Introduction

This book is based on the first part of a graduate course that was given at the Institut Henri Poincaré in 1964. It focuses on the so-called "reduction theory" in a real algebraic group $G_{\mathbb{R}}$, with respect to an arithmetic group Γ. The proofs of the general theorems make extensive use of the theory of linear algebraic groups. However, since the students in the course were not necessarily assumed to be acquainted with this, we first gave direct proofs of some classical cases (after all, these are where the general theory originates from), and we summarized the necessary notions and results on linear algebraic groups, as we needed them. Most of these summaries are in three sections (§7, §10, §11), which also contain some examples and proofs and thus provide, to a certain extent, an introduction to some aspects of the theory of algebraic groups.

By *reduction* in $G_{\mathbb{R}}$, with respect to Γ, we mean, roughly speaking, the search for (open or closed) subsets that meet each orbit of Γ (acting by translations on the right) in at least one point, but never in more than a finite number of points. Such sets are called fundamental sets. (Actually, we will impose precise conditions that are more restrictive (cf. 5.6, 9.6, 15.13).) Alternatively, but equivalently, the problem can be considered in the space $X = K\backslash G_{\mathbb{R}}$ of right cosets of $G_{\mathbb{R}}$ modulo a maximal compact subgroup K. When G is a classical group, we encounter, as a special case, the reduction theory of quadratic forms (and of hermitian forms).

The book has three parts. The first (§1 to §6) is mainly devoted to the reduction of quadratic forms, using methods that will be generalized in a natural way in later sections. We first consider the case where $G = \mathbf{GL}(n, \mathbb{R})$, $\Gamma = \mathbf{GL}(n, \mathbb{Z})$, and $K = \mathbf{O}(n)$, that is, where X is the space of positive-definite quadratic forms on $\mathbb{R}^n$. We show that every orbit of Γ in G meets a suitable Siegel set (1.4) and we deduce a few consequences, including Mahler's criterion for the relative compactness of a subset of the space $\mathbf{GL}(n, \mathbb{R})/\,\mathbf{GL}(n, \mathbb{Z})$ of lattices of $\mathbb{R}^n$, and the finiteness of the volume of $\mathbf{SL}(n, \mathbb{R})/\,\mathbf{SL}(n, \mathbb{Z})$. §2 translates these results into the language of quadratic forms, and establishes some links with Minkowski reduction. §4 shows that a Siegel set only meets a finite number of its translates by elements of $x\,.\Gamma$, for each $x \in \mathbf{GL}(n, \mathbb{Q})$, (4.6). §5 is devoted to the reduction of indefinite quadratic forms, using Hermite's method. Its key point is a finiteness property of "reduced integral" forms, which we will deduce from a more general lemma that is proved in § 6.

The second part (§7, §8, §9) is devoted to two general theorems on arithmetic groups, whose proofs require only a fairly restricted collection of results on algebraic groups (which will be recalled or proved in §7). The two main theorems are: a compactness criterion for the quotient $G_{\mathbb{R}}/\Gamma$ (§8), and a first construction of fundamental sets, which generalizes the method of Hermite. Furthermore, §8 shows that the image of an arithmetic group by an isogeny is also an arithmetic group, and §9 establishes a finiteness theorem for the number of orbits of Γ in the set of integer points of a closed orbit of G, in the space of a linear representation of G. This generalizes the finiteness of the number of classes of quadratic forms of given non-zero determinant (6.4), and also generalizes results of Jordan on the classes of homogeneous forms of degree ≥ 3, (6.5).

The third part (§10 to §17) is devoted to fundamental sets that are usually better than those of §9. Their existence is proved in two very different ways: in §13, where we rely on §9, and in §16, where we apply an extremum principle to a type of function that is studied in §14, and which generalizes, among others, the function $|cz + d|$ of the Poincaré upper half-plane. These sets are the union of a finite number of translates (by elements of $G_{\mathbb{Q}}$) of a set of a simple form, which is called a Siegel set. Finally, §17 describes, in a particular case, the space $K\backslash G_{\mathbb{R}}/\Gamma$ as the interior of a compact manifold with boundary.

The intention to make the first part self-contained, and to appeal to the theory of algebraic groups only when necessary, has led to some redundancies and inconsistencies. For example, §4 is a particular case of §15, but is not assumed in the latter, and the existence of a Bruhat decomposition is proved in §3 for $\mathbf{GL}(n, k)$, while it is assumed without proof in a much more general setting from §11 onward. Consequently, this exposition, which, *grosso modo*, follows the chronological order, and contains some quite extended "reviews," is not the most efficient possible, and reading a particular section does not necessarily require reading all of the preceding ones. Here are a few additional remarks on the interdependence of the various sections, which may serve as a "Leitfaden": §1, up to 1.11, is fundamental for the entire book, but the remainder of that section, and §2 to §5, are not used later, other than for providing concrete examples of the theory; readers who wish to reach the general theorems as quickly as possible may focus their attention on §§1, 8, 12, 14, 15, 16, if they are willing to assume a certain finiteness property whose proof here relies on §13, but which can be established more directly by using the adelic analogue of §§1 and 8 (cf. introduction to §16); finally, §6 plays a crucial role in §5 and §9, and this latter is used in §13, but nowhere else.

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