that a fully extended TQFT is completely determined by its value on a point.

REMARK 1.2.6. Atiyah-Segal's axiomatics is just one of the approaches to describing the locality property of quantum field theory. Another recently developed approach is provided by the language of factorization algebras of Costello and Gwilliam [**32**], inspired by the work of Beilinson-Drinfeld on chiral algebras [**12**]. Here, instead of cobordisms, one evaluates the theory on open subsets of the spacetime and instead of cutting an n-manifold into submanifolds with boundary, one considers open covers (subject to certain technical condition—so-called Weiss covers). The value of the theory $F(U)$ on an open set U is interpreted as the space of local quantum observables on U. The essential piece of data are the structure maps $F(U_1) \otimes \cdots \otimes F(U_k) \to F(U)$ associated to any inclusion of open sets $U_1 \sqcup \cdots \sqcup U_k \hookrightarrow U$. These structure maps provide an elaboration of the notion of operator product expansions (OPEs) in quantum field theory.

1.2.2. Segal's QFT. In Segal's approach to (not necessarily topological) quantum field theory, originating in the seminal paper of G. Segal [**88**] on an axiomatic approach to 2D conformal field theory, one allows manifolds to carry a local geometric structure γ (of the type depending on the particular QFT): Riemannian metric, conformal structure, complex structure, volume form, framing, local system, etc. Atiyah's axioms above have to be appropriately modified to accommodate for the geometric structure. The partition functions are allowed to depend on the geometric structure on the cobordism and the spaces of states may depend on the geometric structure on the codimension 1 manifolds.[12] Denote Geom_M the space of geometric data of the given type on M. For the gluing axiom (1.2.1) to make sense, one must have

$$\mathrm{Geom}_M = \mathrm{Geom}_{M''} \times_{\mathrm{Geom}_{\Sigma_2}} \mathrm{Geom}_{M'}$$

The diffeomorphism equivariance axiom (1.2.2) has to be refined to take care of geometric structures:

$$\begin{tikzcd}
\mathcal{H}_{\Sigma_{\mathrm{in}},\gamma_{\mathrm{in}}} \arrow[r, "Z_{M,\gamma}"] \arrow[d, "\rho(\phi|_{\mathrm{in}})"'] & \mathcal{H}_{\Sigma_{\mathrm{out}},\gamma_{\mathrm{out}}} \arrow[d, "\rho(\phi|_{\mathrm{out}})"] \\
\mathcal{H}_{\Sigma'_{\mathrm{in}},\phi_*\gamma_{\mathrm{in}}} \arrow[r, "Z_{M',\phi_*\gamma}"'] & \mathcal{H}_{\Sigma'_{\mathrm{out}},\phi_*\gamma_{\mathrm{out}}}
\end{tikzcd}$$

with ϕ_* the pushforward of the geometric structure along the diffeomorphism ϕ. In particular, this implies that the partition function of a closed n-manifold is a function on geometric structures modulo diffeomorphisms.

EXAMPLE 1.2.7 (Quantum mechanics). Consider the 1-dimensional Segal's QFT with geometric structure the Riemannian metric on 1-cobordisms. Objects are points with + orientation, to which we assign a vector space $\mathcal{H}$, and points

[12] In the case of γ being the Riemannian metric on n-manifolds, one considers $(n-1)$-manifolds Σ with Riemannian collars—germs of Riemannian metrics on cylinders $\Sigma \times [-\epsilon, \epsilon]$. Then one can glue two Riemannian manifolds (M', g') and (M'', g'') along Σ only if the metrics on the two sides of Σ glue smoothly, i.e., if one can endow Σ with a Riemannian collar compatible with the restriction of the metric from M' and M''.

with $-$ orientation, to which we assign the dual space $\mathcal{H}^*$. Consider an interval of length $t > 0$ (our partition functions depend on a metric on the interval considered modulo diffeomorphisms, thus only on the length), $I_t = [0, t]$. Denote $Z(t) := Z_{I_t} \in \mathrm{End}(\mathcal{H})$. By the gluing axiom (from considering the gluing $[0, t_1] \cup_{\{t_1\}} [t_1, t_1 + t_2] = [0, t_1 + t_2]$), we have the semi-group law $Z(t_1 + t_2) = Z(t_2) \circ Z(t_1)$. It implies in turn that

$$Z(t) = Z\Big(\frac{t}{N}\Big)^N \tag{1.2.4}$$

for N an arbitrarily large integer. Assume that for τ small, we have $Z(\tau) = \mathrm{id} - \frac{i}{\hbar}\hat{H} \cdot \tau + O(\tau^2)$, for $\hat{H} \in \mathrm{End}(\mathcal{H})$ some operator. Then (1.2.4) implies that

$$Z(t) = \exp\Big(-\frac{i}{\hbar}\hat{H}t\Big)$$

This system is the quantum mechanics, with $Z(t)$ the *evolution operator* in time t and $\hat{H}$ the Schrödinger operator (or *quantum Hamiltonian*), describing the infinitesimal evolution of the system.

E.g. the choice $\mathcal{H} = L^2(X)$—the space of square-integrable functions on a Riemannian manifold X, and $\hat{H} = -\frac{\hbar^2}{2m}\Delta_X + U(x)$, with Δ_X the Laplace operator, corresponds to the quantum particle of mass m moving on the manifold X in the force field with potential U. In this case $Z(t) : \psi(x) \mapsto \int_{X \ni y} dy\, Z(t; x, y)\psi(y)$ is the integral operator whose integral kernel $Z(t; x, y)$ is interpreted as the propagation amplitude of the particle from position y to position x in time t.

Example 1.2.8 (Two-dimensional Yang-Mills theory). Fix G a compact Lie group. Consider the following 2-dimensional Segal's QFT on 2-cobordisms endowed with an area form μ. The space of states for a circle is the space of complex-valued square-integrable class functions on G:

$$\mathcal{H}_{S^1} = L^2(G)^G$$

It is a Hilbert space, with the inner product

$$\langle\psi_1, \psi_2\rangle = \int_{G \ni U} dU\, \bar{\psi}_1(U)\, \psi_2(U)$$

It induces an isomorphism $\mathcal{H}_{S^1} \simeq \mathcal{H}^*_{S^1}$ mapping $\psi \mapsto \langle\bar{\psi}, -\rangle$. Here dU is the Haar measure on G. A convenient orthonormal basis in $\mathcal{H}_{S^1}$ is given by the set of characters $\chi_R(U)$ of irreducible unitary representations R of G. The partition function for a surface M of genus g, with m in-circles and n out-circles is the following:

$$Z_M = \sum_R (\dim R)^{\chi(M)} e^{-aC_2(R)} \prod_{j=1}^m \overline{\chi_R(U_j^{\mathrm{in}})} \cdot \prod_{k=1}^n \chi_R(U_k^{\mathrm{out}}) \in \mathrm{Hom}(\mathcal{H}_{S^1}^{\otimes m}, \mathcal{H}_{S^1}^{\otimes n}) \tag{1.2.5}$$

Here:

- The sum is over the irreducible unitary representations R of G; $\dim R$ is the dimension of the representation and $C_2(R)$ is the value of the quadratic Casimir.
- $\chi(M) = 2 - 2g - m - n$ is the Euler characteristic of the surface.
- $a = \int_M \mu > 0$ is the total area of the surface.

- $U_j^{\text{in}}, U_k^{\text{out}} \in G$ are the group variables decorating the boundary circles.
- Expression (1.2.5) is understood as a "matrix element" (or integral kernel) $Z_M(\{U_k^{\text{out}}\}, \{U_j^{\text{in}}\})$; the corresponding linear map $\mathcal{H}_{S^1}^{\otimes m} \to \mathcal{H}_{S^1}^{\otimes n}$ acts by

$$Z_M : \psi(U^{\text{in}}) \quad \mapsto \quad (Z_M\psi)(U^{\text{out}}) = \int_{G^{\times m}} dU^{\text{in}}\, Z_M(U^{\text{out}}, U^{\text{in}}) \cdot \psi(U^{\text{in}})$$

It is a straightforward check that the partition functions (1.2.5) satisfy the Atiyah-Segal gluing axiom (1.2.1). For further details on 2D Yang-Mills theory, we refer the reader to the original papers by Migdal and Witten [**70**, **101**] and to the lectures [**30**].

1.2.3. Example of a TQFT: Dijkgraaf-Witten model. Fix G a finite group and fix $n \geqslant 1$. For M a manifold, consider the moduli space of coverings of M with structure group G (or equivalently the moduli space of G-local systems on M),

$$\mathcal{M}_{M,G} = \mathrm{Hom}(\pi_1(M), G)/G = [M, BG] \tag{1.2.6}$$

—we view it as a finite groupoid (coverings modulo isomorphisms). In the middle, we consider the set of group homomorphisms of the fundamental group into G, modulo conjugation in the target by elements of G —here we assume for simplicity that M is connected; for M disconnected, we take the direct product of expressions above for the connected components.[13] On the right side, $[M, BG]$ is the space of mappings of M into the classifying space BG modulo homotopy.

Dijkgraaf-Witten model is a TQFT which assigns to a closed $(n-1)$-manifold Σ the space of states—the ring of complex-valued functions on the moduli space of G-coverings on Σ:

$$\mathcal{H}_\Sigma = \mathrm{Fun}_{\mathbb{C}}(\mathcal{M}_{\Sigma,G}) = \mathrm{Fun}_{\mathbb{C}}(\mathrm{Hom}(\pi_1(\Sigma), G))^G$$

On the r.h.s. Σ is assumed connected; otherwise, we take a tensor product over connected components. It is a Hilbert space with inner product

$$\langle \psi_1, \psi_2 \rangle = \sum_\gamma \frac{1}{|\mathrm{Stab}(\gamma)|}\, \bar{\psi}_1(\gamma)\, \psi_2(\gamma)$$

Here the sum is over the orbits γ of G-action on $\mathrm{Hom}(\pi_1(\Sigma), G)$, with $|\mathrm{Stab}(\gamma)|$ the number of elements in the stabilizer in G of any representative of γ. Note that $\frac{1}{|\mathrm{Stab}(\gamma)|} = \frac{|\gamma|}{|G|}$ where $|\gamma|$ is the size of the orbit and $|G|$ is the size of the group. Again, as in Example 1.2.8, the inner product determines the isomorphism $\mathcal{H}_\Sigma \simeq \mathcal{H}_\Sigma^*$.

Consider an n-cobordism $\Sigma_{\text{in}} \xrightarrow{M} \Sigma_{\text{out}}$. Restricting a G-covering to the boundary induces a mapping $p : \mathcal{M}_{M,G} \to \mathcal{M}_{\Sigma_{\text{in}},G} \times \mathcal{M}_{\Sigma_{\text{out}},G}$. Dijkgraaf-Witten model assigns to M the partition function $Z_M : \mathcal{H}_{\Sigma_{\text{in}}} \to \mathcal{H}_{\Sigma_{\text{out}}}$ determined by the following matrix elements:

$$Z_M(\gamma_{\text{out}}, \gamma_{\text{in}}) = |\mathrm{Stab}(\gamma_{\text{out}})| \cdot \sum_{\beta \in p^{-1}(\gamma_{\text{in}}, \gamma_{\text{out}})} \frac{1}{|\mathrm{Stab}(\beta)|} \tag{1.2.7}$$

[13] An equivalent model for $\mathcal{M}_{M,G}$ is the set of functors from the fundamental groupoid $\Pi_1(M)$ to the group G (viewed as a category with a single object) modulo natural transformations. This way there is no implicit choice of a base point on M and one does not have to treat the disconnected case separately.

with the sum ranging over the isomorphism classes of G-coverings on M inducing fixed classes $\gamma_{\text{in}}, \gamma_{\text{out}}$ on the in- and out-boundary. The partition function as a linear map between spaces of states is:

$$Z_M : \psi(\gamma_{\text{in}}) \quad \mapsto \quad (Z_M\psi)(\gamma_{\text{out}}) = \sum_{\gamma_{\text{in}}} Z_M(\gamma_{\text{out}}, \gamma_{\text{in}}) \cdot \psi(\gamma_{\text{in}})$$

In particular, the Dijkgraaf-Witten partition function of a closed n-manifold is simply the groupoid volume of the moduli space of G-coverings on M:

$$Z_M = \mathrm{Vol}_{\text{grpd}}(\mathcal{M}_{M,G}) = \frac{|\mathrm{Hom}(\pi_1(M), G)|}{|G|}$$

(Here on the r.h.s. we again assume that M is connected; otherwise, we should take a product over connected components.) In particular, it is a rational number.

One can also modify the construction by choosing a degree n group cocycle of G with coefficients in $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, $\theta \in H^n(BG, S^1) \simeq H^{n+1}(BG, \mathbb{Z})$ and adjusting the formula (1.2.7) as follows:

$$Z_M^\theta(\gamma_{\text{out}}, \gamma_{\text{in}}) = |\mathrm{Stab}(\gamma_{\text{out}})| \cdot \sum_{\beta \in p^{-1}(\gamma_{\text{in}}, \gamma_{\text{out}})} \frac{e^{i\langle [M], \beta^*\theta\rangle}}{|\mathrm{Stab}(\beta)|}$$

Here $[M] \in H_n(M, \partial M)$ is the fundamental class of M relative to the boundary and $\beta^*\theta \in H^n(M, \partial M; S^1)$ is the pullback of the cohomology class θ on BG along the classifying map of the covering.[14] Thus, the twist by θ does not change the spaces of states but deforms the partition functions. Note that before introducing θ we did not need orientations to define the theory; after introducing θ, orientation becomes necessary.

Furthermore, there is an elegant combinatorial "state-sum" construction of this TQFT. Let T be a triangulation of M, with T_k the set of k-simplices. We have the following model for the moduli space of G-coverings: it is the set of "admissible" decorations $U : T_1 \to G$ of edges by group elements, where a decoration is admissible (or "flat"), if for every 2-simplex with boundary edges e_1, e_2, e_3, we have $U_{e_1}U_{e_2}U_{e_3} = 1$. This set of decorations is considered modulo "gauge transformations" $h : T_0 \to G$ acting on a decoration by $U_{vw} \mapsto U'_{vw} = h_v U_e h_w^{-1}$ for any edge e with endpoints v, w.[15] Then, for M a closed n manifold, the θ-twisted Dijkgraaf-Witten partition function can be written as

$$Z_M^\theta = \frac{1}{|G|^{|T_0|}} \sum_{U:T_1\to G \text{ admissible}} \prod_{\sigma=[v_0\cdots v_n]\in T_n} e^{i\,\theta(U_{v_0v_1},\ldots,U_{v_{n-1}v_n})} \tag{1.2.8}$$

Here inside the sum over admissible decorations we have the product over top-dimensional simplices, with vertices $v_0, \ldots, v_n$, and then we evaluate the group cocycle $\theta : \underbrace{G \times \cdots \times G}_{n} \to S^1$ on the group elements decorating n consecutive edges of the simplex. The factor in front of the sum is the inverse size of the group of gauge transformations. In particular, expression (1.2.8) is in fact well-defined—independent of the ordering of vertices of σ and independent of the triangulation T. The extension of (1.2.8) to manifolds with boundary is straightforward: one

[14] This pullback is well-defined as a relative class, since the boundary conditions for the classifying map β are given by fixed classifying maps $\gamma_{\text{in}}, \gamma_{\text{out}}$.

[15] The quotient (as a groupoid) of admissible decorations $U : T_1 \to G$ by gauge transformations $h : T_0 \to G$ yields another equivalent model for the moduli space of G-coverings $\mathcal{M}_{M,G}$.

considers decorations of T restricting to given decorations on the in- and out-boundary.

In the case $n = 2$ and assuming $\theta = 0$, the partition function for a closed surface of genus g can be written in the following form:

$$Z_M = |G|^{-\chi(M)} \sum_R (\dim R)^{\chi(M)}$$

with the sum ranging over irreducible representations of G and with $\chi(M) = 2-2g$ the Euler characteristic. Note that up to a normalization factor, this result looks identical to the result of 2D Yang-Mills (1.2.5) specialized to a closed surface of zero area, with the difference that in one case G is a finite group and in the other case a Lie group.

We refer the reader to the original paper by Dijkgraaf and Witten [**35**] for further details on this model.

1.3. The idea of path integral construction of quantum field theory

1.3.1. Classical field theory data. We start by fixing the data of *classical field theory* on an n-manifold:

- A space of fields $F_M = \Gamma(M, \mathbb{F}_M)$—the space of sections of some sheaf $\mathbb{F}_M$ over M. Typical examples of F_M are:
  - Space of functions $C^\infty(M)$.
  - Space of connections on a principal G-bundle $\mathcal{P}$ over M. (This example is typical for a class of *gauge theories*, e.g., Chern-Simons theory and Yang-Mills theory.)
  - Mapping space $\mathrm{Map}(M, N)$ with N some fixed target manifold. This is typical for so-called *sigma models*.
- The *action functional* $S_M : F_M \to \mathbb{R}$ of the form

  $$S_M(\phi) = \int_M L(\phi, \partial\phi, \partial^2\phi, \ldots)$$

  where L is *the Lagrangian density*—a density on M depending on the value of the field $\phi \in F_M$ and its derivatives (up to fixed finite order) at the point of integration on M. Variational problem of extremization of S (i.e. the critical point equation $\delta S = 0$) leads to Euler-Lagrange PDE on ϕ.

Example 1.3.1 (Free massive scalar field). Let (M, g) be a Riemannian manifold, we set $F_M = C^\infty(M) \ni \phi$ with the action

$$S_M(\phi) = \int_M \left(\frac{1}{2}\langle d\phi, d\phi\rangle_{g^{-1}} + \frac{m^2}{2}\phi^2 \right) d\mathrm{vol}$$

Here $m \geqslant 0$ is a parameter of the theory—the *mass*; $d$vol is the Riemannian volume element on M. The associated Euler-Lagrange equation on ϕ is: $(\Delta - m^2)\phi = 0$.

1.3.2. Idea of path integral quantization. The idea of quantization is to construct the partition function for a closed manifold M as

$$Z_M(\hbar) := \text{“}\int_{F_M} \mathcal{D}\phi\, e^{\frac{i}{\hbar} S_M(\phi)}\text{”} \tag{1.3.1}$$

Here $\hbar$ is a parameter of the quantization (informally, $\hbar$ measures the “distance to the classical theory”); $\mathcal{D}\phi$ is a symbol for a reference measure on the space F_M.

The integral (1.3.1) is problematic to define directly as a measure-theoretic integral, however it can be defined as an asymptotic series in $\hbar \to 0$, as we will discuss below. So far, the right hand side of (1.3.1) is a heuristic expression which has to be made mathematical sense of.

Consider a manifold M with boundary Σ. Denote B_Σ the set of boundary values of fields on M. We have a map $F_M \to B_\Sigma$ of evaluation of the field at the boundary (or pullback by the inclusion $\Sigma \hookrightarrow M$), sending $\phi \mapsto \phi|_\partial$. For the space of states on Σ, we set $\mathcal{H}_\Sigma := \mathrm{Fun}_\mathbb{C}(B_\Sigma)$—complex-valued functions on B_Σ. For the partition function Z_M, we set

$$Z_M(\phi_\Sigma;\hbar) := \int_{\phi\in F_M \text{ s.t. } \phi|_\partial=\phi_\Sigma} \mathcal{D}\phi\; e^{\frac{i}{\hbar}S_M(\phi)} \tag{1.3.2}$$

This path integral gives us a function on $B_\Sigma \ni \phi_\Sigma$ and thus a vector $Z_M(-;\hbar) \in \mathcal{H}_\Sigma$.

1.3.3. Heuristic argument for gluing. Let a closed (for simplicity) n-manifold M be cut by a codimension 1 submanifold Σ into two manifolds M' and M'', i.e. $M = M' \cup_\Sigma M''$. Then the integral (1.3.1) can be performed in the following steps.

(i) Fix ϕ_Σ on Σ.
(ii) Integrate over fields on M' with boundary condition ϕ_Σ on Σ.
(iii) Integrate over fields on M'' with boundary condition ϕ_Σ on Σ.
(iv) Integrate over $\phi_\Sigma \in B_\Sigma$.

This yields

$$Z_M = \int_{B_\Sigma \ni \phi_\Sigma} \mathcal{D}\phi_\Sigma\; Z_{M'}(\phi_\Sigma)\cdot Z_{M''}(\phi_\Sigma)$$

One can recognize in this formula the Atiyah-Segal gluing axiom (1.2.1): M' and M'' yield two vectors in $\mathcal{H}_\Sigma$ which are paired in $\mathcal{H}_\Sigma$ to a number—the partition function for the whole manifold.

1.3.4. How to define path integrals? Let us first look at finite-dimensional *oscillating* integrals: consider X a compact manifold with μ a fixed volume form and $f \in C^\infty(X)$ a function. The asymptotics, as $\hbar \to 0$, of the integral

$$\int_X \mu\; e^{\frac{i}{\hbar}f(x)}$$

is given by the *stationary phase formula*:[16]

$$\int_X \mu\; e^{\frac{i}{\hbar}f(x)} \underset{\hbar\to 0}{\sim} \sum_{x_0\in\{\text{crit. points of } f\}} e^{\frac{i}{\hbar}f(x_0)} |\det f''(x_0)|^{-\frac{1}{2}} e^{\frac{\pi i}{4}\mathrm{sign} f''(x_0)} (2\pi\hbar)^{\frac{\dim X}{2}}$$

The rough idea here is that the rapid oscillations of the integrand cancel out except in the neighborhood of critical points x_0 of f (i.e. points with $df(x_0) = 0$), which are the "stationary phase points" for the integrand—points around which the oscillations slow down.

[16] See e.g. [**39**, **57**, **103**] for the original analytic method and [**37**, **79**] for the field-theoretic context.

This formula can be improved to accommodate corrections in powers of $\hbar$:

$$
\begin{aligned}
&\int_X \mu\, e^{\frac{i}{\hbar}f(x)} \underset{\hbar\to 0}{\sim}\underset{\hbar\to 0}{\sim} \\
&\underset{\hbar\to 0}{\sim} \sum_{x_0\in\{\text{crit. points of } f\}} e^{\frac{i}{\hbar}f(x_0)} |\det f''(x_0)|^{-\frac{1}{2}}\, e^{\frac{\pi i}{4}\mathrm{sign} f''(x_0)}\, (2\pi\hbar)^{\frac{\dim X}{2}} \cdot \\
&\qquad \cdot \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\Phi_\Gamma
\end{aligned}
\tag{1.3.3}
$$

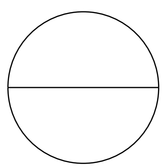
where Γ ranges over graphs with vertices of valence $\geqslant 3$ (possibly disconnected, including $\Gamma = \varnothing$); $\chi(\Gamma) \leqslant 0$ is the Euler characteristic of the graph and $|\mathrm{Aut}(\Gamma)|$ is the order of its automorphism group. Graphs Γ are called the **Feynman diagrams**. Assume that Γ has E edges and V vertices. We decorate all half-edges of Γ with labels $i_1, \ldots, i_{2E}$ each of which can take values $1, 2, \ldots, p := \dim X$. The weight of the graph Γ, Φ_Γ, is defined as follows.

- We assign to every edge e consisting of half-edges h_1, h_2 the decoration $f''(x_0)^{-1}_{i_{h_1} i_{h_2}}$—the matrix element of the inverse Hessian given by the labels of the constituent half-edges.
- We assign to every vertex v of valence k with adjacent half-edges $h_1, \ldots, h_k$ the decoration $\partial_{i_{h_1}} \cdots \partial_{i_{h_k}} f(x_0)$—a k-th partial derivative of f at the critical point.
- We take the product of all the decorations above and sum over all possible values of labels on the half-edges. Φ_Γ is this sum times the factor i^{E+V}.

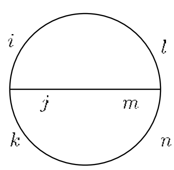
I.e., we have

$$\Phi_\Gamma := i^{E+V} \cdot \sum_{i_1,\ldots,i_{2E}\in\{1,\ldots,p\}} \prod_{\text{edges } e=(h_1h_2)} f''(x_0)^{-1}_{i_{h_1} i_{h_2}} \cdot \prod_{\text{vertices } v} \partial_{i_{h_1}} \cdots \partial_{i_{h_{\mathrm{val}(v)}}} f(x_0)$$

EXAMPLE 1.3.2. Consider the "theta graph"



Note that its Euler characteristic is -1, hence it enters in (1.3.3) in the order $\hbar^1$ (more precisely, it enters with coefficient $\frac{\hbar}{12}$ since the automorphism group of the graph is $\mathbb{Z}_2 \times S_3$ and has order 12). For its weight, we obtain

$\Phi\Bigg($  $\Bigg) =$

$$= i^{3+2} \cdot \sum_{i,j,k,l,m,n\in\{1,\ldots,p\}} f''(x_0)^{-1}_{il} f''(x_0)^{-1}_{jm} f''(x_0)^{-1}_{kn} f'''(x_0)_{ijk} f'''(x_0)_{lmn}$$

Stationary phase formula (1.3.3) replaces, in the asymptotics $\hbar \to 0$, a measure-theoretic integral on the l.h.s. with the purely algebraic expression on the r.h.s., involving only values of derivatives of f at the critical points x_0.

The idea then is to define the path integral (1.3.1) by formally applying the stationary phase formula, as the r.h.s. of (1.3.3), i.e. as a series in $\hbar$ with coefficients given by weights of Feynman diagrams.[17]

We expect that if we start with a classical field theory with S_M invariant under diffeomorphisms of M, the partition functions Z_M coming out of the path integral quantization procedure yield manifold invariants and arrange into a TQFT.

Problem: Stationary phase formula requires critical points of f to be *isolated* (more precisely, we need the Hessian of f at critical points to be non-degenerate). However, diffeomorphism-invariant classical field theories are *gauge theories*, i.e. there is a tangential distribution $\mathcal{E}$ on F_M which preserves the action S_M (in an important class of examples, $\mathcal{E}$ corresponds to an action of a group $\mathcal{G}$—the *gauge group*—on F_M). Thus, critical points of S_M come in $\mathcal{E}$-orbits and therefore are not isolated. Put another way, the Hessian of S_M is degenerate in the direction of $\mathcal{E}$. So, the stationary phase formula cannot be applied to the path integral (1.3.1) in the case of a gauge theory.

The cure for this problem comes from using the Batalin-Vilkovisky construction.

1.3.5. Towards Batalin-Vilkovisky (BV) formalism. Batalin-Vilkovisky construction [**9, 10**] replaces the classical field theory "package" F, S with a new package consisting of:

- A $\mathbb{Z}$-graded supermanifold $\mathcal{F}$ (the "space of BV fields") endowed with an odd-symplectic structure ω of internal degree -1.
- A function S_{BV} on $\mathcal{F}$—the "master action," satisfying the "master equation"
$$\{S_{BV}, S_{BV}\} = 0$$
where $\{-,-\}$ is the degree $+1$ Poisson bracket on $C^\infty(\mathcal{F})$ induced by ω. In particular, this implies that the corresponding Hamiltonian vector field $Q = \{S_{BV}, \bullet\}$ is *cohomological*, i.e. satisfies $Q^2 = 0$. Thus, Q endows $C^\infty(\mathcal{F})$ with the structure of a cochain complex. In other words, $(\mathcal{F}, Q)$ is a *differential graded (dg) manifold*.

The idea is then to replace

$$\int_F e^{\frac{i}{\hbar}S} \to \int_{\mathcal{L}\subset\mathcal{F}} e^{\frac{i}{\hbar}S_{BV}} \tag{1.3.4}$$

with $\mathcal{L} \subset \mathcal{F}$ a Lagrangian submanifold w.r.t. the odd-symplectic structure ω.

The integral on the l.h.s. of (1.3.4) is ill-defined (by means of stationary phase formula) in the case of a gauge theory, whereas the integral on the r.h.s. is well-defined, for a good choice of Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$, and moreover is invariant under deformations of $\mathcal{L}$.

The following remark is due to Stasheff [**90**] (see also [**41, 45**]).

[17] In the path integral case, the labels decorating the half-edges become continuous (they contain the information on the position of a vertex on a manifold, with a coincidence condition for half-edges incident to a given vertex), and the Feynman weight of a graph becomes an integral over the configuration space of V points on M, see Sections 5.2, 5.3 for explicit examples.

REMARK 1.3.3. The ring of functions on the space $\mathcal{F}$ is a two-sided resolution of $C^\infty(F)$ constructed out of:

- Chevalley-Eilenberg resolution for the subspace of gauge-invariant functions of fields $C^\infty(F)^\mathcal{G}$ and
- Koszul-Tate resolution for functions on the space of solutions of Euler-Lagrange equations $C^\infty(EL \subset F)$.

So, coordinates on $\mathcal{F}$ of nonzero degree arise as either Chevalley-Eilenberg generators (in positive degree) or Koszul-Tate generators (in negative degree). In particular, this is the reason why $\mathcal{F}$ has to be a supermanifold (since C-E and K-T generators anti-commute).

REMARK 1.3.4. In the case of a gauge field theory, one could try to remedy the problem of degenerate critical points in the path integral by passing to the integral over the quotient, $\int_F \to \int_{F/\mathcal{G}}$. The latter may indeed have nondegenerate critical points. But the issue is then the following: we know how to make sense of Feynman diagrams for the path integral over the space of sections of a sheaf over M, but the quotient $F/\mathcal{G}$ would not be of this type. In this sense, one may think of the r.h.s. of (1.3.4) as a resolution of the integral over a quotient $F/\mathcal{G}$ by an integral over a locally free object—the space of sections of a sheaf over M.

REMARK 1.3.5. There are finite-dimensional cases where the l.h.s. of (1.3.4) exists as a measure-theoretic integral (despite having non-isolated critical points). Then, under certain assumptions, one has a comparison theorem stating that the l.h.s. and the r.h.s. of (1.3.4) coincide (see Section 4.8.3 and Remark 4.3.7).

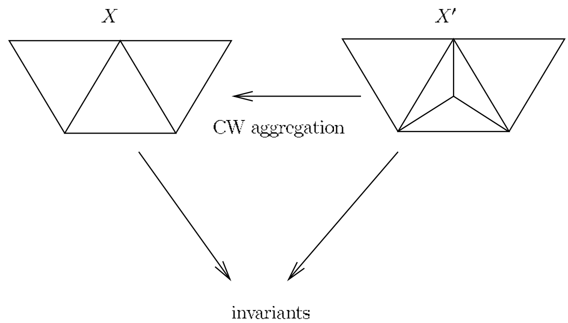
1.4. Plan of the exposition

- Chapter 2: classical Chern-Simons theory—a prototypical example of a diffeomorphism-invariant gauge theory, which we discuss here on the classical level.
- Chapter 3: Feynman diagrams in the context of finite-dimensional integrals.
  - Stationary phase formula.
  - Wick's lemma for moments of a Gaussian integral. Perturbed Gaussian integral. Stationary phase formula with corrections given by Feynman diagrams.
  - Berezin integral over an odd vector space. Feynman diagrams for integrals over a super vector space.
- Chapter 4: introduction to BV formalism.
  - $\mathbb{Z}$-graded supergeometry: odd-symplectic geometry (after [**87**]), differential graded (dg) manifolds, integration on supermanifolds.
  - BV Laplacian, classical and quantum master equation (CME and QME).
  - $\frac{1}{2}$-densities on odd-symplectic manifolds, BV integrals, fiber BV integral as a pushforward of solutions of quantum master equation (following [**20**, **72**]).
  - BV as a solution to the problem of gauge-fixing: Faddeev-Popov construction, BRST (as a homological algebra interpretation of Faddeev-Popov), BV (as a "doubling" of BRST).

– Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) construction of topological field theories in BV formalism [**2**].

In Chapter 5 we discuss several applications of BV formalism.

(I) Section 5.1: we present a model of topological quantum field theory[18] on CW complexes X—the cellular non-abelian BF theory [**26**, **72**].



Here a CW complex X gets assigned a BV package—a space of fields comprised of cellular cochains and chains twisted by a G-local system E, $\mathcal{F}_X = C^\bullet(X,E) \oplus C_\bullet(X,E^*)$ (with certain homological degree shifts which we omitted here); G is a fixed Lie group—the structure group of the theory. The space $\mathcal{F}_X$ carries a natural odd-symplectic structure (coming from pairing chains with cochains). The action is given as a sum, over cells $e \subset X$ of all dimensions, of certain universal local building blocks $\bar{S}_e$ depending only on combinatorial type of the cell and on values of fields restricted to the cell.

One calculates certain invariant $\psi(X)$ of X by pushing forward the BV package to the (cellular) cohomology of X, via a *finite-dimensional* fiber BV integral. If X' is a cellular subdivision of X (then we say that X is an "aggregation" of X'), the pushforward of the BV package on X' to X yields back the package on X, and for the invariant one has $\psi(X') = \psi(X)$. More precisely, one gets a simple-homotopy invariant of CW complexes.[19]

We also discuss here the correspondence between solutions of the QME and infinity algebras (relevant case for this model: unimodular L_∞ algebras) and the interpretation of the fiber BV integral as the homotopy transfer of infinity algebras.

(II) Section 5.2: perturbative Chern-Simons theory (after Axelrod-Singer [**4**, **5**]). Perturbative invariants of 3-manifolds M given in terms of integrals over Fulton-MacPherson-Axelrod-Singer compactifications of configuration spaces of n distinct points on M.

---

[18] "Topological" here means compatibility with cellular subdivisions/aggregations, which is a combinatorial replacement for diffeomorphism invariance in Schwarz's notion of a topological field theory. The model can also be understood as a TQFT in the sense of appropriately modified Atiyah's axioms [**26**], with partition functions assigned to cobordisms (endowed with cellular decompositions) satisfying the gluing axiom.

[19] In fact, ψ contains a "tree part" encoding the rational homotopy type of X for X simply-connected. For X non simply-connected, ψ contains the information on the geometry of the formal neighborhood of a (possibly singular) point E in the moduli space $\mathcal{M}_{X,G}$ of local systems and on the behavior of Reidemester torsion near the singularity.

(III) Section 5.3: Kontsevich's deformation quantization of Poisson manifolds (M, π) [**62**], following [**17, 19**]. Here the problem is to construct a family (parameterized by $\hbar$) of associative non-commutative deformations of the pointwise product on $C^\infty(M)$, of the form

$$f *_\hbar g(x) = f \cdot g(x) - \frac{i\hbar}{2}\{f, g\}_\pi + \sum_{n \geqslant 2} (-i\hbar)^n B_n(f, g)(x) \tag{1.4.1}$$

where B_n are some bi-differential operators (of bounded order depending on n). The idea of the construction (following [**17**]) is to write the star-product as path integral representing certain expectation value for a 2-dimensional topological field theory (the *Poisson sigma model*) on a disk D, with two observables placed on the boundary, at points 0 and 1:

$$f *_\hbar g(x_0) = \int_{X(\infty)=x_0, \eta|_{\partial D}=0} \mathcal{D}X\, \mathcal{D}\eta \quad e^{\frac{i}{\hbar} S_{\mathrm{PSM}}(X,\eta)} f(X(0)) \cdot g(X(1)) \tag{1.4.2}$$

Here the fields X, η are the base and fiber components of a bundle map

$$\begin{tikzcd} TD \arrow[r, "\eta"] \arrow[d] & T^*M \arrow[d] \\ D \arrow[r, "X"'] & M \end{tikzcd}$$

and the action is: $S_{\mathrm{PSM}} = \int_D \langle \eta, dX \rangle + \frac{1}{2}\langle X^*\pi, \eta \wedge \eta \rangle$. This action possesses a rather complicated gauge symmetry (given by a non-integrable distribution on the space of fields) and one needs BV to make sense of the integral (1.4.2). The final result is the explicit construction of operators B_n in (1.4.1) in terms of integrals over compactified configuration spaces of points on the 2-disk D.