CHAPTER 1

Definition and first examples

In this chapter we introduce the category of persistence modules and discuss several important examples.

1.1. Persistence modules

Let us fix a field $\mathbb{F}$.

DEFINITION 1.1.1. A *persistence module of finite type* is a pair (V, π), where V is a collection $\{V_t\}$, $t \in \mathbb{R}$, of finite-dimensional vector spaces over $\mathbb{F}$, and π is a collection $\{\pi_{s,t}\}$ of linear maps $\pi_{s,t} : V_s \to V_t$ for all $s \leq t$ in $\mathbb{R}$ such that

(1) (*Persistence*) For any $s \leq t \leq r$ one has $\pi_{s,r} = \pi_{t,r} \circ \pi_{s,t}$, i.e., the following diagram commutes:

$$V_s \xrightarrow[\pi_{s,t}]{} V_t \xrightarrow[\pi_{t,r}]{} V_r, \qquad V_s \xrightarrow{\pi_{s,r}} V_r$$

(2) For all but a finite number of points $t \in \mathbb{R}$ there exists a neighborhood U of t such that $\pi_{s,r}$ is an isomorphism for any $s < r$ in U.

(3) (*Semicontinuity*) For any $t \in \mathbb{R}$ and any $s \leq t$ sufficiently close to t, the map $\pi_{s,t}$ is an isomorphism.

(4) There exists an $s_- \in \mathbb{R}$, such that $V_s = 0$ for any $s \leq s_-$.

Let us elaborate on the various conditions in Definition 1.1.1. The *persistence* condition (1) is the heart of the definition, and some authors take it as the sole condition in the definition of a persistence module. Conditions (2) and (4) are sometimes called "finite-type" assumptions, and they greatly simplify the presentation. As we will see, adopting these restrictions still allows for interesting examples of persistence modules of finite type, although they are sometimes relaxed in favor of a more general definition (see Section 2.4 and Chapter 9). Finally, condition (3) is superfluous, and is included simply to allow uniqueness of decomposition of persistence modules into basic "blocks" (see the Normal Form Theorem 2.1.2).

REMARKS 1.1.2. (1) Note that by conditions (1) and (3), $\pi_{t,t} = \mathbb{1}_{V_t}$ for any $t \in \mathbb{R}$.

(2) One may check that by condition (2) there is $s_+ \in \mathbb{R}$, such that for any $t > s \geq s_+$, $\pi_{s,t} : V_s \to V_t$ is an isomorphism, i.e., the collection $\{V_t\}$ stabilizes starting at some s_+. We will use the notation V_∞ when referring to this "terminal" vector space, i.e., $V_\infty = V_t$ for t large enough. Note also that V_∞ is the direct limit of the system $\{V_t, \pi_{s,t}\}$.

Later on, in Section 2.4, we introduce a slightly more general class of persistence modules of *locally finite type*. For the moment, unless otherwise stated, we assume that all persistence modules are of finite type, and call them simply *persistence modules*.

Let us present now two fundamental examples that will reappear in the exposition.

EXAMPLE 1.1.3 (Morse theory). Let X be a closed manifold (i.e., a smooth compact manifold without boundary) and let $f : X \to \mathbb{R}$ be a Morse function. Fix $0 \le k \in \mathbb{Z}$ and put $V_t = H_k(\{f < t\})$ (taking homology with coefficients in $\mathbb{F}$ throughout the text, where $\mathbb{F}$ is an arbitrary fixed field, unless stated otherwise). Consider the natural inclusion $\{f < s\} \xhookrightarrow[i_{s,t}]{} \{f < t\}$ for $s \le t$. It induces the map $\pi_{s,t} := (i_{s,t})_* : V_s \to V_t$ in homology, and one can verify that we get a persistence module.

REMARK 1.1.4. Later on, we will also write $V_t = H_*(\{f < t\})$, referring to homology of some arbitrary degree $*$.

EXAMPLE 1.1.5 (Finite metric spaces, Rips complex). Let (X, d) be a finite metric space. For $0 < \alpha \in \mathbb{R}$ define the simplicial complex $R_\alpha(X)$, called the *Rips complex*, as follows: the vertices of $R_\alpha(X)$ are the points of X, and $k+1$ points in X determine a k-simplex $\sigma = [x_0, \dots, x_k]$ if $d(x_i, x_j) < \alpha$ for all i, j.

This construction is illustrated in Figure 1 (see also the discussion in the Preface). Note that in fact the Rips complex is completely determined by its 1-skeleton, it is in fact a *flag complex*. Due to this feature, the Rips complex is relatively easy to compute, which on the other hand might result in loss of information regarding the original space (as opposed to other complexes that can be attached to (X, d), see Section 5.2).

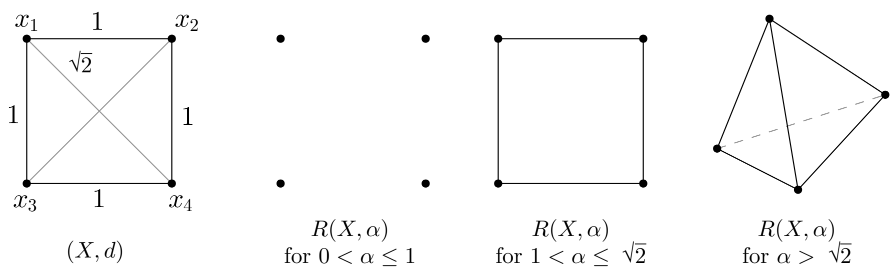

FIGURE 1. An example of Rips complex of a metric space consisting of four points.

Note that for $0 < \alpha \le \min_{x,y \in X,\ x \ne y} d(x, y)$ the complex $R_\alpha(X)$ is a finite collection of points, while for $\alpha > \mathrm{diam}(X)$, $R_\alpha(X)$ is a simplex of dimension $|X|-1$. For $\alpha \le \beta$, there is a natural simplicial map $i_{\alpha,\beta} : R_\alpha(X) \to R_\beta(X)$. Thus, taking $V_\alpha(X) = H_*(R_\alpha(X))$ and $\pi_{\alpha,\beta} = (i_{\alpha,\beta})_*$, we get a persistence module, which will be referred to as the *Rips module*.

Let us mention that Rips complexes were first introduced by Vietoris in [**101**]. Rips reintroduced them in order to study hyperbolic groups. We will follow Gromov [**44**] and stick to the name Rips, although they are sometimes called *Vietoris* or *Vietoris-Rips* complexes, see [**49**].

DEFINITION 1.1.6. Let (V, π) be a persistence module. The collection of spaces $P_{s,t} = \mathrm{im}(\pi_{s,t})$ will be called the *persistent homology* of V. Note that in fact, by condition (2) in Definition 1.1.1, it would be enough to record only a finite number of such spaces $P_{s,t}$, since there are finitely many "jump" points at which $\pi_{s,t} \neq \mathbb{1}$.

1.2. Morphisms

Let (V, π) and (V', π') be two persistence modules.

DEFINITION 1.2.1. A *morphism* $A : (V, \pi) \to (V', \pi')$ is a family of linear maps $A_t : V_t \to V'_t$ such that the following diagram commutes for all $s \leq t$:

$$\begin{array}{ccc} V_s & \xrightarrow{\pi_{s,t}} & V_t \\ {\scriptstyle A_s}\downarrow & & \downarrow{\scriptstyle A_t} \\ V'_s & \xrightarrow[\pi'_{s,t}]{} & V'_t \end{array}$$

Thus, one can now speak of the category of persistence modules.

In particular, we have the notion of an *isomorphism*: two persistence modules (V, π) and (V', π') are *isomorphic* if there exists two morphisms $A : V \to V'$ and $B : V' \to V$ such that both compositions $A \circ B$ and $B \circ A$ are the identity morphisms on the corresponding persistence module. (The *identity morphism* on V is the identity on V_t for all t.)

EXAMPLE 1.2.2 (Shift). For a persistence module (V, π) and $\delta \in \mathbb{R}$, define a persistence module $(V[\delta], \pi[\delta])$ by taking $(V[\delta])_t = V_{t+\delta}$ and $(\pi[\delta])_{s,t} = \pi_{s+\delta,t+\delta}$. This new persistence module is called the *δ-shift* of V. For $\delta > 0$, the map $\Phi^\delta : (V, \pi) \to (V[\delta], \pi[\delta])$ defined by $\Phi^\delta_t = \pi_{t,t+\delta}$ is a morphism of persistence modules (it will be referred to as *δ-shift morphism*). Also, if we have a morphism $F : V \to W$ between two persistence modules, we denote by $F[\delta] : V[\delta] \to W[\delta]$ the corresponding morphism between their δ-shifts.

EXERCISE 1.2.3. Prove that Φ^δ is indeed a morphism.

DEFINITION 1.2.4. Let (V, π) be a persistence module. A *persistence submodule* $(W, \tilde{\pi})$ of V is a collection of subspaces $W_s \subseteq V_s$ for all $s \in \mathbb{R}$, such that the maps $\tilde{\pi}_{s,t} := \pi_{s,t}\big|_{W_s} : W_s \to W_t$ are well-defined for all $s \leq t$, and yield a persistence module $(W, \tilde{\pi})$.

EXERCISE 1.2.5. Let $\Phi : V \to V'$ be a morphism between two persistence modules (V, π) and (V', π'). We can define the kernel and the image of Φ as follows. The kernel $(\ker \Phi, \pi^{\ker \Phi})$ is the collection of the vector spaces $\{\ker \Phi_t\}_t$ for all $t \in \mathbb{R}$, equipped with the collection of linear maps $\pi_{s,t}\big|_{\ker \Phi_s}$ for all $s \leq t$. Similarly, the image $(\mathrm{im}\, \Phi, \pi^{\mathrm{im}\, \Phi})$ of Φ is the collection $\{\mathrm{im}\, \Phi_t\}_t$ of vector spaces, $t \in \mathbb{R}$, equipped with the collection of linear maps $\pi'_{s,t}\big|_{\mathrm{im}\, \Phi_s}$ for all $s \leq t$ in $\mathbb{R}$. Prove that $\ker \Phi$ and $\mathrm{im}\, \Phi$ are persistence submodules of V and V', respectively.

CONVENTION 1.2.6. We will use the notation $(a, b]$ with $-\infty < a < b \leq +\infty$, meaning either a bounded interval when $b < \infty$, or the ray $(a, +\infty)$, when $b = +\infty$.

EXAMPLE 1.2.7 (Interval modules). For an interval $(a, b]$ (with $b \leq +\infty$), define a persistence module $\mathbb{F}(a, b]$ as follows:

$$\mathbb{F}(a,b]_t = \begin{cases} \mathbb{F}, & \text{if } t \in (a,b], \\ 0, & \text{otherwise,} \end{cases} \qquad \pi_{s,t} = \begin{cases} \mathbb{1}, & \text{if } s,t \in (a,b], \\ 0, & \text{otherwise.} \end{cases}$$

Such persistence modules will be called *interval modules.*

Consider the natural inclusions $\mathbb{F}(1,2] \to \mathbb{F}(1,3]$ and $\mathbb{F}(2,3] \to \mathbb{F}(1,3]$. Are they morphisms? As one can check, the first one is not a morphism, while the second one is. (See Figure 2.)

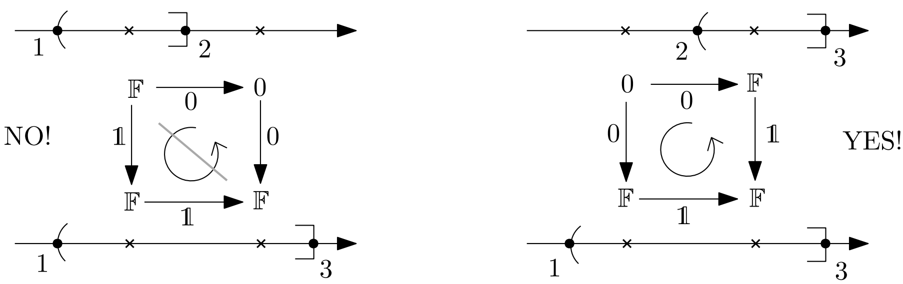

FIGURE 2. Comparison of two situations.

EXERCISE 1.2.8. More generally, check that for two intersecting intervals $(a, b]$ and $(c, d]$, there is a non-zero morphism $\mathbb{F}(a, b] \to \mathbb{F}(c, d]$ if and only if $c \leq a$ and $a < d \leq b$. (Moreover, any morphism between $\mathbb{F}(a, b]$ and $\mathbb{F}(c, d]$ is given by multiplication by some element $\lambda \in \mathbb{F}$.)

DEFINITION 1.2.9. Let (V, π) and (V', π') be two persistence modules. Their direct sum (W, θ) is the persistence module whose underlying vector spaces are $W_t = V_t \oplus V'_t$ (direct sum of vector spaces) and accordingly, $\theta_{s,t} = \pi_{s,t} \oplus \pi'_{s,t}$.

Following the example illustrated in Figure 2, let us note that in general for $a \leq b \leq c$, $\mathbb{F}(a, b] \cong \mathbb{F}(a, c]/\mathbb{F}(b, c]$ (as vector spaces for each t), and we have an exact sequence of persistence modules

$$0 \to \mathbb{F}(b, c] \to \mathbb{F}(a, c] \to \mathbb{F}(a, b] \to 0.$$

However, one can check that $\mathbb{F}(a, c]$ is not isomorphic to $\mathbb{F}(a, b] \oplus \mathbb{F}(b, c]$ (see Figure 3), which is equivalent to the statement that the short exact sequence above does not split. Indeed, denote by $p : \mathbb{F}(a, c] \to \mathbb{F}(a, b]$ the map in the short exact sequence above. Since any morphism $i : \mathbb{F}(a, b] \to \mathbb{F}(a, c]$ is zero, the composition $p \circ i \neq \mathbb{1}_{\mathbb{F}(a,b]}$. We also conclude that $\mathbb{F}(b, c]$ is a submodule of $\mathbb{F}(a, c]$, but it is not isomorphic to a direct summand.

EXAMPLE 1.2.10. Let us give a concrete example of a δ-shift persistence module and a δ-shift morphism. Consider $V = \mathbb{F}(0, 1]$ and $\delta = \frac{1}{3}$. Then $V[\delta] = \mathbb{F}(-\frac{1}{3}, \frac{2}{3}]$, but $\operatorname{im} \Phi^\delta = \mathbb{F}(0, \frac{2}{3}]$. (See Figure 4, and the definition of the image of a morphism in Exercise 1.2.5.). So Φ^δ in fact "chops" V by δ from the right.

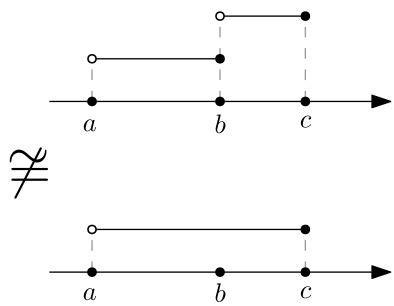

FIGURE 3. $\mathbb{F}(a,b] \oplus \mathbb{F}(b,c] \ncong \mathbb{F}(a,c]$.

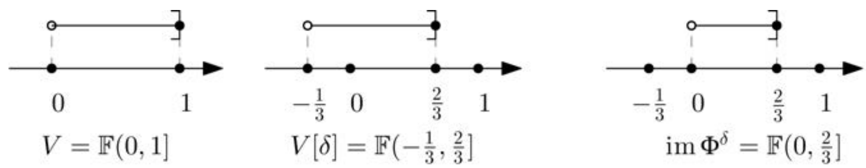

FIGURE 4. Φ^δ "chops" V from the right.

1.3. Interleaving distance

We would like to have a metric, or at least a pseudo-metric, on the space of persistence modules.

DEFINITION 1.3.1. Given a $\delta > 0$, we say that two persistence modules (V, π) and (W, θ) are *δ-interleaved* if there exist two morphisms $F : V \to W[\delta]$ and $G : W \to V[\delta]$, such that the following diagrams commute:

$$V \xrightarrow{F} W[\delta] \xrightarrow{G[\delta]} V[2\delta], \quad \Phi_V^{2\delta} : V \to V[2\delta] \qquad W \xrightarrow{G} V[\delta] \xrightarrow{F[\delta]} W[2\delta], \quad \Phi_W^{2\delta} : W \to W[2\delta]$$

where $\Phi_V^{2\delta}$ and $\Phi_W^{2\delta}$ are the shift morphisms (see Example 1.2.2). We will also refer to such a pair of morphisms F and G as *δ-interleaving morphisms.*

EXERCISE 1.3.2.
(1) Show that two persistence modules (V, π) and (W, θ) are δ-interleaved with finite δ if and only if $\dim V_\infty = \dim W_\infty$ (see definition in Remarks 1.1.2.).
(2) Prove that if V, W are δ-interleaved, then they are δ'-interleaved for any $\delta' > \delta$.
(3) Prove that if V, W are δ_1-interleaved and W, Z are δ_2-interleaved, then V, Z are $(\delta_1 + \delta_2)$-interleaved.

DEFINITION 1.3.3. For two persistence modules (V, π) and (W, θ), define the interleaving distance between them to be

$$d_{\text{int}}(V, W) = \inf \{\delta > 0 \mid (V, \pi) \text{ and } (W, \theta) \text{ are } \delta\text{-interleaved}\}.$$

(For brevity, we use the notation $d_{\text{int}}(V, W)$, writing just V instead of (V, π) and similarly for (W, θ), unless there is a danger of confusion.)

Note that in this way we get a pseudo-metric on isomorphism classes of persistence modules with the same V_∞. A priori, it might happen that $d_{\text{int}}(V, W)$ vanishes for non-isomorphic V and W.

However, because of the semicontinuity condition we impose on persistence modules, we will be able to show that d_{int} is a genuine metric, i.e., that it is non-degenerate (see Theorem 2.2.8 and Exercise 2.2.10).

1.3.1. First example: interval modules.

CLAIM 1.3.4. Fix $a, b, c, d < \infty$, with $a < b$, $c < d$, and consider $d_{\text{int}}(\mathbb{F}(a, b], \mathbb{F}(c, d])$, between the persistence modules $\mathbb{F}(a, b]$ and $\mathbb{F}(c, d]$ defined as in Example 1.2.7. Then

$$(1) \qquad d_{\text{int}}\big(\mathbb{F}(a,b], \mathbb{F}(c,d]\big) \leq \min\left(\max\Big(\frac{b-a}{2}, \frac{d-c}{2}\Big), \max\big(|a-c|, |b-d|\big)\right).$$

We will see later that in fact equality holds. For now, let us prove this inequality by exploring two strategies of interleaving $\mathbb{F}(a, b]$ and $\mathbb{F}(c, d]$.

I. Take $\delta = \max(|a - c|, |b - d|)$. We want to show that $\mathbb{F}(a, b]$ and $\mathbb{F}(c, d]$ are δ-interleaved. By definition, $a - 2\delta \leq c - \delta \leq a$ and $b - 2\delta \leq d - \delta \leq b$. In view of Exercise 1.2.8, one can take the morphisms $F : \mathbb{F}(a, b] \to \mathbb{F}(c - \delta, d - \delta]$ and $G : \mathbb{F}(c, d] \to \mathbb{F}(a - \delta, b - \delta]$. They may be zero, e.g., if $d - \delta < a$, then $F = 0$ (see Figure 5).

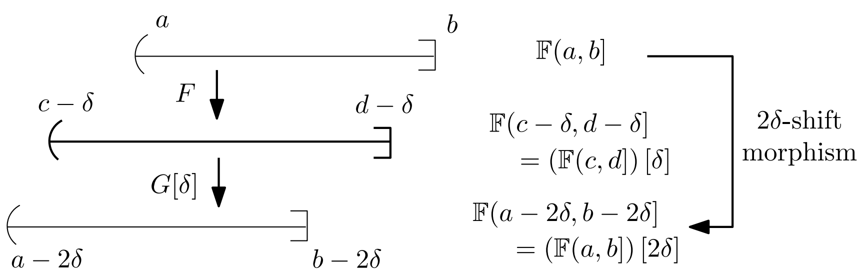

FIGURE 5. First method of interleaving $\mathbb{F}(a, b]$ and $\mathbb{F}(c, d]$: by $\delta = \max(|a - c|, |b - d|)$.

II. Put this time $\delta = \max\left(\frac{b-a}{2}, \frac{d-c}{2}\right)$. Note that the shift morphism by 2δ vanishes for both modules, see Figure 6 (e.g., the shift between $\mathbb{F}(a, b]$ and $\mathbb{F}(a-2\delta, b-2\delta]$ vanishes, because $b - 2\delta \leq a$, i.e., $(a, b] \cap (a - 2\delta, b - 2\delta] = \emptyset$). Taking the interleaving morphisms to be 0 concludes the proof.

EXERCISE 1.3.5. For two infinite intervals, $d_{\text{int}}\big(\mathbb{F}(a, \infty), \mathbb{F}(c, \infty)\big) = |a - c|$.

EXAMPLE 1.3.6. In order to get the flavor of this bound, let us list some concrete examples (we write δ_{I} and δ_{II} for δ taken as in the cases I and II of Claim 1.3.4, respectively):

(1) For $\mathbb{F}(1, 2]$ and $\mathbb{F}(1, 3]$, $\delta_{\text{I}} = \delta_{\text{II}} = 1$, so $d_{\text{int}} \leq 1$.
(2) For $\mathbb{F}(1, 2]$ and $\mathbb{F}(2, 3]$, $\delta_{\text{I}} = 1$, $\delta_{\text{II}} = \frac{1}{2}$, so $d_{\text{int}} \leq \frac{1}{2}$.
(3) For $\mathbb{F}(1, 4]$ and $\mathbb{F}(2, 5]$, $\delta_{\text{I}} = 1$, $\delta_{\text{II}} = \frac{3}{2}$, so $d_{\text{int}} \leq 1$.

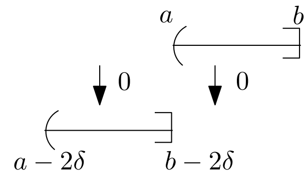

FIGURE 6. Second method of interleaving. The shift morphism vanishes.

As we remarked above regarding the inequality (1) in Claim 1.3.4, these bounds are in fact the exact values of d_{int}.

1.4. Morse persistence modules and approximation

Take a closed manifold M and a Morse function $f : M \to \mathbb{R}$. Put $\|f\| = \max |f|$ (the uniform norm of f).

As before, we define a persistence module $V(f)$ by setting $V_t(f) = H_*(\{f < t\})$. Note that in these notations $V(f - \delta) = V(f)[\delta]$. Also, if $g : M \to \mathbb{R}$ is another Morse function and $f \leq g$, then $\{g < t\} \subset \{f < t\}$, and we get a natural morphism $F : V(g) \to V(f)$.

For any $f, g : M \to \mathbb{R}$ we have $f - \|f - g\| \leq g$. Denote $\delta = \|f - g\|$. By the above considerations, since $\{g < t\} \subseteq \{f - \delta < t\}$, there is a natural morphism $F : V(g) \to V(f)[\delta]$. Similarly, $g - \delta \leq f$, hence we have another morphism $G : V(f) \to V(g)[\delta]$. Combining these two inequalities, we obtain $f - 2\delta \leq g - \delta \leq f$, that is, we actually have three natural morphisms, yielding the following commutative diagram:

$$V(f) \xrightarrow{G} V(g)[\delta] \xrightarrow{F[\delta]} V(f)[2\delta], \qquad \Phi^{2\delta}_{V(f)} : V(f) \to V(f)[2\delta]$$

where $\Phi^{2\delta}_{V(f)}$ stands for the 2δ-shift morphism of $V(f)$. By a symmetric argument, we get the second diagram required by Definition 1.3.1, hence

$$d_{\text{int}}\big(V(f), V(g)\big) \leq \delta = \|f - g\|. \tag{2}$$

Note that for any $\varphi \in \text{Diff}(M)$, the persistence modules $V(f)$ and $V(\varphi^* f)$ are isomorphic, hence

$$d_{\text{int}}\big(V(f), V(g)\big) \leq \inf_{\varphi \in \text{Diff}(M)} \|f - \varphi^* g\|. \tag{3}$$

Let us have a closer look at this inequality by considering a sub-example. Take a Morse function $f : S^2 \to \mathbb{R}$. How well can it be C^0-approximated by a Morse function with exactly two critical points? (See Figure 7.) For such functions illustrated in Figure 7, we shall calculate the lower bound given in (3) in Example 4.2.6 below.

In Chapter 6 we discuss further applications of persistence modules to function theory and to approximation.

1.5. Rips modules and the Gromov-Hausdorff distance

Let X, Y be finite sets. A *surjective correspondence* $C : X \rightrightarrows Y$ between X and Y is a subset $C \subset X \times Y$ such that $\text{proj}_X(C) = X$ and $\text{proj}_Y(C) = Y$. The

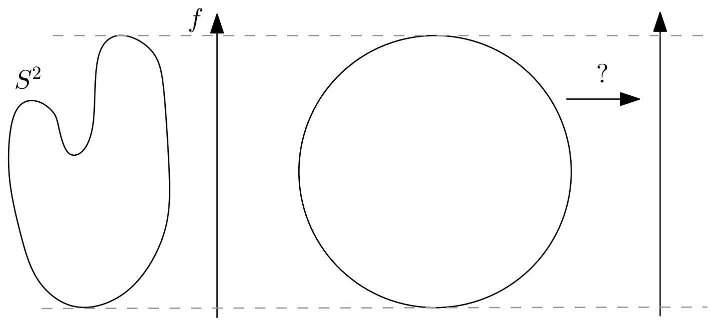

FIGURE 7. Approximation question

inverse correspondence $C^{\mathrm{T}} : Y \rightrightarrows X$ is defined by $C^{\mathrm{T}} = \{(y,x) : \ (x,y) \in C\}$. Note that C is a surjective correspondence if and only if there exist $f : X \to Y$ and $g : Y \to X$, such that $\mathrm{graph}(f) \subset C$ and $\mathrm{graph}(g) \subset C^{\mathrm{T}}$.

DEFINITION 1.5.1. Assume now that (X,ρ), (Y,r) are finite metric spaces. The *distortion* of a surjective correspondence $C : X \rightrightarrows Y$ is

$$\mathrm{dis}(C) = \max_{(x,y),(x',y')\in C} |\rho(x,x') - r(y,y')|. \tag{4}$$

For instance, if we take C to be a graph of a map $f : X \to Y$, then $(x,y) \in C$ means $y = f(x)$, and so

$$\mathrm{dis}(C) = \max_{x,x'\in X} |\rho(x,x') - r(f(x),f(x'))|.$$

In particular, $\mathrm{dis}(C) = 0$ if and only if f is an isometry.

Let us adopt the following notion of distance between metric spaces:

DEFINITION 1.5.2. The Gromov-Hausdorff distance between two finite metric spaces (X,ρ) and (Y,r) is

$$d_{\mathrm{GH}}\big((X,\rho),(Y,r)\big) = \frac{1}{2}\min_C \mathrm{dis}(C),$$

where the min is taken over all the surjective correspondences between X and Y.

EXERCISE 1.5.3. Prove that d_{GH} is a distance between isometry classes of finite metric spaces.

For the finite metric space (X,ρ), consider its Rips complex $R_t(X)$ and accordingly the Rips persistence module $V_t(X) = H_*\big(R_t(X)\big)$. (See Example 1.1.5.)

THEOREM 1.5.4 (See [**21**]).

$$d_{\mathrm{GH}}\big((X,\rho),(Y,r)\big) \geq \frac{1}{2} d_{\mathrm{int}}\big(V(X,\rho),V(Y,r)\big).$$

PROOF. Take a surjective correspondence $C : X \rightrightarrows Y$ and any $\delta > \mathrm{dis}(C)$. We need to show that $V(X)$ and $V(Y)$ are δ-interleaved.

Pick any $f : X \to Y$ with $\mathrm{graph}(f) \subset C$. Note that since $\delta > \mathrm{dis}(C)$, we have $r\big(f(x),f(x')\big) < \rho(x,x') + \delta$, so f induces a simplicial map $F : R_t(X) \to R_{t+\delta}(Y)$. Let $F_* : V_t(X) \to V_{t+\delta}(Y) = \big(V(Y)[\delta]\big)_t$ be the induced map on homology. Similarly, taking $g : Y \to X$ for which $\mathrm{graph}(g) \subset C^{\mathrm{T}}$, we get a map $G : R_t(Y) \to R_{t+\delta}(X)$, which induces a map $G_* : V_t(Y) \to \big(V(X)[\delta]\big)_t$ in homology.

We claim that the maps F_* and G_* are δ-interleaving morphisms. To prove it, we have to show that the following diagram and a similar diagram for the converse composition both commute:

$$\begin{array}{c} V(X) \xrightarrow{F_*} V(Y)[\delta] \xrightarrow{G_*[\delta]} V(X)[2\delta] \\ \underbrace{\qquad\qquad\qquad\qquad\qquad}_{i_*} \end{array}$$

where $i : R_t(X) \to R_{t+2\delta}(X)$ is the natural inclusion.

We recall that two simplicial maps $H, H' : K \to L$ (between simplicial complexes K, L) are called *contiguous* if for any simplex $\sigma \in K$, $H(\sigma) \cup H'(\sigma)$ is a simplex in L. For contiguous maps H and H', one has $H_* = H'_*$ (see [**71**, Theorem 12.5]).

Let us show that $G \circ F$ and i are contiguous as maps $R_t(X) \to R_{t+2\delta}(X)$. Choose any simplex $[x_0, \ldots, x_k] \in R_t(X)$. Note that $i(x_j) = x_j$. We have to check that $[gf(x_0), \ldots, gf(x_k), x_0, \ldots, x_k]$ is a simplex in $R_{t+2\delta}(X)$.

By the definition of the distortion of C, we know that for any $x, x' \in X$, $y, y' \in Y$ that satisfy $(x, y), (x', y') \in C$, we have $|\rho(x, x') - r(y, y')| \le \mathrm{dis}(C) < \delta$. So for all $0 \le i, j \le k$,

$$\rho\big(gf(x_i), x_j\big) < r\big(f(x_i), f(x_j)\big) + \delta < \rho(x_i, x_j) + 2\delta < t + 2\delta.$$

Here the first inequality holds since $\big(gf(x_i), f(x_i)\big), \big(x_j, f(x_j)\big) \in C$ for all i, j, the second one follows, as again $(x_i, f(x_i)), (x_j, f(x_j)) \in C$, and the last one is by the definition of $R_t(X)$. Similarly, we get that

$$\rho\big(gf(x_i), gf(x_j)\big) < r\big(f(x_i), f(x_j)\big) + \delta < t + 2\delta.$$

So $G \circ F$ and i are contiguous, and the result follows. □

See Chapter 5 for further discussion on persistence modules associated to Rips complexes.