CHAPTER 2

Generalized Riemannian Geometry

On the road to understanding Riemannian geometry from the modern point of view, one begins with the notion of a smooth manifold, and derives various objects canonically associated to the smooth structure, such as the tangent and cotangent bundles, and the Lie bracket. Geometric structure enters by assigning a Riemannian metric, which measures the lengths of vectors in the tangent bundle. This way, one can associate an energy to a smooth curve, thought of as the trajectory of a point particle moving along the manifold, which leads to geodesics and Jacobi fields.

Generalized geometry changes this story in a fundamental way. Whereas vectors and the tangent bundle have primacy in Riemannian geometry, the starting point of generalized geometry is to put vectors and covectors on equal footing, and to treat sections of $T \oplus T^*$ as the fundamental object of geometry. This leads to a new bracket structure on such sections, the Dorfman bracket, which involves a delicate combination of the classical operators of differential geometry. This point of view also inspires the definition of a generalized Riemannian metric, which measures sections of $T \oplus T^*$. Such generalized metrics are equivalent to a choice of a classical Riemannian metric g together with a skew-symmetric two-form b. Similar to Riemannian metrics, a generalized metric can be used to define an energy for a surface mapping into the manifold, thought of as the trajectory of a 'string' moving along the target space.

2.1. Courant algebroids

The fundamental object underlying generalized Riemannian geometry is a Courant algebroid. These first arose in the work of Courant [**54**] and Dorfman [**57**], partly in an effort to describe the geometry of equivariant moment map level sets, and was later axiomatized by Liu, Weinstein, and Xu [**129**]. Before giving the general definition of Courant algebroid we begin by describing the main example, with further examples to follow. The linear algebra of $T \oplus T^*$ plays a central role in the subject, and we begin with some basic definitions.

Definition 2.1. Given V a vector space of dimension n, define an inner product on $V \oplus V^*$ via

$$\langle X + \xi, Y + \eta \rangle := \tfrac{1}{2} \left(\xi(Y) + \eta(X) \right). \tag{2.1}$$

The inner product $\langle , \rangle$ is obviously symmetric and has signature (n, n), and thus determines a copy of $O(n, n)$ consisting of endomorphisms preserving this product.

Note that for an n-dimensional vector space V there is a natural decomposition

$$\Lambda^{2n}(V \oplus V^*) = \Lambda^n V \otimes \Lambda^n V^*.$$

As there is the natural determinant pairing between $\Lambda^n V$ and $\Lambda^n V^*$, this induces a canonical identification of $\Lambda^{2n}(V \oplus V^*)$ with $\mathbb{R}$, and hence the positive real numbers correspond to a canonical orientation. The Lie group preserving both the symmetric product and this orientation will be isomorphic to $SO(n,n)$.

DEFINITION 2.2. Given M a smooth manifold, we denote $T \oplus T^* := TM \oplus T^*M$. The *Dorfman bracket* on sections of $T \oplus T^*$ is defined by

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - i_Y d\xi. \tag{2.2}$$

Note that this bracket restricts to the Lie bracket when acting on tangent vectors. However, some of the fundamental properties of the Lie bracket do not extend to the Dorfman bracket. We first observe that the familiar Leibniz rule still holds for the first-order differential operator $[X+\xi, \cdot]$ acting on sections of $T \oplus T^*$. Before stating this we define the natural projection

$$\pi\colon\ T \oplus T^* \to T, \qquad X+\xi \mapsto X,$$

which will be called the *anchor map.*

LEMMA 2.3. *Given $a, b \in \Gamma(T \oplus T^*)$ and $f \in C^\infty(M)$, the Dorfman bracket satisfies*

$$[a, fb] = f[a,b] + \pi(a) fb.$$

PROOF. Setting $a = X+\xi, b = Y+\eta$, we directly compute using the properties of Lie derivatives,

$$\begin{aligned}
[X+\xi, f(Y+\eta)] &= [X, fY] + L_X f\eta - i_{fY} d\xi \\
&= f[X+\xi, Y+\eta] + (Xf)Y + (Xf)\eta \\
&= f[X+\xi, Y+\eta] + (Xf)(Y+\eta),
\end{aligned}$$

as required. □

The differential operator $[X+\xi, \cdot]$ is furthermore compatible with the inner product $\langle , \rangle$, in the following sense:

LEMMA 2.4. *Given $a, b, c \in \Gamma(T \oplus T^*)$, the Dorfman bracket satisfies*

$$\pi(a)\langle b, c\rangle = \langle [a,b], c\rangle + \langle b, [a,c]\rangle .$$

PROOF. Setting $a = X+\xi, b = Y+\eta, b = Z+\zeta$ and using the identity $i_{[X,Y]} = [L_X, i_Y]$ we compute

$$\begin{aligned}
\langle [a,b], c\rangle + \langle b, [a,c]\rangle &= \tfrac{1}{2}(i_{[X,Y]}\zeta + i_Z(L_X\eta - i_Y d\xi) + i_{[X,Z]}\eta + i_Y(L_X\zeta - i_Z d\xi)) \\
&= \tfrac{1}{2}(L_X i_Y \zeta + L_X i_Z \eta) \\
&= \tfrac{1}{2} i_X d(\zeta(Y) + \eta(Z)),
\end{aligned}$$

as required. □

The main structural property of the Dorfman bracket is the Jacobi identity:

LEMMA 2.5. *Given $a, b, c \in \Gamma(T \oplus T^*)$ one has*

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

PROOF. Setting $a = X + \xi,\ b = Y + \eta,\ c = Z + \zeta$, one has, using the identities $i_{[X,Y]} = [L_X, i_Y]$ and $L_{[X,Y]} = L_X L_Y - L_Y L_X$,

$$\begin{aligned}
&[[a,b],c] + [b,[a,c]] \\
&\quad = [[X,Y] + L_X\eta - i_Y d\xi, Z + \zeta] + [Y + \eta, [X,Z] + L_X\zeta - i_Z d\xi] \\
&\quad = [[X,Y],Z] + L_{[X,Y]}\zeta - i_Z d(L_X\eta - i_Y d\xi) \\
&\qquad + [Y,[X,Z]] + L_Y(L_X\zeta - i_Z d\xi) - i_{[X,Z]} d\eta \\
&\quad = [X,[Y,Z]] + L_X L_Y\zeta - L_X i_Z d\eta - L_Y i_Z d\xi + i_Z d i_Y d\xi \\
&\quad = [X,[Y,Z]] + L_X(L_Y\zeta - i_Z d\eta) - i_{[Y,Z]} d\xi \\
&\quad = [a,[b,c]],
\end{aligned}$$

as claimed. □

Despite these favorable properties, unlike the Lie bracket of vector fields, the Dorfman bracket is not skew-symmetric. In fact, one can easily show that

$$[a,b] + [b,a] = 2d\,\langle a,b\rangle\,. \tag{2.3}$$

The skew-symmetrization of the Dorfman bracket, called the Courant bracket, will be useful for some calculations.

DEFINITION 2.6. The *Courant bracket* on sections of $T \oplus T^*$ is defined by

$$[a,b]_c = \tfrac{1}{2}([a,b] - [b,a]).$$

Notice that

$$[a,b]_c = [a,b] - d\,\langle a,b\rangle\,. \tag{2.4}$$

Furthermore, a straightforward calculation leads to the following explicit formula:

$$[X+\xi, Y+\eta]_c = [X,Y] + L_X\eta - L_Y\xi + \tfrac{1}{2}d\,(\xi(Y) - \eta(X))\,.$$

We summarize the main properties of the Courant bracket in the following lemma.

LEMMA 2.7. *The following hold:*

(1) *Given $a, b \in \Gamma(T \oplus T^*)$ and $f \in C^\infty(M)$, the Courant bracket satisfies*

$$[a, fb]_c = f[a,b]_c + \pi(a)fb - \langle a,b\rangle\,df.$$

(2) *Given $a,b,c \in \Gamma(T \oplus T^*)$, one has*

$$\begin{aligned}
&[a,[b,c]_c]_c + [c,[a,b]_c]_c + [b,[c,a]_c]_c \\
&\qquad = \tfrac{1}{3}d\,(\langle [a,b]_c, c\rangle + \langle [b,c]_c, a\rangle + \langle [c,a]_c, b\rangle)
\end{aligned}$$

PROOF. We set $a = X + \xi,\ b = Y + \eta,\ c = Z + \zeta$. To prove (1), we directly compute as in Lemma 2.3

$$\begin{aligned}
&[X+\xi, f(Y+\eta)]_c \\
&\quad = [X, fY] + L_X f\eta - L_{fY}\xi + \frac{1}{2}d\,(f\xi(Y) - f\eta(X)) \\
&\quad = f[X+\xi, Y+\eta]_c + (Xf)Y + (Xf)\eta - \xi(Y)df + \tfrac{1}{2}\,(\xi(Y) - \eta(X))\,df \\
&\quad = f[X+\xi, Y+\eta]_c + (Xf)(Y+\eta) - \tfrac{1}{2}\,(\xi(Y) + \eta(X))\,df \\
&\quad = f[X+\xi, Y+\eta]_c + (Xf)(Y+\eta) - \langle X+\xi, Y+\eta\rangle\,df,
\end{aligned}$$

as required.

As for (2), using Lemma 2.3 and the fact that $[\xi, c] = 0$ for any closed one-form ξ, we obtain

$$\begin{aligned} [[a,b]_c, c]_c &= [[a,b]_c, c] - d\langle [a,b]_c, c\rangle \\ &= [([a,b] - d\langle a,b\rangle), c] - d\langle [a,b]_c, c\rangle \\ &= [[a,b],c] - d\langle [a,b]_c, c\rangle . \end{aligned}$$

Using these properties we compute

$$\begin{aligned} &[[a,b]_c, c]_c + [[c,a]_c, b]_c + [[b,c]_c, a]_c \\ &\quad = \tfrac{1}{4}\left([[a,b],c] - [c,[a,b]] - [[b,a],c] + [c,[b,a]] + \text{cyclic}\right) \\ &\quad = \tfrac{1}{4}\left([a,[b,c]] - [b,[a,c]] - [c,[a,b]]\right. \\ &\qquad\qquad \left. -[b,[a,c]] + [a,[b,c]] + [c,[b,a]] + \text{cyclic}\right) \\ &\quad = \tfrac{1}{4}\left([a,[b,c]] - [b,[a,c]] + \text{cyclic}\right) \\ &\quad = \tfrac{1}{4}\left([[a,b],c] + \text{cyclic}\right) \\ &\quad = \tfrac{1}{4}\left([[a,b]]_c c]_c + d\langle [a,b]_c, c\rangle + \text{cyclic}\right) \\ &\quad = \tfrac{1}{4}\left([[a,b]_c, c]_c + [[c,a]_c, b]_c + [[b,c]_c, a]_c\right. \\ &\qquad\qquad \left. +d\left(\langle [a,b]_c, c\rangle + \langle [b,c]_c, a\rangle + \langle [c,a]_c, b\rangle\right)\right), \end{aligned}$$

from which the claim follows. □

The structure $(T \oplus T^*, \langle,\rangle, [,], \pi)$ was formalized into a general definition, that of a Courant algebroid, first given in [**129**].

Definition 2.8. A *Courant algebroid* [1] is a vector bundle $E \to M$ together with a nondegenerate symmetric bilinear form $\langle,\rangle$ a bracket $[,]$ on $\Gamma(E)$, and a bundle map $\pi : E \to TM$ such that, given $a, b, c \in \Gamma(E)$ and $f \in C^\infty(M)$, one has

(1) $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$
(2) $\pi[a,b] = [\pi a, \pi b]$
(3) $[a, fb] = f[a,b] + \pi(a) fb$
(4) $\pi(a)\langle b,c\rangle = \langle [a,b], c\rangle + \langle b, [a,c]\rangle$
(5) $[a,b] + [b,a] = \mathcal{D}\langle a,b\rangle$,

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ denotes the composition of three maps: the exterior differential d acting on functions, the natural map $\pi^* : T^* \to E^*$, and the isomorphism $E^* \to E$ provided by the symmetric product.

In the sequel, we will abuse notation and denote by π^* the composition of $\pi^* : T^* \to E^*$ with the natural isomorphism $E^* \to E$ provided by the symmetric product. The fact that our explicit structure $(T \oplus T^*, \langle,\rangle, [,], \pi)$ fulfills the conditions of a Courant algebroid follows from Lemma 2.3, Lemma 2.4, Lemma 2.5, equation (2.3), and the definition of π. Note that, in this explicit situation, we have that

$$\mathcal{D}f = 2df.$$

We leave as an **exercise** to check from the previous axioms that the image of π^* is isotropic. To finish this section, we unravel Definition 2.8 for the particular class of Courant algebroids which is of primary interest for the present book.

[1] We retain the term Courant algebroid despite the fact that all brackets appearing in this definition are Dorfman brackets, not Courant brackets.

DEFINITION 2.9. A Courant algebroid is *exact* if it fits into an exact sequence of vector bundles

$$0 \longrightarrow T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \longrightarrow 0.$$

For exact Courant algebroids notice that, by definition, the image of π^* coincides with the kernel of π. Further consider an isotropic splitting $\sigma\colon T \to E$ of π. Such a splitting can be constructed by choosing an arbitrary splitting $\sigma_0\colon T \to E$, and setting

$$\sigma = \sigma_0 - \tfrac{1}{2}\pi^*\tau, \tag{2.5}$$

where $\tau \in \mathrm{Sym}^2 T^*$ is defined by $\tau(X,Y) = \langle \sigma_0 X, \sigma_0 Y\rangle$. Using these structures we can give a classification result for exact Courant algebroids, incorporating the first appearance of a closed three-form H, which features prominently in generalized geometry.

PROPOSITION 2.10. *Given an exact Courant algebroid E with isotropic splitting σ, the map $F\colon T \oplus T^* \to E$ defined by*

$$F(X+\xi) = \sigma X + \tfrac{1}{2}\pi^*\xi$$

is an isomorphism of orthogonal bundles, for the symmetric product (2.1)*. Via this isomorphism the anchor map is given by $\pi(X+\xi) = X$ and the Dorfman bracket is*

$$[X+\xi, Y+\eta]_H := [X+\xi, Y+\eta] + i_Y i_X H, \tag{2.6}$$

where $H \in \Lambda^3 T^*$ is a closed three-form defined by

$$H(X,Y,Z) = 2\left\langle [\sigma X, \sigma Y], \sigma Z\right\rangle. \tag{2.7}$$

PROOF. We first note that

$$\left\langle \sigma X + \tfrac{1}{2}\pi^*\xi, \sigma Y + \tfrac{1}{2}\pi^*\eta\right\rangle = \tfrac{1}{2}(\xi(\pi\sigma Y) + \eta(\pi\sigma X)) = \tfrac{1}{2}(\xi(Y) + \eta(X)),$$

noting that $\langle \sigma X, \sigma Y\rangle = 0$, which proves that F is an isometry. Next we transport the Courant algebroid structure from E to $T \oplus T^*$ using F. It follows from the definition of F that the induced anchor map on $T \oplus T^*$ is $\pi_E(X+\xi) := \pi F(X+\xi) = X$ and that $\mathcal{D}_E = \pi^* d = 2d$. The induced bracket, denoted $[,]_E$, is defined via

$$\begin{aligned} F[X+\xi, Y+\eta]_E &:= [F(X+\xi), F(Y+\eta)] \\ &= [\sigma X, \sigma Y] + \tfrac{1}{2}[\sigma X, \pi^*\eta] + \tfrac{1}{2}[\pi^*\xi, \sigma Y] + \tfrac{1}{4}[\pi^*\xi, \pi^*\eta]. \end{aligned} \tag{2.8}$$

It is an **exercise** using the axioms of a Courant algebroid to show that $[\pi^*\xi, \pi^*\eta] = 0$ for any ξ, η. For the second term on the right hand side we first observe that

$$\pi[\sigma X, \pi^*\eta] = [\pi\sigma X, \pi\pi^*\eta] = [\pi\sigma X, 0] = 0.$$

Thus $[X,\eta]_E \in T^*$, and we can compute, using axiom (5),

$$\begin{aligned} \langle [\sigma X, \pi^*\eta], \sigma Z\rangle =\ & \langle \sigma X, \mathcal{D}\langle \pi^*\eta, \sigma Z\rangle\rangle - \langle \pi^*\eta, [\sigma X, \sigma Z]\rangle \\ =\ & X\langle \pi^*\eta, Z\rangle - \eta(\pi[\sigma X, \sigma Z]) \\ =\ & X\eta(Z) - \eta([X,Z]) \\ =\ & d\eta(X,Z) + Z\eta(X). \end{aligned}$$

We conclude that

$$[X,\eta]_E = L_X\eta.$$

The third term of (2.8) is similar, where using axiom (5) we conclude

$$[\xi, Y]_E = -[Y, \xi]_E + 2d\langle \xi, Y \rangle = -L_Y \xi + d\xi(Y) = -i_Y d\xi.$$

Finally, we decompose the first term on the right hand side of (2.8) into tangent and cotangent pieces with respect to F. From axiom (2) it follows that $\pi[\sigma X, \sigma Y] = [X, Y]$. We define a tensor $H \in (T^*)^{\otimes 3}$ via

$$H(X, Y, Z) = 2\langle [\sigma X, \sigma Y], \sigma Z \rangle ,$$

and then it follows that

$$[\sigma X, \sigma Y] = \sigma[X, Y] + \tfrac{1}{2}\pi^* H(X, Y)$$

It is an **exercise** to show that H is indeed a tensor, and moreover is totally skew-symmetric. Putting the above observations back into (2.8) shows that

$$[X + \xi, Y + \eta]_E = [X, Y] + L_X \eta - i_Y d\xi + i_Y i_X H,$$

as claimed. The Jacobi identity of axiom (1) for $[X + \xi, Y + \eta]_E$ ensures that $dH = 0$. □

2.2. Symmetries of the Dorfman bracket

A classic fact in smooth manifold theory is that the only automorphisms of the tangent bundle preserving the Lie bracket structure are given by the tangent maps associated to diffeomorphisms of the underlying manifold. In this section we prove an analogous result for $T \oplus T^*$ equipped with the symmetric product (2.1) and the Dorfman bracket, and use this to obtain Ševera's classification of exact Courant algebroids [**149**]. In particular the symmetry group of the Dorfman bracket is enlarged to include so-called B-field transformations. These extra symmetries play a central role throughout generalized geometry. With the group of *generalized diffeomorphisms* at hand, we provide a different interpretation of this group by means of a two-dimensional variational problem. This leads to a first-principles derivation of the Dorfman bracket and a conceptual explanation for some of its most basic features collected in Definition 2.8.

Recall that a bundle automorphism of $T \oplus T^*$ is given by a pair (f, F), where $f \in \mathrm{Diff}(M)$ is diffeomorphism of M and $F \colon T \oplus T^* \to T \oplus T^*$ is a vector bundle automorphism, which fit into a commutative diagram

$$\begin{array}{ccc} T \oplus T^* & \xrightarrow{F} & T \oplus T^* \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

Here, the vertical arrows denote the bundle projections.

Definition 2.11. Given M a smooth manifold, a bundle automorphism (f, F) of $T \oplus T^*$ is said to be a *Courant automorphism* if F is orthogonal and the natural action of (f, F) on sections of $T \oplus T^*$ preserves the Dorfman bracket.

Given a diffeomorphism f of M, we can define an orthogonal bundle automorphism via $(f, \overline{f})$, where

$$\overline{f} := \begin{pmatrix} f_* & 0 \\ 0 & (f^*)^{-1} \end{pmatrix}. \tag{2.9}$$

It follows from naturality of the various operations in formula (2.2) that $\overline{f}$ preserves the Dorfman bracket. We next discuss a natural class of Courant automorphisms, given by (closed) B-field transformations, which are not associated to diffeomorphisms of the manifold.

DEFINITION 2.12. Given M a smooth manifold and $B \in \Lambda^2 T^*$, define the associated *B-field transformation* via

$$e^B(X+\xi) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = X + \xi + i_X B.$$

LEMMA 2.13. *Given M a smooth manifold and $B \in \Lambda^2 T^*$, the map e^B is orthogonal with respect to $\langle , \rangle$.*

PROOF. Given $X+\xi, Y+\eta \in T \oplus T^*$ we directly compute

$$\begin{aligned} \left\langle e^B(X+\xi), e^B(Y+\eta) \right\rangle &= \left\langle X+\xi+i_X B, Y+\eta+i_Y B \right\rangle \\ &= \tfrac{1}{2}\left(\eta(X) + i_X i_Y B + \xi(Y) + i_Y i_X B\right) \\ &= \tfrac{1}{2}\left(\eta(X) + \xi(Y)\right) \\ &= \left\langle X+\xi, Y+\eta \right\rangle. \end{aligned}$$

□

PROPOSITION 2.14. *Given M a smooth manifold and $B \in \Lambda^2 T^*$, we have*

$$[e^B(X+\xi), e^B(Y+\eta)] = e^B[X+\xi, Y+\eta] + i_Y i_X dB, \tag{2.10}$$

for any $X+\xi, Y+\eta \in \Gamma(T \oplus T^*)$. Consequently, the map e^B is an automorphism of the Dorfman bracket if and only if $dB = 0$.

PROOF. Fix $X+\xi, Y+\eta \in \Gamma(T \oplus T^*)$ and $B \in \Gamma(\Lambda^2 T^*)$, and then compute using the Cartan formula

$$\begin{aligned} [e^B(X+\xi), e^B(Y+\eta)] &= [X+\xi+i_X B, Y+\eta+i_Y B] \\ &= [X+\xi, Y+\eta] + [X, i_Y B] + [i_X B, Y] \\ &= [X+\xi, Y+\eta] + L_X i_Y B - i_Y d(i_X B) \\ &= [X+\xi, Y+\eta] + L_X i_Y B - i_Y L_X B + i_Y i_X dB \\ &= [X+\xi, Y+\eta] + i_{[X,Y]} B + i_Y i_X dB \\ &= e^B[X+\xi, Y+\eta] + i_Y i_X dB. \end{aligned}$$

The result follows. □

Having exhibited that closed B-field transformations are Courant automorphisms of the form (Id, e^B) in Lemma 2.13 and Proposition 2.14, we now show that these, together with the differentials of diffeomorphisms $(f, \overline{f})$, give all possible Courant automorphisms.

PROPOSITION 2.15. *Let M be a smooth manifold and suppose (f, F) is a Courant automorphism. Then $F = \overline{f} \circ e^B$, that is, F can be expressed as a composition of a diffeomorphism and a closed B-field transformation.*

PROOF. Since the given f is a diffeomorphism, we can define a bundle automorphism via $(f, \overline{f})$, as in (2.9). Setting $\Phi = \overline{f}^{-1} \circ F$, we obtain that (Id, Φ) is also a bundle automorphism. Fixing sections $a, b \in \Gamma(T \oplus T^*)$ and a smooth function p, we obtain, using Lemma 2.3, that

$$\begin{aligned}\Phi([pa, b]) &= \Phi\left(p[a, b] - ((\pi b)p)\, a\right)\\ &= p\Phi([a, b]) - ((\pi b)p)\,\Phi(a).\end{aligned}$$

On the other hand, again using Lemma 2.3, one has

$$[\Phi(pa), \Phi(b)] = p[\Phi(a), \Phi(b)] - (\pi\Phi(b)p)\Phi(a).$$

As Φ preserves the Courant bracket, the two quantities above are equal, and hence $\pi \circ \Phi = \pi$ since a, b and p are arbitrary. On the other hand, the equality

$$2\left\langle b, \Phi^{-1} dp\right\rangle = 2\left\langle \Phi(b), dp\right\rangle = \pi\Phi(b)p = \pi b p = 2\left\langle b, dp\right\rangle$$

implies that Φ^{-1} (and hence Φ) acts as the identity on T^*. Together these facts imply that

$$\Phi = \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix},$$

where $B \in (T^*)^{\otimes 2}$. Since Φ is also orthogonal, it follows that $B \in \Lambda^2 T^*$. Lastly, since Φ preserves the Dorfman bracket, Proposition 2.14 implies that B is closed. Thus $F = \overline{f} \circ e^B$, as claimed. □

Notice that the proof of Proposition 2.15 implies that a Courant automorphism (f, F) is automatically compatible with the anchor map, in the sense that

$$\pi \circ F = \overline{f} \circ \pi.$$

Proposition 2.10 and Proposition 2.14 make clear an elementary way to construct new Courant algebroids, by 'twisting' the standard Dorfman bracket by a three-form H. This is key to Ševera's classification of exact Courant algebroids in Theorem 2.21 below.

Definition 2.16. Given M a smooth manifold and $H \in \Lambda^3 T^*$, we define the H-twisted Dorfman bracket on sections of $T \oplus T^*$ via

$$[X + \xi, Y + \eta]_H := [X + \xi, Y + \eta] + i_Y i_X H.$$

Proposition 2.17. *Given M a smooth manifold and $H \in \Lambda^3 T^*$ such that $dH = 0$, the triple $(T \oplus T^*, \langle , \rangle, [,]_H, \pi)$ defines a Courant algebroid.*

Proof. Most of the axioms in the definition of Courant algebroid are routine to check. The main difficulty is to verify axiom (1). Since H is closed and the computation is local, we can assume $H = dB$. We note that we can rephrase equation (2.10) as

$$\begin{aligned}e^{-B}[e^B(X + \xi), e^B(Y + \eta)] &= [X + \xi, Y + \eta] + e^{-B} i_Y i_X dB\\ &= [X + \xi, Y + \eta] + i_Y i_X dB\\ &= [X + \xi, Y + \eta]_H.\end{aligned}$$

Using this, Lemma 2.5, and Lemma 2.13 we compute for any $a, b, c \in \Gamma(T \oplus T^*)$

$$\begin{aligned}&[[a, b]_H, c]_H + [[c, a]_H, b]_H + [[b, c]_H, a]_H\\ &\quad = e^{-B}[e^B[a, b]_H, e^B c] + e^{-B}[e^B[c, a]_H, e^B b] + e^{-B}[e^B[b, c]_H, e^B a]\\ &\quad = e^{-B}[[e^B a, e^B b], e^B c] + e^{-B}[[e^B c, e^B a], e^B b] + e^{-B}[[e^B b, e^B c], e^B a] = 0.\end{aligned}$$

The proposition follows. □

Our next goal is to provide a classification of exact Courant algebroids due to Ševera [**149**], which follows from Proposition 2.10 and Proposition 2.17. We first give the following abstract definition.

Definition 2.18. Given M a smooth manifold and E and E' Courant algebroids over M, an *isomorphism* between E and E' is a pair (f, F), where f is a diffeomorphism of M and $F\colon E \to E'$ is an orthogonal Courant automorphism, which fit into a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

We say that (f, F) is *small* if f lies in the identity component of $\mathrm{Diff}(M)$. We say that E and E' are in the same *small isomorphism class* if E and E' are isomorphic via a small isomorphism.

Theorem 2.19. *Given M a smooth manifold, the small isomorphism classes of exact Courant algebroids on M are in one-to-one correspondence with the cohomology group $H^3(M, \mathbb{R})$.*

Proof. Let E be an exact Courant algebroid over M. By Proposition 2.10 we can choose an isotropic splitting σ of E, such that E is isomorphic via a small isomorphism to $(T \oplus T^*, \langle , \rangle, [,]_{H_\sigma}, \pi)$, for H_σ the closed three-form on M given by (2.7). To the small isomorphism class of E we want to associate the *Ševera class*

$$[H_\sigma] \in H^3(M, \mathbb{R}).$$

Let us check that this is well-defined. First, if σ' is a different choice of isotropic splitting, via the orthogonal automorphism $E \cong T \oplus T^*$ induced by σ we can identify

$$\sigma'(X) = e^B(X)$$

for some $B \in \Lambda^2 T^*$. Therefore, from (2.10)

$$H_{\sigma'} = H_\sigma + dB,$$

and the class $[H_\sigma]$ is independent of the choice of isotropic splitting.

Let E' be another exact Courant algebroid over M, that we can identify with $(T \oplus T^*, \langle , \rangle, [,]_{H'}, \pi)$ for a closed three-form H' on M. If E' is isomorphic to E there exists $(f, F)\colon E \to E'$ which exchanges the brackets on E and E'. Without loss of generality, we identify E with $(T \oplus T^*, \langle , \rangle, [,]_H, \pi)$, where we set $H = H_\sigma$. We denote $\Phi = \overline{f}^{-1} \circ F$, and arguing as in the proof of Proposition 2.15 we have that $\pi \circ \Phi = \pi$ and that Φ acts as the identity on T^*. Thus, $\Phi = (\mathrm{Id}, e^B)$ for some $B \in \Lambda^2 T^*$. Since (f, F) exchanges the brackets we obtain

$$e^B [a, b]_H = \overline{f}^{-1} [\overline{f} e^B a, \overline{f} e^B b]_{H'}.$$

Combined with

$$\overline{f}^{-1} [\overline{f} a, \overline{f} b]_{H'} = [a, b]_{f^* H'}$$

for any $a, b \in \Gamma(T \oplus T^*)$, we are led to

$$f^* H' = H - dB. \tag{2.11}$$

If E' is in the same small isomorphism class as E, then f is in the identity component of $\mathrm{Diff}(M)$ and by definition there exists a one-parameter group of diffeomorphisms f_t generated by $X \in \Gamma(T)$ such that $f = f_1$. Thus, using that H is closed

$$f^*H - H = \int_0^1 \frac{d}{dt} f_t^* H dt = \int_0^1 f_t^*(di_X H) dt = db,$$

where $b = \int_0^1 \frac{d}{dt} f_t^*(i_X H) dt \in \Lambda^2 T^*$, and we conclude that

$$H' = H - dB',$$

for $B' = f_*(B + b)$, which implies $[H'] = [H]$.

By Proposition 2.17 the map that associates to any small isomorphism class of exact Courant algebroids its Ševera class is surjective. Finally, if E and E' yield the same cohomology class, taking isotropic splittings as before we obtain $H = H' + dB$ and therefore by Proposition 2.14 we have that E and E' are isomorphic via the B-field transformation e^B, and consequently their small isomorphism classes coincide. □

As an immediate consequence of the proof of the previous theorem we obtain the following complete classification of exact Courant algebroids on a smooth manifold.

Corollary 2.20. *Given M a smooth manifold, denote by $\mathrm{Diff}_0(M)$ the component of the identity in $\mathrm{Diff}(M)$ and set $\Gamma_M = \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ the corresponding mapping class group. Then, the isomorphism classes of exact Courant algebroids on M are in one-to-one correspondence with the quotient $H^3(M,\mathbb{R})/\Gamma_M$.*

Proof. By the proof of Theorem 2.19, the isomorphism classes of exact Courant algebroids on M are in one-to-one correspondence with $H^3(M,\mathbb{R})/\mathrm{Diff}(M)$. Since the action of $\mathrm{Diff}_0(M)$ on $H^3(M,\mathbb{R})$ is trivial, the proof follows. □

Another direct consequence of the proof of Theorem 2.19 is an explicit characterization of the group of automorphisms $(f,F)\colon E \to E$ of an exact Courant algebroid E over a smooth manifold M. By a choice of an isotropic splitting of E, we can identify E with the twisted Courant algebroid $(T \oplus T^*, \langle,\rangle, [,]_H, \pi)$ in Proposition 2.17. To state the result, we abuse notation and denote a pair (f,F) simply by F.

Proposition 2.21. *Given M a smooth manifold and $H \in \Lambda^3 T^*$ a closed three-form, the group of automorphisms $F\colon E \to E$ of the twisted Courant algebroid $(T \oplus T^*, \langle,\rangle, [,]_H, \pi)$ is given by*

$$\mathrm{Aut}(E) = \{\overline{f} \circ e^B : f \in \mathrm{Diff}(M), B \in \Lambda^2 T^* \text{ such that } f^*H = H - dB\},$$

together with the product given, for $F = \overline{f} \circ e^B$ and $F' = \overline{f'} \circ e^{B'}$, by

$$F \circ F' = \overline{f \circ f'} \circ e^{B' + f'^* B}.$$

Proof. The characterization of the group follows from (2.11) and the proof of Theorem 2.19. Following the notation in the statement, the composition law can be obtained from

$$H = f_*(f'_*(H - dB') - dB).$$

□

Using the previous result, we can also give an explicit description of the Lie algebra of infinitesimal automorphisms of an exact Courant algebroid E.

COROLLARY 2.22. *Given M a smooth manifold and $H \in \Lambda^3 T^*$ a closed three-form, the Lie algebra of the group of automorphisms of the twisted Courant algebroid $(T \oplus T^*, \langle , \rangle, [,]_H, \pi)$ is given by*

$$\operatorname{Lie}\operatorname{Aut}(E) = \{X + B \mid X \in T,\ B \in \Lambda^2 T^*,\ L_X H = -dB\},$$

together with the Lie bracket given by

$$[X + B, X' + B'] = [X, X'] + L_X B' - L_{X'} B. \tag{2.12}$$

PROOF. Let $F_t = \overline{f}_t \circ e^{B_t}$ be a one-parameter family in $\operatorname{Aut}(E)$ with $F_0 = \operatorname{Id}_E$. Taking derivatives in $f_t^* H = H - dB_t$ at $t = 0$, it follows that

$$X := \left(\frac{\partial}{\partial t} f_t\right)_{|t=0}, \qquad B = \left(\frac{\partial}{\partial t} B_t\right)_{|t=0}$$

satisfies $L_X H = -dB$. Conversely, given $X + B \in T \oplus \Lambda^2 T^*$ satisfying $L_X H = -dB$, we define

$$F_t = \overline{f}_t \circ e^{\overline{B}_t} \in \operatorname{Aut}(E)$$

where f_t is the one-parameter family of diffeomorphisms generated by X and

$$\overline{B}_t = \int_0^t f_s^* B_s ds.$$

Notice that

$$d\overline{B}_t = \int_0^t f_s^* dB_s ds = -\int_0^t f_s^* L_{X_s} H ds = H - f_t^* H$$

and hence F_t is well-defined.

To calculate the Lie bracket, by Proposition 2.21 we have

$$F^{-1} \circ F_t \circ F = \overline{f^{-1} \circ f_t \circ f} \circ e^{B + f_* B_t - (f^{-1} f_t f)^* B},$$

and hence the adjoint action is

$$F^{-1}(X' + B') = f^* X' + f^* B' - L_{f^* X'} B.$$

Setting $F = F_t$ and taking derivatives in this expression we obtain (2.12). Observe that we use the convention of right invariant vector fields for the Lie bracket. □

We next provide a different view on the group $\operatorname{Aut}(E)$ in Proposition 2.21. As we will see, this is an infinite-dimensional Lie group which arises naturally from a 2-dimensional variational problem [**149**]. Here, we will avoid entering into any details about the theory of infinite-dimensional manifolds and Lie groups, but rather focus on formal aspects of the construction. Let M be a smooth manifold. Let Σ be a smooth compact surface (without boundary, say). To be consistent with our notation, we will denote the tangent and cotangent bundles of M by T and T^*, while those of Σ will be denoted $T\Sigma$ and $T^*\Sigma$. Consider the infinite-dimensional space of smooth maps from Σ to M, denoted $C^\infty(\Sigma, M)$. Given a closed three-form $H \in \Lambda^3 T^*$ on M representing an integral cohomology class $[H] \in H^3(M, \mathbb{Z})$, we can define a functional

$$S_H : C^\infty(\Sigma, M) \to \mathbb{R}/\mathbb{Z}, \qquad \varphi \mapsto \int_Y \overline{\varphi}^* H, \tag{2.13}$$

where Y is a three-manifold with boundary Σ and $\overline{\varphi}\colon Y \to M$ is any smooth extension of φ to Y. Since H is assumed to have integral periods over the integer homology $H_3(M, \mathbb{Z})$ of M, this functional is well-defined. Notice also that when H is exact, by Stokes' Theorem the functional reduces to

$$S_{dB}(\varphi) = \int_\Sigma \varphi^* B.$$

This functional is known in the mathematical physics literature as the *Wess-Zumino term* in the action of a two-dimensional Σ-model (see e.g. [**79**]), and lies at the core of the relation between generalized geometry and string theory.

Using that H is closed and applying Stokes' Theorem again, the variation of S_H is given by

$$\delta S_H(X) = \int_Y \overline{\varphi}^* L_{\overline{X}} H = \int_\Sigma \varphi^* i_X H,$$

where $X \in \Gamma(\varphi^* T)$ is identified with a vector in the tangent space of $C^\infty(\Sigma, M)$ at φ, and $\overline{X} \in \Gamma(\overline{\varphi}^* T)$ denotes any extension of X along $\overline{\varphi}$. Observe that the formula for δS_H makes sense even if H does not have integral periods. Solutions of the variational problem, that is, critical points of S_H, are given by maps φ such that for all $X \in \Gamma(\varphi^* T)$ there exists a one-form $\xi \in \Gamma(T^*\Sigma)$ such that

$$\varphi^*(i_X H) = d\xi.$$

In particular, if $X + \xi \in \Gamma(T \oplus T^*)$ is a global section satisfying

$$i_X H = d\xi, \tag{2.14}$$

by pulling-back to Σ via φ we obtain tangent directions on the space of smooth maps along which S_H is constant.

A natural question is whether these 'flat directions' of S_H arise from some symmetries of the functional. As is customary in variational problems, we will try to provide an answer by looking at the space of *Lagrangian densities*, given in this case by the space of closed three-forms on M

$$\Omega^3_{cl}(M) := \{H \in \Lambda^3 T^* : dH = 0\}.$$

Note the group of diffeomorphisms $\mathrm{Diff}(M)$ acts on $C^\infty(\Sigma, M)$ via $\varphi \mapsto f \circ \varphi$, for $f \in \mathrm{Diff}(M)$ and we have

$$S_H(f \circ \varphi) = S_{f^* H}(\varphi).$$

This suggests a left action of $\mathrm{Diff}(M)$ on $\Omega^3_{cl}(M)$ given by push-forward. Consider now the semi-direct product of $\mathrm{Diff}(M)$ by the space of two-forms on M

$$\mathcal{G} = \mathrm{Diff}(M) \ltimes \Gamma(\Lambda^2 T^*),$$

with group structure

$$(f, B) \cdot (f', B') = (f \circ f', B' + f'^* B).$$

Then, the left action of $\mathrm{Diff}(M)$ on $\Omega^3_{cl}(M)$ extends to a $\mathcal{G}$-action defined by

$$(f, B) \cdot H = f_*(H - dB).$$

In terms of the corresponding functionals, setting $H' = f_*(H - dB)$ we have

$$S_{H'}(\varphi) = S_H(f^{-1} \circ \varphi) - \int_\Sigma \varphi^*(f_* B).$$

Notice that our formula for the $\mathcal{G}$-action reproduces the change of a closed three-form under an isomorphism of Courant algebroids in (2.11) and, in fact, we have the following.

PROPOSITION 2.23. *Given M a smooth manifold and a closed three-form H, the isotropy group $\mathcal{G}_H \leq \mathcal{G}$ of H is isomorphic to the group of automorphisms of the twisted Courant algebroid $(T \oplus T^*, \langle , \rangle, [,]_H, \pi)$.*

PROOF. The proof is straightforward from Proposition 2.21 and the definition of the $\mathcal{G}$-action on Ω^3_{cl}. □

By the previous result, a Lagrangian density for our variational problem, given by a closed three-form H, determines a geometric gadget, the exact Courant algebroid with twisted bracket, such that symmetries of S_H are realized as automorphisms of $(T \oplus T^*, \langle , \rangle, [,]_H, \pi)$. In order to interpret condition (2.14) in the present setup, we need to regard elements in $\Gamma(T \oplus T^*)$ as symmetries. For this, it is better to think in terms of the Lie algebra

$$\operatorname{Lie}\mathcal{G} = \Gamma(T \oplus \Lambda^2 T^*)$$

with Lie bracket (see Corollary 2.22)

$$[X_1 + B_1, X_2 + B_2] = [X_1, X_2] + L_{X_1} B_2 - L_{X_2} B_1.$$

Given $H \in \Omega^3_{cl}$, the Lie algebra of its isotropy group is

$$\operatorname{Lie}\mathcal{G}_H = \{X + B : d(i_X H + B) = 0\}.$$

PROPOSITION 2.24. *Given M a smooth manifold and a closed three-form H, the following defines an ideal of* $\operatorname{Lie}\mathcal{G}_H$*:*

$$\operatorname{Lie}\mathcal{G}^0_H = \{X + d\xi - i_X H : \xi \in T^*\}.$$

PROOF. Using the identity $i_{[X,Y]} = [L_X, i_Y]$ we calculate

$$\begin{aligned}
[X + d\xi - i_X H, Y + B] &= [X, Y] + L_X B - L_Y(d\xi - i_X H) \\
&= [X, Y] + d i_X B + i_X d(B + i_Y H) - i_{[X,Y]} H - d(L_Y \xi) \\
&= [X, Y] + d(i_X B - L_Y \xi) - i_{[X,Y]} H
\end{aligned}$$

for any $Y + B \in \operatorname{Lie}\mathcal{G}_H$ and $X + d\xi - i_X H \in \operatorname{Lie}\mathcal{G}^0_H$, as required. □

To finish, we note that the ideal $\operatorname{Lie}\mathcal{G}^0 \subset \operatorname{Lie}\mathcal{G}_H$ is parametrized by elements in $\Gamma(T \oplus T^*)$. This vector space provides a natural representation for $\mathcal{G}$, via the action

$$(f, B) \cdot (X + \xi) = f_*(X + \xi + i_X B)$$

and there is a (right) $\mathcal{G}_H$-equivariant map

$$\Psi \colon \Gamma(T \oplus T^*) \to \operatorname{Lie}\mathcal{G}^0_H \colon X + \xi \mapsto X + d\xi - i_X H. \tag{2.15}$$

We have now that the twisted Dorfman bracket is recovered via the infinitesimal action

$$\Psi(X + \xi) \cdot (Y + \eta) = [X, Y] + L_X \eta - i_X d\xi + i_Y i_X H,$$

and the Jacobi identity in Lemma 2.5 is a direct consequence of the $\mathcal{G}_H$-equivariance of the map Ψ. The map Ψ determines special elements $X + \xi \in \Gamma(T \oplus T^*)$ such that $\Psi(X + \xi) = X$, that is, $i_X H = d\xi$. The geometric content of this condition is that the classical infinitesimal action of the vector field X on $T \oplus T^*$ via the Lie derivative is realized through the twisted Dorfman bracket via the differential operator $[X + \xi, \cdot]_H$. Going back to our original variational problem, elements $X + \xi \in \Gamma(T \oplus T^*)$ such that $i_X H = d\xi$ provide tangent directions in the space of smooth maps $C^\infty(\Sigma, M)$ along which S_H is constant. The condition $i_X H = d\xi$

appears also naturally in the context of reduction of Courant algebroids by the action of a Lie group in [**31**].

2.3. Generalized metrics

Recall the definition of a classical Riemannian metric, which is a smooth choice of positive definite inner product on the tangent spaces of a smooth manifold. The geometric kernel of this fundamental object is clear, as it allows one to assign the size of a vector tangent to a manifold, and hence by integration along a curve one obtains concepts of length, and eventually notions of differentiation, curvature, etc. The forthcoming definition of a generalized metric plays a similar role, when we substitute curves–regarded as the trajectories of point particles–by maps from a surface into the manifold–regarded as the trajectories of strings.

To start, we introduce generalized metrics seen through the lens of structure groups. In particular, recall that a classical Riemannian metric is equivalent to a reduction of the structure group of the tangent bundle to $O(n)$. If we now take the bundle $T \oplus T^*$ with its neutral inner product instead of the tangent bundle, then reductions of its structure group should render interesting geometries. As discussed in §2.1, the neutral inner product already reduces the structure group to $O(n, n)$, and as it turns out a further reduction to $O(n) \times O(n)$ will yield the right notion of "generalized metric". Unwinding this reduction in terms of tensor fields (see Lemma 2.37) finally yields the fundamental notion.

2.3.1. Linear algebra. We shall first consider the linear algebra of generalized metrics. Let V and W be real vector spaces which fit into a short exact sequence

$$0 \longrightarrow V^* \longrightarrow W \xrightarrow{\pi} V \longrightarrow 0. \tag{2.16}$$

We assume that W is endowed with a metric $\langle , \rangle$ of split signature (n, n) such that the image of $\pi^* : V^* \to W^*$ is isotropic. Furthermore, we assume that the first arrow is given by the composition of $\frac{1}{2}\pi^* : V^* \to W^*$ with the isomorphism $W \cong W^*$ given by the metric. We denote by $O(n, n)$ the group of orthogonal transformations of W. Of course, (2.16) corresponds to the exact sequence in Definition 2.9 for a fixed point on the base manifold.

Definition 2.25. A *generalized metric* on W is an endomorphism $\mathcal{G}$ of W satisfying

1. $\langle \mathcal{G}a, \mathcal{G}b\rangle = \langle a, b\rangle$,
2. $\langle \mathcal{G}a, b\rangle = \langle a, \mathcal{G}b\rangle$,
3. The bilinear pairing $\langle \mathcal{G}a, b\rangle$ is symmetric and positive definite.

As a basic consequence of the definitions, it follows that $\mathcal{G}^2 = \mathrm{Id}$.

Lemma 2.26. *A generalized metric on W is equivalent to either*

1. *a choice of maximal compact subgroup $O(n) \times O(n) \leq O(n, n)$,*
2. *an orthogonal decomposition $W = V_+ \oplus V_-$ such that the restriction of $\langle , \rangle$ to V_+ is positive definite.*

Proof. A choice of maximal compact subgroup $O(n) \times O(n)$ inside $O(n, n)$ requires a choice of plane on which the neutral inner product is positive definite. Call such a choice V_+, and let V_- denote the neutral orthogonal complement of V_+, and note that the neutral inner product is negative definite on V_-. Thus (1) implies (2). Given the orthogonal decomposition $W = V_+ \oplus V_-$ as in (2), we can define a positive definite inner product on W via

$$G(a,b) = \langle a_+, b_+ \rangle_{V_+} - \langle a_-, b_- \rangle_{V_-} ,$$

where $a = a_+ + a_- \in V_+ \oplus V_-$ and similarly for b. Using the neutral inner product we can identify W with its dual, and so identify G with an endomorphism $\mathcal{G}$ of W, which is easily seen to satisfy the axioms of a generalized metric.

Conversely, given a generalized metric, since $\mathcal{G}$ is real and $\mathcal{G}^2 = \mathrm{Id}$ it follows that the only possible eigenvalues for $\mathcal{G}$ are ± 1, with eigenspaces $V_\pm$. It follows that V_+ is a maximal plane on which the neutral inner product is positive definite, yielding a maximal compact $O(n) \times O(n)$ given by orthogonal transformations which commute with $\mathcal{G}$. Note that of course the two constructions are inverses, with the spaces $V_\pm$ playing the same role in each. □

A generalized metric determines a series of useful isomorphisms and projections which we will employ in the sequel, and which clarify the relationship between the generalized metric and the classical data to be determined below.

Definition 2.27. Given $\mathcal{G}$ a generalized metric on W, define projection maps $\pi_\pm : W \to V_\pm$ via

$$\pi_\pm a := \tfrac{1}{2} (a \pm \mathcal{G} a) .$$

Since the pairing $\langle , \rangle$ vanishes on $\pi^* V^*$ it follows that $V_\pm \cap V^* = \{0\}$, and hence the map $\pi : V_\pm \to V$ has no kernel. Since both are vector spaces of dimension n it follows that $\pi : V_\pm \to V$ is an isomorphism.

Definition 2.28. Given $\mathcal{G}$ a generalized metric on W, define isomorphisms

$$\sigma_\pm := \pi^{-1}_{|V_\pm} : V \to V_\pm. \tag{2.17}$$

Using now either σ_+ or σ_-, we can define an isotropic splitting of (2.16) (see (2.5))

$$\sigma = \sigma_\pm - \tfrac{1}{2} \pi^* \tau_\pm, \tag{2.18}$$

where $\tau_\pm \in \mathrm{Sym}^2 V^*$ is defined by $\tau_\pm(X,Y) = \langle \sigma_\pm X, \sigma_\pm Y \rangle$. Recall that, similar to Proposition 2.10, the isotropic splitting σ determines an isometry $F \colon V \oplus V^* \to W$ defined by

$$F(X + \xi) = \sigma X + \tfrac{1}{2} \pi^* \xi$$

for the symmetric product (2.1).

Lemma 2.29. *A generalized metric $\mathcal{G}$ on W is equivalent to pair (g, σ), where g is a positive definite metric on V and $\sigma : V \to W$ is an isotropic splitting of* (2.16)*, such that*

$$F^{-1} \mathcal{G} F = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \qquad F^{-1}(V_\pm) = \{ X \pm g(X) : X \in V \}. \tag{2.19}$$

We refer to the generalized metric determined by (g, σ) as $\mathcal{G}(g, \sigma)$.

Proof. Given a generalized metric $\mathcal{G}$, we can construct σ as in (2.18). Notice that

$$\left\langle \sigma_+(X) - \tfrac{1}{2} \pi^* \tau_+(X) - \sigma_-(X) + \tfrac{1}{2} \pi^* \tau_-(X), a \right\rangle = 0$$

for all $a \in W$, and therefore σ is independent of the choice of $\sigma_\pm$. We also have

$$0 < \langle \mathcal{G} \sigma_+ X, \sigma_+ X \rangle = \langle \sigma_+ X, \sigma_+ X \rangle = \tau_+(X, X)$$

for $X \neq 0$, and therefore $g = \tau_+$ is a positive definite metric on V. Thus,

$$F(X + g(X)) = \sigma X + \tfrac{1}{2}\pi^* g(X) = \sigma_+ X$$

and therefore $F^{-1}(V_+) = \{X + g(X) : X \in V\}$. Finally, since F is an isometry and V_- is orthogonal to V_+, (2.19) follows. Conversely, given a pair (g, σ), where g is a positive definite metric on V and $\sigma : V \to W$ is an isotropic splitting, it is easy to verify that $\mathcal{G}$ defined by (2.19) is a generalized metric on W. □

Our next goal is to make our previous discussion more explicit, writing a generalized metric in terms of tensors on the vector space V in (2.16). This will help us to determine the classical data which is equivalent to the specification of a generalized metric, in the geometric situation below. We fix an isotropic splitting $\sigma_0 : V \to W$, inducing an identification

$$W = V \oplus V^*,$$

so that $\sigma_0(X) = X$ and the neutral metric on W is given by (2.1). Given another isotropic splitting $\sigma : V \to V \oplus V^*$, it follows that $b = \sigma - \sigma_0 \colon V \to V^*$, which can be identified with an element $b \in V^* \otimes V^*$. The isotropic condition implies further that $b \in \Lambda^2 V^*$ and therefore

$$\sigma(X) = X + b(X) = e^b(X).$$

Thus, we can identify the isometry $F \colon V \oplus V^* \to V \oplus V^*$ with e^b, that is,

$$F(X + \xi) = e^b(X + \xi) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = X + \xi + i_X b.$$

The next proposition follows now as an immediate consequence of Lemma 2.29.

Proposition 2.30. *A generalized metric $\mathcal{G}$ on $V \oplus V^*$ is equivalent to a pair (g, b) consisting of a positive definite metric g on V and $b \in \Lambda^2 T^*$, such that*

$$\mathcal{G} = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}. \tag{2.20}$$

We refer to the generalized metric determined by (g, b) as $\mathcal{G}(g, b)$. Furthermore, given $k \in \Lambda^2(V^*)$, we obtain a new generalized metric via $\mathcal{G}_k := e^k \mathcal{G} e^{-k}$, with associated pair $(g, b + k)$.

In terms of the previous explicit description of a generalized metric $\mathcal{G} = \mathcal{G}(g, b)$ on $V \oplus V^*$, we have that

$$V_\pm = \{X + (b \pm g)X,\ X \in V\},$$

that the projections $\pi_\pm \colon V \oplus V^* \to V_\pm$ in Definition 2.27 are given by

$$\pi_\pm(X + \xi) = \tfrac{1}{2}(X + (b \pm g)X) + \tfrac{1}{2}(\xi \pm bg^{-1}\xi \pm g^{-1}\xi),$$

and that the linear sections $\sigma_\pm \colon V \to V_\pm$ in Definition 2.28 are

$$\sigma_\pm(X) = X + (b \pm g)X.$$

To conclude our study of the linear theory, we next characterize the tangent space to the space of generalized metrics. It will be convenient to go back to the abstract setup at the beginning of this section (see (2.16)). We denote

$$\mathcal{G}(W) \subset \mathrm{End}(W)$$

the space of generalized metrics on W.

LEMMA 2.31. *The tangent space at $\mathcal{G}$ to the space of generalized metrics $\mathcal{G}(W)$ is given by*

$$\{\mathcal{G}\Phi \in \mathrm{End}(W) \mid \Phi \in \mathfrak{o}(W), \quad \Phi\mathcal{G} + \mathcal{G}\Phi = 0\},$$

where $\mathfrak{o}(W) = \{\Phi \in \mathrm{End}(W) : \langle \Phi, \rangle = -\langle , \Phi\rangle\}$.

PROOF. Let $\mathcal{G}_t$ be a one-parameter family of generalized metrics on W, with $\mathcal{G}_0 = \mathcal{G}$. Denote

$$\dot{\mathcal{G}} = \frac{d}{dt}\mathcal{G}_t\Big|_{t=0} \in \mathrm{End}(W).$$

Setting $\Phi = \mathcal{G}\dot{\mathcal{G}}$, by Definition 2.25 it follows that

$$\Phi \in \mathfrak{o}(W), \qquad \Phi\mathcal{G} + \mathcal{G}\Phi = 0. \tag{2.21}$$

It remains to show that an element $\Phi \in \mathrm{End}(W)$ satisfying (2.21) can be realized through a curve of generalized metrics. For this, consider $\Phi_t = e^{-\frac{t}{2}\Phi} \in O(W)$ and define

$$\mathcal{G}_t = \Phi_t \mathcal{G} \Phi_{-t}.$$

It is easy to see that $\mathcal{G}_t$ defines a path of generalized metrics and

$$\dot{\mathcal{G}} = \frac{d}{dt}\mathcal{G}_t\Big|_{t=0} = -\tfrac{1}{2}\Phi\mathcal{G} + \tfrac{1}{2}\mathcal{G}\Phi = \mathcal{G}\Phi,$$

as required. □

By the second condition in (2.21), a tangent vector Φ at $\mathcal{G}$ induces well-defined maps

$$\Phi_{|V_-} : V_- \to V_+, \qquad \Phi_{|V_+} : V_+ \to V_-.$$

Furthermore, the first condition in (2.21) implies that $\Phi_{|V_+}$ is uniquely determined by $\Phi_{|V_-}$, thus leading us to the following result.

LEMMA 2.32. *The map $\Phi \to \Phi_{|V_-}$ defines a linear isomorphism between the tangent space at $\mathcal{G}$ and* $\mathrm{Hom}(V_-, V_+)$.

PROOF. By Lemma 2.31, the linear map $\Phi \to \Phi_{|V_-}$ is well-defined and injective. Given $R_+ \in \mathrm{Hom}(V_-, V_+)$, we can define $R_- \in \mathrm{Hom}(V_+, V_-)$ by

$$\langle R_- a_+, b_- \rangle := -\langle a_+, R_+ b_- \rangle$$

for any $a_+ \in V_+$ and $b_- \in V_-$. Then it follows that $\Phi = R_+ + R_- \in \mathrm{End}(W)$ is tangent to $\mathcal{G}$, and furthermore $\Phi_{|V_-} = R_+$. □

Using Lemma 2.29 and Proposition 2.30, we unravel our previous discussion in terms of the vector space V in (2.16). By Lemma 2.29, the space of generalized metrics $\mathcal{G}(W)$ is in one-to-one correspondence with

$$\mathcal{M}(V) \times \mathcal{S}(W),$$

where $\mathcal{M}(V)$ denotes the space of positive definite metrics on V and $\mathcal{S}(W)$ denotes the space of isotropic splittings $\sigma \colon V \to W$ of the sequence (2.16). Upon a choice of an element $(g, \sigma) \in \mathcal{M}(V) \times \mathcal{S}(W)$, the space $\mathcal{G}(W)$ is bijective to

$$\mathrm{GL}(V)/O(V) \times \Lambda^2 V^*,$$

where $O(V)$ denotes the space of orthogonal transformations of (V, g). Thus, the tangent space at $\mathcal{G} = \mathcal{G}(g, \sigma)$ can be canonically identified with

$$\mathrm{Sym}^2 V^* \oplus \Lambda^2 V^* \cong V^* \otimes V^*.$$

LEMMA 2.33. *Let $\mathcal{G} = \mathcal{G}(g, \sigma)$ be a generalized metric, and consider the induced identification $W = V \oplus V^*$ as in Lemma 2.29. Given a one-parameter family of generalized metrics $\mathcal{G}_t = \mathcal{G}(g_t, \sigma_t)$ with $\mathcal{G}_0 = \mathcal{G}$,*

$$\dot{\mathcal{G}} = \frac{d}{dt}\mathcal{G}_t\Big|_{t=0} = [\mathbf{1} - e^{\dot{b}}, \mathcal{G}_0] + \begin{pmatrix} 0 & -g^{-1}\dot{g}g^{-1} \\ \dot{g} & 0 \end{pmatrix} = \begin{pmatrix} -g^{-1}\dot{b} & -g^{-1}\dot{g}g^{-1} \\ \dot{g} & \dot{b}g^{-1} \end{pmatrix}, \tag{2.22}$$

where

$$\frac{d}{dt}g_t\Big|_{t=0} = \dot{g} \in \mathrm{Sym}^2 V^*, \qquad \frac{d}{dt}\sigma_t\Big|_{t=0} = \dot{b} \in \Lambda^2 V^*.$$

In particular, the associated tangent vector to the space of generalized metrics in Lemma 2.31 is

$$\Phi = \mathcal{G}\dot{\mathcal{G}} = \begin{pmatrix} g^{-1}\dot{g} & g^{-1}\dot{b}g^{-1} \\ -\dot{b} & -\dot{g}g^{-1} \end{pmatrix}. \tag{2.23}$$

PROOF. Using the identification $W = V \oplus V^*$ induced by σ, following the notation in Lemma 2.30 we can write $\mathcal{G}_t = \mathcal{G}(g_t, b_t)$ for

$$(g_t, b_t) \in \mathcal{M}(V) \times \Lambda^2 V^*$$

and $b_0 = 0$. Then, (2.22) follows easily taking derivatives in expression (2.20). We leave (2.23) to the reader. □

REMARK 2.34. Given $\Phi \in \mathrm{End}(W)$ satisfying (2.21) it is an **exercise** to verify that there exists $(\dot{g}, \dot{b}) \in \mathrm{Sym}^2 V^* \oplus \Lambda^2 V^*$ such that (2.23) holds.

Given a generalized metric $\mathcal{G}$ and a tangent vector Φ, using the isomorphism $\sigma_- \colon V \to V_-$ in (2.17) combined with the map $\pi \colon V_+ \to V$ we can regard the homomorphism $\Phi_{|V_-}$ in Lemma 2.32 as an element

$$\pi \circ \Phi_{|V_-} \circ \sigma_- \in \mathrm{End}(V).$$

Then, using (2.23) we have the following.

LEMMA 2.35. *Let $\mathcal{G} = \mathcal{G}(g, \sigma)$ be a generalized metric. Given a tangent vector Φ as in (2.23), we have*

$$\pi \circ \Phi_{|V_-} \circ \sigma_-(X) = g^{-1}(\dot{g}(X) - \dot{b}(X)). \tag{2.24}$$

PROOF. Using the identification $W = V \oplus V^*$ induced by σ we have $\sigma_-(X) = X - g(X)$, and the proof follows from (2.23). □

2.3.2. Generalized metrics on manifolds.

We next turn to the geometry, by introducing the notion of generalized metric on a smooth manifold M. Consider the exact Courant algebroid $(T \oplus T^*, \langle , \rangle, [,], \pi)$ studied in §2.1.

DEFINITION 2.36. Given M a smooth manifold, a *generalized metric* on $T \oplus T^*$ is an endomorphism $\mathcal{G} \in \Gamma(\mathrm{End}\, T \oplus T^*)$ satisfying

(1) $\langle \mathcal{G}(X + \xi), \mathcal{G}(Y + \eta) \rangle = \langle X + \xi, Y + \eta \rangle$,
(2) $\langle \mathcal{G}(X + \xi), Y + \eta \rangle = \langle X + \xi, \mathcal{G}(Y + \eta) \rangle$,
(3) The bilinear pairing $\langle \mathcal{G}(X + \xi), Y + \eta \rangle$ is symmetric and positive definite.

Relying on Lemma 2.26, a generalized metric provides a reduction of the $O(n,n)$-bundle of frames of $T \oplus T^*$ to a maximal compact $O(n) \times O(n)$, thus showing that generalized metrics are natural analogues of ordinary metrics (regarded as reductions of the frame bundle of M to the orthogonal group $O(n)$). The details of the following lemma are left to the reader.

LEMMA 2.37. *A generalized metric on $T \oplus T^*$ is equivalent to either*

(1) *a reduction of the bundle of frames of $T \oplus T^*$ to a maximal compact subgroup $O(n) \times O(n) \le O(n,n)$,*
(2) *an orthogonal bundle decomposition $T \oplus T^* = V_+ \oplus V_-$ such that the restriction of $\langle , \rangle$ to V_+ is positive definite.*

We will use the same notation for the bundle maps $\pi_\pm, \sigma_\pm, \sigma$ as in the linear theory. The next proposition is fundamental to the study of generalized geometry, indicating the classical data which is equivalent to the specification of a generalized metric. In particular, a generalized metric is seen equivalent to a classical Riemannian metric g, together with a b-field. We have already seen closed b-fields playing a role as symmetries of the Courant bracket in §2.2, whereas here the associated b-field is not closed, and plays a nontrivial role in determining relevant associated geometric quantities. The proof of the following is straightforward from Proposition 2.30.

PROPOSITION 2.38. *Given M a smooth manifold, a generalized metric $\mathcal{G}$ on $T \oplus T^*$ uniquely determines a pair (g, b) consisting of a Riemannian metric on M and two-form $b \in \Gamma(\Lambda^2 T^*)$. Conversely, such a pair (g, b) determines $\mathcal{G}$ via*

$$\mathcal{G} = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}.$$

We refer to the generalized metric determined by (g, b) as $\mathcal{G}(g, b)$. Furthermore, given $k \in \Gamma(\Lambda^2 T^*)$, we obtain a new generalized metric via $\mathcal{G}_k := e^k \mathcal{G} e^{-k}$, with associated pair $(g, b + k)$.

By Proposition 2.29, a generalized metric $\mathcal{G}$ determines an isotropic splitting on $T \oplus T^*$ given by

$$\sigma(X) = e^b(X) = X + b(X),$$

and therefore a preferred presentation of the Dorfman bracket on $\Gamma(T \oplus T^*)$:

$$e^{-b}[e^b(X + \xi), e^b(Y + \eta)] = [X, Y] + L_X \eta - i_Y d\xi + i_Y i_X db.$$

We also obtain a preferred representative db of the (trivial) Ševera class of the exact Courant algebroid $(T \oplus T^*, \langle , \rangle, [,], \pi)$. This situation extends naturally to the case of generalized metrics on general exact Courant algebroids, as considered in Definition 2.9.

DEFINITION 2.39. Given M a smooth manifold and an exact Courant algebroid E over M, a *generalized metric* on E is a bundle endomorphism $\mathcal{G} \in \Gamma(\operatorname{End} E)$ satisfying

(1) $\langle \mathcal{G}a, \mathcal{G}b \rangle = \langle a, b \rangle$,
(2) $\langle \mathcal{G}a, b \rangle = \langle a, \mathcal{G}b \rangle$,
(3) The bilinear pairing $\langle \mathcal{G}a, b \rangle$ is symmetric and positive definite.

The analogue of Proposition 2.38 for the case of a general exact Courant algebroid is provided by the following proposition. The proof follows directly from Lemma 2.29. Recall that an isotropic splitting $\sigma \colon T \to E$ determines an isometry $F \colon T \oplus T^* \to E$ defined by

$$F(X + \xi) = \sigma X + \tfrac{1}{2} \pi^* \xi$$

for the symmetric product (2.1).

PROPOSITION 2.40. *Given M a smooth manifold and an exact Courant algebroid E over M, a generalized metric $\mathcal{G}$ is equivalent to pair (g, σ), where g is a positive definite metric on M and $\sigma : T \to E$ is an isotropic splitting of* (2.16), *such that*

$$F^{-1}\mathcal{G}F = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \qquad F^{-1}(V_{\pm}) = \{X \pm g(X) : X \in T\}.$$

Consequently, a generalized metric determines a preferred presentation of the Dorfman bracket on $\Gamma(E)$,

$$F^{-1}[F(X+\xi), F(Y+\eta)] = [X,Y] + L_X\eta - i_Y d\xi + i_Y i_X H,$$

where H is a preferred representative of the Ševera class of E, given by

$$H(X,Y,Z) = 2\left\langle [\sigma X, \sigma Y], \sigma Z \right\rangle.$$

Given a generalized metric $\mathcal{G}$ on an exact Courant algebroid E over M, our next goal is to characterize the *group of generalized isometries*, that is, the group of Courant automorphisms $F \colon E \to E$ which preserve the generalized metric. Using the isotropic splitting induced by $\mathcal{G}$, E is isomorphic to the twisted Courant algebroid $(T \oplus T^*, \langle , \rangle, [,]_H, \pi)$, whose automorphism group has been characterized in Proposition 2.21.

PROPOSITION 2.41. *Let $\mathcal{G} = \mathcal{G}(g,\sigma)$ be a generalized metric on an exact Courant algebroid E over M. The* group of generalized isometries *of $\mathcal{G}$, defined as the group of Courant automorphisms $F \colon E \to E$ which preserve the generalized metric, is given by*

$$\{\overline{f} : f \in \mathrm{Diff}(M), \text{ such that } f^*H = H, \ f^*g = g\} \subset \mathrm{Aut}(E)$$

where $H \in \Omega^3_{cl}(M)$ is the preferred representative of the Ševera class of E determined by $\mathcal{G}$.

PROOF. Consider the isomorphism $E \cong (T \oplus T^*, \langle , \rangle, [,]_H, \pi)$ given by the generalized metric $\mathcal{G}$. With this identification, by Lemma 2.29) we have

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

By Proposition 2.21, a general element $\overline{f} \circ e^B \in \mathrm{Aut}(E)$ preserves $\mathcal{G}$, that is,

$$\overline{f} \circ e^B \circ \mathcal{G} = \mathcal{G} \circ \overline{f} \circ e^B,$$

if and only if

$$(f^*g)^{-1}(B(X) + \xi) = g^{-1}\xi, \qquad (f^*g)(X) = B(g^{-1}\xi) + g(X)$$

for all $X + \xi \in \Gamma(T \oplus T^*)$. Setting e.g. $\xi = 0$ this implies $f^*g = g$, and consequently $B = 0$, as claimed. Returning again to Proposition 2.21 it follows that $f^*H = H$. □

REMARK 2.42. With the notation in Proposition 2.38, the action of $F = \overline{f} \circ e^B$ on $\mathcal{G}$ is

$$F \circ \mathcal{G} \circ F^{-1} = \mathcal{G}(f_*g, f_*B).$$

In particular, by Lemma 2.33 the infinitesimal action of $X + B \in \mathrm{Lie}\,\mathrm{Aut}(E)$ is given by

$$(X+B) \cdot \mathcal{G} = \begin{pmatrix} -g^{-1}B & g^{-1}(L_X g)g^{-1} \\ -L_X g & Bg^{-1} \end{pmatrix}.$$

To finish this section, we show that a generalized metric $\mathcal{G}$ determines an energy functional $S_{\mathcal{G}}$ on the space of smooth maps $C^\infty(\Sigma, M)$ from a surface Σ into our target manifold M. This functional can be thought of as an analogue of the energy functional which associates the length squared to a parametrized curve on M, leading to the notion of distance and geodesics in Riemannian geometry. Morally, $S_{\mathcal{G}}$ can be similarly used to define a pseudo-metric on the space of loops $C^\infty(S^1, M)$ of M, by extremizing the energy with prescribed boundary components. We will follow our discussion at the end of §2.2.

Let M be an oriented smooth manifold. Let Σ be a smooth compact surface (oriented and without boundary, say). Consider the infinite-dimensional space of smooth maps $C^\infty(\Sigma, M)$. Let $(E, \mathcal{G})$ be an exact Courant algebroid over M endowed with a generalized metric $\mathcal{G}$. By Lemma 2.40, $\mathcal{G}$ determines a Riemannian metric g and a closed three-form $H \in \Lambda^3 T^*$ on M. We will assume that $[H] \in H^3(M, \mathbb{Z})$. We fix a Riemannian metric η on Σ with volume dV_η. Then, we can define a functional

$$S_{\mathcal{G}} \colon C^\infty(\Sigma, M) \to \mathbb{R}/\mathbb{Z}, \qquad \varphi \mapsto \frac{1}{2}\int_\Sigma |d\varphi|^2 dV_\eta + \int_Y \overline{\varphi}^* H. \tag{2.25}$$

Here, $|d\varphi|^2$ denotes the norm squared of $d\varphi \in \Gamma(\Sigma, T^*\Sigma \otimes \varphi^* T)$ with respect to the metric $\eta^* \otimes \varphi^* g$ and Y is a three-manifold with boundary Σ and $\overline{\varphi} \colon Y \to M$ is any smooth extension of φ to Y. This functional is known in the mathematical physics literature as the *Polyakov action with Wess-Zumino term* in the action of a two-dimensional Σ-model (see e.g. [**79**]). Note that the second summand in the definition of $S_{\mathcal{G}}$ corresponds to the functional S_H in §2.2. Our next result shows that the critical points of $S_{\mathcal{G}}$ provide an interesting generalization of the harmonicity condition for maps $\varphi \in C^\infty(\Sigma, M)$.

PROPOSITION 2.43. *The critical points $\varphi \in C^\infty(\Sigma, M)$ of the functional $S_{\mathcal{G}}$ are the solutions of the equation*

$$\nabla^*\nabla\varphi + \left\langle \varphi^* g^{-1} H, dV_\eta \right\rangle_\eta = 0,$$

where $\nabla^*\nabla$ is the rough Laplacian for the product of Levi-Civita connections of η and g, and $\left\langle \varphi^* g^{-1} H, dV_\eta \right\rangle_\eta \in \Gamma(\varphi^* T)$ denotes the product of $\varphi^* g^{-1} H \in \Gamma(\varphi^* T \otimes \Lambda^2 T^*\Sigma)$ with dV_η with respect to η.

PROOF. A calculation in coordinates shows that the functional $S_{\mathcal{G}}$ can be alternatively written as

$$S_{\mathcal{G}}(\varphi) = \frac{1}{2}\int_\Sigma |\nabla\varphi|^2 dV_\eta + \int_Y \overline{\varphi}^* H.$$

This follows simply from the fact that action of the Levi-Civita connection on functions coincides with the exterior differential. Then, the variation of $S_{\mathcal{G}}$ along $X \in \Gamma(\varphi^* T)$, identified with a vector in the tangent space of $C^\infty(\Sigma, M)$ at φ, is given by

$$\begin{aligned}
\delta S_{\mathcal{G}}(X) &= \int_\Sigma \langle \nabla\varphi, \nabla X \rangle\, dV_\eta + \int_\Sigma \varphi^* i_X H \\
&= \int_\Sigma \langle \nabla^*\nabla\varphi, X \rangle\, dV_\eta + \int_\Sigma \left\langle \varphi^* g^{-1} H, X \otimes dV_\eta \right\rangle dV_\eta,
\end{aligned}$$

as claimed. □

2.4. Divergence operators

As we have seen in the previous section, natural notions in generalized geometry are described in terms of the $O(n,n)$-bundle of frames of $T \oplus T^*$. Starting with its definition as a reduction to a maximal compact subgroup, a generalized metric $\mathcal{G}$ is *a posteriori* seen to be equivalent to a classical Riemannian metric g together with a b-field. Since the neutral metric on $T \oplus T^*$ is fixed in this geometry, how does generalized geometry keep track of conformal rescalings of the classical metric associated to $\mathcal{G}$? In this section we answer this question, by means of the following device introduced in [**69**].

DEFINITION 2.44. Let E be an exact Courant algebroid over a smooth manifold M. A *divergence operator* on E is a differential operator $\operatorname{div}\colon \Gamma(E) \to C^\infty(M)$ satisfying

$$\operatorname{div}(fe) = \pi(e)(f) + f \operatorname{div}(e), \tag{2.26}$$

for all $e \in \Gamma(E)$ and $f \in C^\infty(M)$.

As a consequence of the Leibniz rule (2.26), the divergence operators on E form an affine space modeled on $\Gamma(E)$, since if we fix a divergence operator div, any other divergence operator div$'$ is given by

$$\operatorname{div}' = \operatorname{div} + \langle e, \cdot \rangle$$

for some $e \in \Gamma(E)$. The following result shows that this space is always non-empty.

LEMMA 2.45. *Let E be an exact Courant algebroid over M. Given a connection ∇ on T, one obtains a divergence operator on E via*

$$\operatorname{div}^\nabla(e) = \operatorname{tr} \nabla \pi(e),$$

where $\pi\colon E \to T$ is the anchor map.

PROOF. This follows from a simple calculation:

$$\operatorname{div}^\nabla(fe) = f \operatorname{div}^\nabla(e) + \operatorname{tr} df \otimes \pi(e) = f \operatorname{div}^\nabla(e) + \pi(e)(f).$$

□

More explicitly, consider an isotropic splitting $\sigma\colon T \to E$ and the induced identification $E = T \oplus T^*$, with anchor map

$$\pi(X + \xi) = X.$$

Then, it follows that, for $e = X + \xi$,

$$\operatorname{div}^\nabla(e) = \operatorname{tr} \nabla X.$$

Assuming now that ∇ is torsion-free, so that $\operatorname{tr} \nabla X = \operatorname{tr}(\nabla_X - L_X)$, and that there exists a density $\mu \in \Gamma(|\det T^*|)$ preserved by ∇, we obtain

$$\operatorname{div}^\nabla(e)\mu = L_X \mu,$$

which recovers the standard notion of divergence for the vector field $\pi(e) = X$. In particular, for any choice of metric g on M we can choose its Levi-Civita connection ∇ and the associated Riemannian density μ^g, obtaining a divergence operator

$$\operatorname{div}^g(e)\mu^g = L_X \mu^g.$$

Definition 2.46. Given a generalized metric $\mathcal{G}$ on an exact Courant algebroid E, define its *Riemannian divergence* as

$$\mathrm{div}^{\mathcal{G}}(e) = \frac{L_{\pi(e)}\mu^g}{\mu^g},$$

where g is the Riemannian metric associated to $\mathcal{G}$ in Lemma 2.40, and μ^g is the associated Riemannian density.

The relation between divergence operators and conformal geometry appears via the notion of *Weyl structure* [**66**]. Recall that a conformal structure $\mathcal{C}$ on a smooth manifold M is a set of Riemannian metrics, such that if $g, g' \in \mathcal{C}$ there exists a smooth function $f \in C^\infty(M)$ such that

$$g' = e^f g.$$

Definition 2.47. Let $\mathcal{C}$ be a conformal structure on a smooth manifold M. A *Weyl structure* is a map

$$W \colon \mathcal{C} \to \Gamma(T^*)$$

such that $W(e^f g) = W(g) - df$ for all $g \in \mathcal{C}$ and $f \in C^\infty(M)$.

Let us fix an exact Courant algebroid E over M, and denote by $\mathcal{G}(E)$ the space of generalized metrics on E. By Lemma 2.40, there is a bijection

$$\mathcal{G}(E) \cong \mathcal{M}(T) \times \mathcal{S}(E),$$

where $\mathcal{M}(T)$ and $\mathcal{S}(E)$ denote the space of Riemannian metrics on M and the space of isotropic splittings of E, respectively. Given $(g,\sigma) \in \mathcal{M}(T) \times \mathcal{S}(E)$, we denote $\mathcal{G}(g,\sigma)$ the associated generalized metric. Our next result shows that a divergence operator provides a natural analogue of a Weyl structure in the context of generalized geometry.

Lemma 2.48. *Let* div *be a divergence operator on* E. *Define*

$$W \colon \mathcal{G}(E) \to \Gamma(E) \colon \mathcal{G} \mapsto \mathrm{div} - \mathrm{div}^{\mathcal{G}},$$

where $\mathrm{div}^{\mathcal{G}}$ *is the Riemannian divergence of* $\mathcal{G}$ *(see Definition* 2.46*). Then,*

$$W(\mathcal{G}(e^f g, \sigma)) = W(\mathcal{G}(g,\sigma)) - n\,\langle df, \rangle$$

for any $(g,\sigma) \in \mathcal{M}(T) \times \mathcal{S}(E)$ *and any* $f \in C^\infty(M)$.

Proof. Consider the identification $E = T \oplus T^*$ given by $\mathcal{G} = \mathcal{G}(g,\sigma)$. Then, setting $\mathcal{G}' = \mathcal{G}(e^f g, \sigma)$ we have $\mu^{e^f g} = e^{\frac{n}{2}f}\mu^g$ and therefore

$$\mathrm{div}^{\mathcal{G}'}(X+\xi) = \frac{L_X(e^{\frac{n}{2}f}\mu^g)}{e^{\frac{n}{2}f}\mu^g} = \mathrm{div}^{\mathcal{G}}(X+\xi) + \tfrac{n}{2}df(X) = \mathrm{div}^{\mathcal{G}}(X+\xi) + n\,\langle df, X\rangle\,.$$

□

Our next goal is to understand a natural compatibility condition between a generalized metric and a divergence operator, which plays an important role in the definition of the generalized Ricci flow. Given a generalized metric $\mathcal{G}$ on E and $e \in \Gamma(E)$ we define an endomorphism

$$[e, \mathcal{G}] \in \mathrm{End}(E),$$

by

$$[e,\mathcal{G}](e') := [e, \mathcal{G}e'] - \mathcal{G}[e,e'].$$

It is an easy **exercise**, using axiom (3) of the Dorfman bracket in Definition 2.8, to check that $[e,\mathcal{G}]$ indeed defines a tensor.

DEFINITION 2.49. Given a generalized metric $\mathcal{G}$ on E, we say that $e \in \Gamma(E)$ is an *infinitesimal isometry* if $[e,\mathcal{G}]=0$.

LEMMA 2.50. *Let $\mathcal{G}$ be a generalized metric on E. Then, $e \in \Gamma(E)$ is an infinitesimal isometry if and only if*

$$L_X g = 0, \qquad d\xi = i_X H,$$

where $e = X+\xi$ via the identification $E = T \oplus T^*$ given by $\mathcal{G}$.

PROOF. Consider the identification $E = T\oplus T^*$ given by $\mathcal{G} = \mathcal{G}(g,\sigma)$, so that $e = X+\xi$. Using the notation in §2.2, we have (see (2.15) and Remark 2.42)

$$[e,\mathcal{G}] = \Psi(X+\xi)\cdot\mathcal{G} = \begin{pmatrix} -g^{-1}(d\xi - i_X H) & g^{-1}(L_X g)g^{-1} \\ -L_X g & (d\xi - i_X H)g^{-1}\end{pmatrix},$$

and the result follows. □

REMARK 2.51. Following the discussion in §2.2, an infinitesimal isometry provides a flat direction for the Wess-Zumino functional S_H in (2.13).

We introduce the following compatibility condition.

DEFINITION 2.52. Let E be an exact Courant algebroid over M. A pair $(\mathcal{G},\operatorname{div})$ is *compatible* if $W(\mathcal{G})$ is an infinitesimal isometry of $\mathcal{G}$, that is,

$$[W(\mathcal{G}),\mathcal{G}] = 0.$$

As a straightforward consequence of Lemma 2.40 and Lemma 2.50 we obtain the following characterization of compatible pairs $(\mathcal{G},\operatorname{div})$.

PROPOSITION 2.53. *Let E be an exact Courant algebroid over M. Then, $(\mathcal{G},\operatorname{div})$ is a compatible pair if and only if*

$$\operatorname{div} = \operatorname{div}^{\mathcal{G}} + \left\langle \sigma(X) + \tfrac{1}{2}\pi^*\xi, \cdot \right\rangle$$

where $\mathcal{G} = \mathcal{G}(g,\sigma)$ and

$$L_X g = 0, \qquad d\xi = i_X H.$$

An interesting class of infinitesimal isometries, and hence of compatible pairs, is provided by closed one-forms, that is, given by $X=0$ and $d\xi = 0$. More invariantly, this condition can be expressed in terms of the anchor map as

$$[W(\mathcal{G}),\mathcal{G}] = 0, \qquad \pi(W(\mathcal{G})) = 0.$$

This class of compatible pairs, associated to closed one-forms on the manifold, will be formally introduced in Definition 3.40 (see also Definition 3.48), and further studied in §3.6 and §3.7.