

Preface

The theory of persistence modules is an emerging field of algebraic topology, which originated in topological data analysis and which lies at the crossroads of several disciplines, including metric geometry and the calculus of variations. Persistence modules were introduced by G. Carlsson and A. Zomorodian [110] in 2005 as an abstract algebraic language for dealing with persistent homology, a version of homology theory invented by H. Edelsbrunner, D. Letscher, and A. Zomorodian [32] at the beginning of the millennium, aimed, in particular, at extracting robust information from noisy topological patterns. We refer to the articles by H. Edelsbrunner and J. Harer [31], R. Ghrist [39], G. Carlsson [17], S. Weinberger [104], and U. Bauer and M. Lesnick [8] and the monographs by H. Edelsbrunner [30], S. Oudot [76], and F. Chazal, V. de Silva, M. Glisse, and S. Oudot [20] for various aspects of this rapidly developing subject. In the past few years, the theory of persistence modules expanded its “sphere of influence” within pure mathematics, exhibiting fruitful interactions with function theory and symplectic geometry. The purpose of these notes is to provide an introduction to this field and to give an account on some of the recent advances, emphasizing applications to geometry and analysis. We tried to minimize algebraic prerequisites so that the material should be accessible to readers with a basic background in algebraic and differential topology. More advanced preliminaries in geometry and function theory will be reviewed.

Topological data analysis deals with data clouds modeled by finite metric spaces. Its main motto is

$$\text{geometry} + \text{scale} = \text{topology}.$$

In the case when a finite metric space appears as a discretization of a Riemannian manifold M , the above relation enables one to infer the topology of M , provided one knows the mesh. In general, given a scale $t > 0$, one can associate to any abstract metric space (X, d) a topological space R_t , called *the Vietoris-Rips complex*. By definition, R_t is a subcomplex of the full simplex Σ formed by the points of X , where $\sigma \subset X$ is a simplex of R_t whenever the diameter of σ is $< t$. For instance, the Rips complex for the vertices of the unit square in the plane is presented in Figure 1 a). Thus we get a filtered topological space, a.k.a., a collection of topological spaces R_t , $t \in \mathbb{R}$, with $R_s \subset R_t$ for $s < t$. Let us mention that R_t is empty for $t \leq 0$ and $R_t = \Sigma$ for $t > \text{diam}(X, d)$. Some authors call this structure *a topological signature* of the data cloud (X, d) . Rips complexes, which originated in geometric group theory [11], also play an important role in detecting low-dimensional topological patterns in big data, which nowadays is an active area of applied mathematics (see e.g. [73]).

The calculus of variations studies critical points and critical values of functionals, the simplest case being smooth functions on manifolds. The sublevel sets

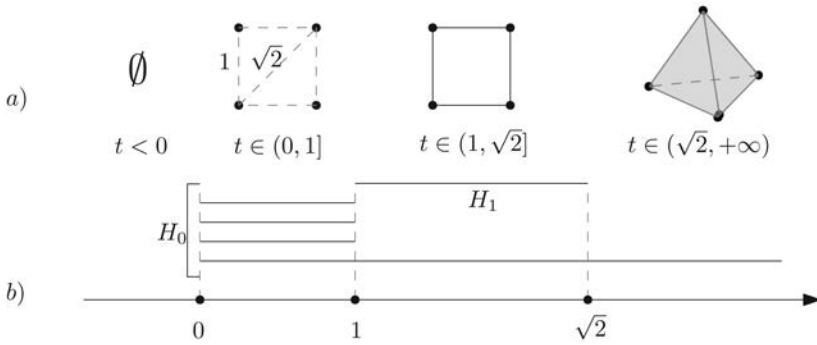


FIGURE 1. The Rips complex of a square and the corresponding barcode.

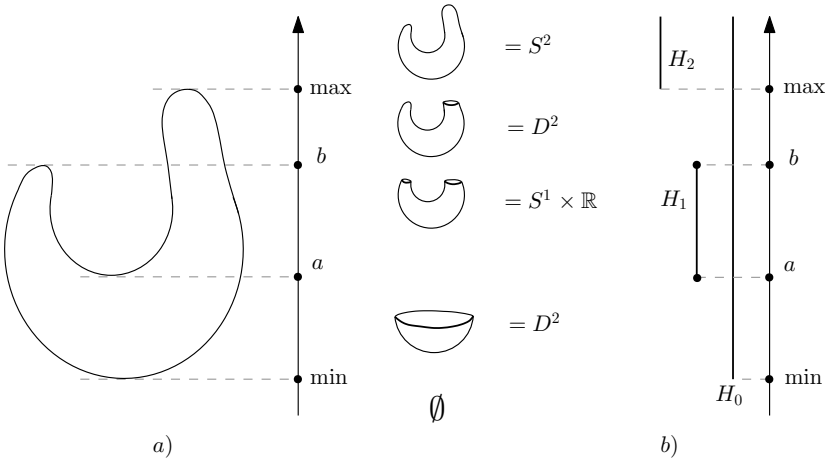


FIGURE 2. The height function on the (topological) sphere and the corresponding barcode.

$R_t := \{f < t\}$ of a function f on a closed manifold M induce a structure of a filtered topological space. According to Morse theory, the topology of R_t , $t \in \mathbb{R}$, changes exactly when the parameter t hits a critical value of f . Note that $R_t = \emptyset$ when $t \leq \min f$ and $R_t = M$ when $t > \max f$. See Figure 2 a) illustrating sublevel sets of a function on the two-dimensional sphere with two local maxima and one local minima.

We are going to study filtered topological spaces by using algebraic tools. Fix a field \mathbb{F} and look at the homology $V_t := H(R_t, \mathbb{F})$ of the spaces R_t as above with coefficients in \mathbb{F} . The family of vector spaces V_t , $t \in \mathbb{R}$ together with the morphisms $V_s \rightarrow V_t$, $s < t$ induced by the inclusions, form an algebraic object called a *persistence module*, which plays a central role in the present notes.

Let us discuss the contents of the book in more detail. Part 1 lays foundations of the theory of persistence modules and introduces basic examples. It turns out that persistence modules (which are defined in Chapter 1) are classified by simple combinatorial objects, the barcodes, which are defined as collections of intervals and rays in \mathbb{R} with multiplicities. While the real meaning of barcodes will be clarified

later on, some intuition can be gained by looking at Figures 1 b) and 2 b). In these figures, for illustrative purposes, the bars are equipped with an additional decoration corresponding to the degree of homology they represent. The number of bars in degree k over a point $t \in \mathbb{R}$ equals the k -th Betti number of the space R_t . For instance, the bar in degree 1 manifests that the spaces R_t possess non-trivial first homology for $t \in (1, \sqrt{2}]$ in Figure 1 b) and for $t \in (a, b]$ in Figure 2 b). Look also at the bars in degree 0 in Figure 1 b), that is $(0, 1)$ taken with multiplicity 3 and $(0, +\infty)$. This carries the following information: $H_0(R_s) = \mathbb{F}^4$ for $s \in (0, 1]$, $H_0(R_t) = \mathbb{F}$ for $t > 1$, and the map $H_0(R_s) \rightarrow H_0(R_t)$ does not vanish. Very roughly speaking, this means that one (and only one) of the four generators of $H_0(R_s)$ persists when s increases and hits the value 1. In Chapter 2 we will make this intuitive picture rigorous.

A highlight of the theory of persistence modules is an isometry between the space of persistence modules equipped with a certain algebraic distance, which naturally appears in applications but is hard to calculate, and the space of barcodes equipped with a user-friendly *bottleneck distance* of a combinatorial nature. This is a profound fact discovered by F. Chazal, D. Cohen-Steiner, M. Glisse, L. Guibas, and S. Oudot in [19]. It will be proved below (see Chapters 2 and 3) following the approach by U. Bauer and M. Lesnick [8].

Thus one can associate to a Morse function on a closed manifold or to a finite metric space a barcode. Remarkably, this correspondence is stable, or, more precisely, Lipschitz with respect to the uniform norm on functions and the Gromov-Hausdorff distance on metric spaces. This fundamental phenomenon was discovered by D. Cohen-Steiner, H. Edelsbrunner, and J. Harer [24] for functions and by F. Chazal, V. de Silva, and S. Oudot [21] for metric spaces. In particular, metric spaces whose barcodes are remote in the bottleneck distance are far from being isometric, and a small C^0 -perturbation of a function cannot significantly change its barcode. The stability of barcodes with respect to C^0 -perturbations of functions paves way to applications of persistence modules to topological function theory, a theme we develop in Chapter 6.

In Chapter 4 we discuss some natural Lipschitz functionals on the space of barcodes, which yield interesting numerical invariants of functions and metric spaces. They include, for instance, the end-points of infinite rays, which in the case of functions correspond to the homological min-max. Another example is given by the length of the longest finite bar in the barcode, which is called the *boundary depth*, an invariant introduced by M. Usher in [97]. The boundary depth gives rise to a non-negative functional on smooth functions on a manifold which is Lipschitz in the uniform norm, invariant under the action of diffeomorphisms on functions, and sends each function to the difference between a pair of its critical values. The very existence of such a functional different from $f \mapsto \max f - \min f$ is not at all obvious. We conclude with the multiplicity function, an invariant that appears in the study of representations of finite groups on persistence modules and which will be useful for applications to symplectic geometry in Chapter 8.

In Part 2 of the book we elaborate applications of persistence modules to metric geometry and function theory. Chapter 5 focuses on Rips complexes. After reviewing their origins in geometric group theory (here our exposition closely follows a book by M. Bridson and A. Haefliger [11]), we discuss the appearance of

Rips complexes in data analysis. We present a toy version of manifold learning motivated by a seminal paper [73] by P. Niyogi, S. Smale, and S. Weinberger.

Chapter 6 deals with topological function theory, which studies features of smooth functions on a manifold that are invariant under the action of the diffeomorphism group. The theory of persistence modules provides a wealth of invariants coming from the homology of the sublevel sets of a function. We shall focus, roughly speaking, on the “size” of the barcode, which can be considered as a useful measure of oscillation of a function. We prove bounds on this size in terms of norms of a function and its derivatives and discuss links to approximation theory. This chapter is mostly based on the papers [25] by D. Cohen-Steiner, H. Edelsbrunner, J. Harer, and Y. Mileyko, [85] by L. Polterovich and M. Sodin, and [78] by I. Polterovich, L. Polterovich, and V. Stojisavljević. In the course of exposition we present also an algorithm for finding a canonical normal form of filtered complexes with a preferred basis due to S. Barannikov [6].

In Part 3, after a crash-course on symplectic geometry and Hamiltonian dynamics (see Chapter 7), we discuss their interactions with the theory of persistence modules. Here instead of functions on a finite-dimensional manifold the object of interest is the classical action functional on the loop space of a symplectic manifold. It was a great insight due to A. Floer [36] that by using the theory of elliptic PDEs and Gromov’s theory of pseudo-holomorphic curves in symplectic manifolds [43] one can properly define a Morse-type homology theory for sublevel sets of the action functional. L. Polterovich and E. Shelukhin [83] and M. Usher and J. Zhang [100] showed that filtered Floer homology gives rise to persistence modules and barcodes. We shall elaborate this construction in two different contexts: Hamiltonian diffeomorphisms of symplectic manifolds (Chapter 8) and star-shaped domains of Liouville manifolds (Chapter 9). The group of Hamiltonian diffeomorphisms is equipped with Hofer’s bi-invariant metric introduced by H. Hofer in 1990 [51], which has played a central role in symplectic topology for almost three decades, while the space of star-shaped domains also has a natural structure of a metric space with respect to a non-linear analogue of the Banach-Mazur classical distance on convex bodies (Chapter 9). The exploration of the non-linear Banach-Mazur distance, which has been introduced following unpublished ideas of Y. Ostrover and L. Polterovich circa 2015 with an important modification by M. Usher and J. Gutt [47], nowadays is making its very first steps; see the papers [95] by V. Stojisavljević and J. Zhang and [99] by M. Usher. We shall outline the proof of symplectic stability theorems stating that the correspondence sending a Hamiltonian diffeomorphism (resp., a star-shaped domain) to its barcode is Lipschitz with respect to Hofer’s (resp., the non-linear Banach-Mazur) distance.

Barcodes of Hamiltonian diffeomorphisms carry some interesting information. For instance, one can read from them spectral invariants introduced by C. Viterbo [102], M. Schwarz [90], and Y.-G. Oh [74], as well as the above-mentioned boundary depth [97]. Furthermore, the natural action by conjugation of a diffeomorphism on the Floer homology of its p -th power gives rise to a basic representation theory of the cyclic group \mathbb{Z}_p on Floer’s barcodes, yielding in turn applications to geometry and dynamics. In Chapter 8 we discuss some of these advances due to L. Polterovich and E. Shelukhin [83], J. Zhang [109], and L. Polterovich, E. Shelukhin, and V. Stojisavljević [84].

Persistence modules associated to star-shaped domains have applications to embedding problems in symplectic topology. We illustrate this by presenting a proof of M. Gromov's famous non-squeezing theorem [43] in Chapter 9.

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