Introduction

The Axiom of Determinacy (AD) is the statement that all length-ω integer games of perfect information are determined (see Section 1.1 for a more precise definition). If AD holds then it holds in any inner model of ZF containing the real numbers. In particular it holds in $L(\mathbb{R})$, the smallest model of ZF containing the reals and the ordinals. There may be larger models of AD, however. For instance, assuming the stronger axiom $\mathsf{AD}_{\mathbb{R}}$ (which says that all length-ω real games of perfect information are determined), there are infinitely many models of the theory $\mathsf{ZF} + V{=}L(\mathcal{P}(\mathbb{R}))$ containing the reals and the ordinals. Wadge's Theorem (Theorem 2.4) shows, assuming only AD, that the models of $\mathsf{ZF} + V{=}L(\mathcal{P}(\mathbb{R}))$ containing the reals and the ordinals are linearly ordered by containment.

The axiom AD^+ is an extension, due to W. Hugh Woodin, of the Axiom of Determinacy. Assuming AD, AD^+ holds in an initial segment of the models in the linear order just described. Whether it holds in all such models, i.e., whether it is equivalent to AD, is an open question. As we discuss below, AD^+ was designed to capture the theory which reflects from certain larger models of AD to smaller ones. In practice it provides a useful general setting for the study of models of determinacy. In particular, models of AD^+ are the natural setting for Woodin's $\mathbb{P}_{\max}$ forcing [**52**]. They also play an important role in contemporary inner model theory (see, for instance, the recent books [**49**] and [**43**, in preparation]).

We define AD^+ as the conjunction of three statements.

0.1 Definition. The axiom system AD^+ consists of the following three statements.

(1) $\mathsf{DC}_{\mathbb{R}}$.
(2) Every subset of ω^ω is ∞-Borel.
(3) ($<\Theta$-$\mathsf{Determinacy}$) For all $\lambda < \Theta$, every $A \subseteq \omega^\omega$ and every continuous function $\pi\colon \lambda^\omega \to \omega^\omega$, $\pi^{-1}(A)$ is determined.

The axiom $\mathsf{DC}_{\mathbb{R}}$ is a weak form of the Axiom of Choice, defined in Section 0.3. Our definition of AD^+ differs from Woodin's, as he defines AD^+ to be the conjunction of items (2) and (3) above. As AD^+ is usually considered only in the context of $\mathsf{DC}_{\mathbb{R}}$ (which is essential for much of the basic theory of determinacy) the choice of definition does not make a significant difference. One difference between $\mathsf{DC}_{\mathbb{R}}$ and the other two parts of AD^+ as we have defined it is that instances of $\mathsf{DC}_{\mathbb{R}}$ are witnessed by real numbers, so, assuming the Axiom of Choice, $\mathsf{DC}_{\mathbb{R}}$ holds in any inner model containing the reals. The other two parts of AD^+, however, are witnessed by bounded subsets of Θ.

The ∞-Borel sets are defined in Definition 8.1 for subsets of 2^ω, and in Remark 8.9 for subsets of ω^ω. We define $<\Theta$-$\mathsf{Determinacy}$ (which Woodin calls Ordinal

Determinacy) in Chapter 7. The restriction of $<\Theta$-Determinacy to the case where π is the identity function on ω^ω is exactly AD. The notion of continuity in the statement of $<\Theta$-Determinacy refers to the product topology induced by the discrete topology on λ (as opposed to the interval topology).

It is an open question whether AD implies any or all of the parts of AD^+, and also whether AD plus any two parts of AD^+ imply the third. It is also open whether $\mathsf{AD}_\mathbb{R}$ implies AD^+. The issue in this case is whether $\mathsf{AD}_\mathbb{R}$ implies $<\Theta$-Determinacy, as $\mathsf{DC}_\mathbb{R}$ is easily seen to follow from $\mathsf{AD}_\mathbb{R}$, and the second part of AD^+ follows from $\mathsf{AD}_\mathbb{R}$ (moreover, from AD + Uniformization) by Theorem 11.19.

If $M \subseteq N$ are models of AD with the same reals, and every set of reals in M is Suslin in N, then $M \models \mathsf{AD}^+$ (this is Theorem 8.22). This is the context which the axiom AD^+ was designed to describe; its original name was "within scales" (a scale being a certain means of coding a Suslin representation). Since (assuming AD) the set of $A \subseteq \omega^\omega$ for which $L(A, \mathbb{R}) \models \mathsf{AD}^+$ is a nonempty initial segment of the Wadge hierarchy (see the first two items in the list below), AD^+ is also useful for a bottom-up analysis of models of determinacy.

The results proved in this book (all due to Woodin, except where noted) include the following:

- In $L(\mathbb{R})$, AD implies AD^+. This is Corollary 8.20.
- If AD^+ holds, then every inner model containing the reals satisfies AD^+. This is Theorem 8.22
- If AD holds and every true Σ^2_1 sentence is witnessed by a Suslin, co-Suslin set, then AD^+ holds. This is Corollary 8.21.
- If AD holds, then for all $A \subseteq \omega^\omega$, either $\mathcal{P}(\omega^\omega) \subseteq L(A, \mathbb{R})$ or $A^\#$ exists. This is Theorem 10.10.
- Assuming $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, a subset of ω^ω is ∞-Borel if and only if there is a set of ordinals S such that $A \in L(S, \mathbb{R})$. This follows from part (1) of Corollary 10.24.
- Assuming $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, if $A \subseteq \omega^\omega$ is ∞-Borel, then so is every set of reals in $L(A, \mathbb{R})$. See Remark 10.26. Moreover, if $A^\#$ exists, then $A^\#$ is also ∞-Borel. This is Corollary 10.27.
- If $<\Theta$-Determinacy and Uniformization hold, then AD^+ holds and every subset of ω^ω is Suslin. The first conclusion follows from Remark 6.25 and Theorem 11.19. The second then follows from Theorem 11.13.
- If AD^+ holds, then the set of Suslin cardinals is closed below Θ. This is Corollary 11.23.
- If AD^+ holds and the ultrapower $\mathrm{Ord}^{\mathcal{D}_\emptyset}/\mu_\emptyset$ is wellfounded (which in fact follows from AD^+ by later results of Woodin not proved in this book), then $\mathsf{AD}_\mathbb{R}$ fails if and only if there is a set of ordinals T such that $L(\mathcal{P}(\mathbb{R})) = L(T, \mathbb{R})$. The reverse direction of this implication follows from Theorem 6.28. The forward direction follows from Corollaries 6.19 and 11.23 (which together give that there is a largest Suslin cardinal if there is subset of ω^ω which is not Suslin), along with Theorems 13.1 and 13.19.

The following restatement of Theorem 13.18 gives a number of equivalences of $\mathsf{AD}_\mathbb{R}$ under the assumption of AD^+.

THEOREM 0.2. *Assuming* AD^+, $\mathsf{AD}_{\mathbb{R}}$ *is equivalent to each of the following.*

(1) *There is no* $\leq_{\mathcal{D}}$*-maximal set of ordinals.*
(2) *The Suslin cardinals are cofinal in* Θ.
(3) *Every subset of* ω^ω *is Suslin.*
(4) Uniformization.
(5) *The Solovay sequence has limit length.*

In addition, the following theorem illustrates the relationships among AD, $\mathsf{AD}_{\mathbb{R}}$, the Suslin property and Uniformization. The first sentence of the theorem is due to Woodin, aside from the implication from (1) to (3), which is due independently to Donald Martin. The implication from (4) to (1) under DC combines results of Woodin and Becker (see Chapter 12).

THEOREM 0.3. *Each of the following statements implies the ones below it, and the first two statements are equivalent. If* DC *holds, then all four statements are equivalent.*

(1) AD + *"every subset of* ω^ω *is Suslin".*
(2) AD^+ + *"every subset of* ω^ω *is Suslin".*
(3) $\mathsf{AD}_{\mathbb{R}}$.
(4) AD + Uniformization.

PROOF. Since AD is the restriction of $<\Theta$-Determinacy to the case $\lambda = \omega$, (2) implies (1). Theorem 6.24 says that Suslin sets can be uniformized, and Remark 6.25 shows that Uniformization implies $\mathsf{DC}_{\mathbb{R}}$. Remark 8.3 and Theorem 8.7 each show that Suslin sets are ∞-Borel. The implication from (1) to (2) then follows from Corollary 7.3. Theorem 13.1 says that (1) implies (3). The implication from (3) to (4) is covered in Remark 6.25. That (4) implies (1) under DC is Theorem 12.12. □

Many of the results listed above were stated in [**52**].

Part 1 of this book collects various facts (primarily from the Cabal seminar) which will be needed in Part 2. It is not intended as a comprehensive introduction to the topics discussed there (for which, see [**23–26**, **40**]). A natural way to read the book might be to start at the beginning of Part 2 and refer back to Part 1 as needed. Part 1 also significantly overlaps Jackson's [**11**]. Other parts of the book overlap with Jackson's [**12**], Ketchersid's [**28**] and Koellner and Woodin's [**29**]. Much of Chapters 7–11 is based on lectures given in the 1990s by Woodin at UC Berkeley, using my own notes and those taken by Joel Hamkins, and lectures given by John Steel (around the same time) at UCLA, using notes taken by Sarah Reznikoff. Chapters 12 and 13 are mostly based on more recent conversations with Woodin, along with [**2**, **9**] in the former case.

0.4 Remark. For the most part, the material in this book covers the development of the theory of AD^+ up to some point in the early 1990s. We plan to continue the story in future volumes, starting with a book on derived models and reversals. In particular, the following results of Woodin are not included in this book, although we hope to prove them in the next volume.

(1) Assuming $\mathsf{AD}^+ + V{=}L(\mathcal{P}(\mathbb{R}))$, every true statement of the form $\exists A \subseteq \omega^\omega \phi(A)$, where ϕ is a projective formula, is witnessed by an A which is Suslin and co-Suslin.

(2) AD^+ implies that the ultrapower of V by the Turing measure is wellfounded.
(3) If $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ holds, AD^+ fails, and Δ is the pointclass of Suslin, co-Suslin sets, then
   - $L(\Delta, \mathbb{R}) \models \mathsf{AD}_\mathbb{R}$,
   - $\mathcal{P}(\omega^\omega) \cap L(\Delta, \mathbb{R}) = \mathcal{P}(\omega^\omega) \cap \Delta$, and
   - $\mathcal{P}(\omega^\omega) \cap \Delta$ is the set of $A \subseteq \omega^\omega$ for which $L(A, \mathbb{R}) \models \mathsf{AD}^+$.
(4) If $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ holds and there is a largest Suslin cardinal, then AD^+ holds.
(5) If $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ holds and $A \subseteq \omega^\omega$ is Suslin, then $L(A, \mathbb{R}) \models \mathsf{AD}^+$.
(6) If $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ holds, then for each ordinal $\chi < \Theta$, every element of $\mathcal{P}(\omega^\omega)$ in $L(\mathcal{P}(\chi))$ is ∞-Borel.

The statement that the ultrapower of V by the Turing measure is wellfounded was originally one of the axioms of AD^+, before Woodin proved that it follows from the other axioms. This statement and its variants, which are used often in Chapters 10–12, are easily seen to follow from the assumption that Turing Determinacy and DC hold, which in turn follows from the assumption that Turing Determinacy and $\mathsf{DC}_\mathbb{R}$ hold, along with the statement that there is some set X with the property that every set is definable from X, a real and an ordinal. Assuming DC can then often be avoided by passing to an inner model of $\mathsf{DC}_\mathbb{R}$ in which this latter statement holds. This is the approach taken in this book, although in some instances we include the wellfoundedness of the Turing ultrapower or some other related ultrapower as an explicit assumption.

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1. Notation

We reserve the symbol $\mathbb{R}$ for the real line, which is never used directly. However, we use traditional notation such as $\mathsf{AD}_\mathbb{R}$, $\mathsf{DC}_\mathbb{R}$, $L(\mathbb{R})$ and so on, as these terms are equivalent to more relevant forms such as $\mathsf{AD}_{\omega^\omega}$, $\mathsf{DC}_{\omega^\omega}$ and $L(\omega^\omega)$. We informally use the word "real" to mean an element of the Baire space ω^ω.

We use the following notational conventions.

- We write Ord for the class of ordinals.
- We write X^Y to mean the set of functions from Y to X, and, when γ is an ordinal, $X^{<\gamma}$ to mean $\bigcup_{\alpha\in\gamma} X^\alpha$.
- For an ordinal α and a set of ordinals X, $[X]^\alpha$ denotes the collection of subsets of X of ordertype α, and $[X]^{<\alpha}$ denotes the collection of subsets of X of ordertype less than α.
- Given a set X, we write $\exists^X$ and $\forall^X$ for existential and universal quantification over X, respectively.

- A *preorder* on a set is a binary relation which is reflexive and transitive. A preorder is *wellfounded* if every nonempty subset of its domain has a minimal element.
- If $\leq$ is a wellfounded preorder on a set X, the *canonical rank function* $\text{rank}_{\leq}$ associated to $\leq$ assigns to each $x \in X$ the least ordinal α such that $\alpha > \text{rank}_{\leq}(y)$ for all $y \in X$ such that $y \leq x$ and $x \not\leq y$.
- In any preorder, especially the natural order on the ordinals, the *strict supremum* of a set (if it exists) is the least element in the domain of the order which is strictly greater than every element of the set.
- Formally, a partial order $\mathbb{P}$ is a pair $(\text{dom}(\mathbb{P}), \leq_{\mathbb{P}})$, where $\text{dom}(\mathbb{P})$ is the domain or underlying set of $\mathbb{P}$, and $\leq_{\mathbb{P}}$ is a partial order on $\text{dom}(\mathbb{P})$.
- Given a partial order $\mathbb{P}$, a *nice* $\mathbb{P}$-name for an element of ω^ω is a set τ such that
  - each element of τ is a pair $(p,(n,m))$, where $p \in \mathbb{P}$ and $n, m \in \omega$ (formally we should write $(p, \check{x})$, where x is the pair (n,m));
  - for each $p \in \mathbb{P}$ and each $n \in \omega$, there exist $q \leq p$ and $m \in \omega$ such that $(q,(n,m)) \in \tau$;
  - for all $p, p' \in \mathbb{P}$ and all $n, m, m' \in \omega$, if $(p,(n,m))$ and $(p',(n,m'))$ are both in τ, and p and p' are compatible, then $m = m'$.
- Given a nice $\mathbb{P}$-name τ and a set $g \subseteq \mathbb{P}$, we let τ_g be the set of pairs (n,m) for which there exists a $p \in g$ with $(p,(n,m)) \in \tau$.
- Given a partial order $\mathbb{P}$ and a set (or class) B, we say that a set $g \subseteq \mathbb{P}$ is *B-generic* if g intersects every dense open subset of $\mathbb{P}$ in B.
- We let $\leq_{\text{Gö}}$ be the Gödel order, i.e., the order on pairs of ordinals defined as follows : $(\alpha, \beta) \leq_{\text{Gö}} (\delta, \gamma)$ if any of the following holds:
  - $\max\{\alpha, \beta\} < \max\{\delta, \gamma\}$;
  - $\max\{\alpha, \beta\} = \max\{\delta, \gamma\}$ and $\alpha < \delta$;
  - $\max\{\alpha, \beta\} = \max\{\delta, \gamma\}$, $\alpha = \delta$ and $\beta \leq \gamma$.
- Given ordinals α and β, we write $\prec\alpha, \beta\succ$ for the ordinal ordertype of the set of predecessors of (α, β) in the order $\leq_{\text{Gö}}$.
- For a set X, we let $\text{TC}(X)$ denote the transitive closure of X.
- Given a cardinal κ, $H(\kappa)$ denotes the collection of sets of hereditary cardinality less than κ, i.e., the set of x for which $|\text{TC}(x)| < \kappa$.

We sometimes write HF for $H(\aleph_0)$. The members of HF are said to be *hereditarily finite.* A set $x \subseteq \text{HF}$ is *semi-recursive* if x is Σ_1-definable over HF, and *recursive* if x and $\text{HF} \setminus x$ are both semi-recursive. Given $A, B \subseteq \text{HF}$, we say that A is *Turing reducible* to B, and write $A \leq_{\text{Tu}} B$ if there is a semi-recursive function $f\colon \text{HF} \to \text{HF}$ such that $A = f^{-1}[B]$. We say that A is *Turing equivalent* to B if A and B are Turing-reducible to each other. We write $\leq_{\text{Tu}}$ and $\equiv_{\text{Tu}}$ respectively for the restrictions of Turing reducibility and Turing equivalence to ω^ω.

We also sometimes write HC for $H(\aleph_1)$. The members of HC are said to be *hereditarily countable.* We say that a set $x \in \omega^\omega$ HC-*codes* a set $y \in \text{HC}$ if the structure

$$(\{i \in \omega : x(2^i) = 0\}, \{(j,k) \in \omega \times \omega : x(2^j) = x(2^k) = 0, x(3^j 5^k) = 0\})$$

is isomorphic to $(\text{TC}(\{y\}), \in)$.

2. Prerequisites

We expect the reader to be familiar with Zermelo-Fraenkel set theory, Gödel numbering of formulas, relative constructibility, ordinal definability, inner models of the form HOD_X, ultrapowers, sharps and forcing, along with some elementary notions from descriptive set theory (e.g., Borel sets, projective sets and tree representations), all of which can be found in [**14**, **31**, **44**]. Chapter 1 contains a quick review of the classical theory of determinacy, which can be found in more detail in [**14**, **16**]. We do not expect the reader to be familiar with everything in the books [**14**, **16**, **20**, **31**, **40**, **44**], but they make good references.

3. Forms of Choice

Our base theory in this book is Zermelo-Fraenkel set theory (ZF). Additional axioms will be stated as used. Although we will sometimes consider models of the Axiom of Choice (AC), our main interest is in models of the Axiom of Determinacy, which contradicts AC. Weak forms of AC can (and do) hold in models of AD, however.

Given a set X, the principle of Dependent Choice for X (DC_X) is the statement that whenever $T \subseteq X^{<\omega}$ is a nonempty tree (i.e., a subset of $X^{<\omega}$ closed under initial segments) with the property that every element of T has a proper extension in T, there exists an infinite path through T. The principle of Countable Choice for X (CC_X) is the statement that for all countable sets Y, if $\langle A_y : y \in Y\rangle$ is a sequence of nonempty subsets of X, then there exists a function $f\colon Y \to X$ such that $f(y)$ is in A_y, for each $y \in Y$. The principle CC_X follows immediately from DC_X. A classical argument (due to Mycielski; see Remark 1.2) shows that $\mathsf{CC}_{\mathbb{R}}$ follows from AD. Whether or not $\mathsf{DC}_{\mathbb{R}}$ follows from AD is an open question. Note however that if $\mathsf{DC}_{\mathbb{R}}$ holds, then any inner model containing $\mathcal{P}(\omega)$ satisfies $\mathsf{DC}_{\mathbb{R}}$.

The following theorem is part of the result that, in $L(\mathbb{R})$, AD implies AD^+.

THEOREM 0.5 (Kechris [**18**]). *Assuming $V{=}L(\mathbb{R})$, AD implies $\mathsf{DC}_{\mathbb{R}}$.*

The axiom of Dependent Choice (DC) asserts that DC_X holds for every set X. Similarly, the axiom of Countable Choice (CC) asserts that CC_X holds for every set X.

0.6 Remark. We make frequent use of the following standard facts (for all sets X and Y).

- If there is surjection from X to Y, then DC_X implies DC_Y.
- If DC_X holds and α is an ordinal, then $\mathsf{DC}_{X\times\alpha}$ holds.
- If every set is definable from an ordinal and a member of X, then DC_X implies DC.

4. Partial orders

We list here some classical partial orders which are used as forcing notions throughout the book.

- $\mathrm{Col}(\kappa, X)$, where κ is an infinite cardinal and X is a set. Conditions are functions $f\colon \alpha \to X$, where $\alpha < \kappa$. The order is extension.
- $\mathrm{Col}^*(\kappa, X)$, where κ is an infinite cardinal and X is a set. Conditions are injective functions $f\colon \alpha \to X$, where $\alpha < \kappa$. The order is extension.

- Given a set X consisting of infinite subsets of ω, the classical *almost-disjoint coding* forcing for X (due to Jensen and Solovay [**15**]) consists of pairs (a, B), where a is a finite subset of ω and B is a finite subset of X, with the order $(a, B) \leq (c, D)$ if
  - c is either the emptyset or $a \cap (\max(c) + 1)$,
  - $D \subseteq B$, and
  - $(a \setminus c) \cap r = \emptyset$ for all $r \in D$.

  This partial order is c.c.c. and adds a subset of ω having finite intersection with each member of X and infinite intersection with each element of $\mathcal{P}(\omega)$ not contained mod-finite in the union of a finite subset of X.