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IMPROVEMENT OF THE BERNSTEIN-TYPE THEOREM FOR SPACE-LIKE ZERO MEAN CURVATURE GRAPHS IN LORENTZ-MINKOWSKI SPACE USING FLUID MECHANICAL DUALITY

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ABSTRACT. Calabi's Bernstein-type theorem asserts that *a zero mean curvature entire graph in Lorentz-Minkowski space* $\boldsymbol{L}^3$ *which admits only space-like points is a space-like plane*. Using the fluid mechanical duality between minimal surfaces in Euclidean 3-space $\boldsymbol{E}^3$ and maximal surfaces in Lorentz-Minkowski space $\boldsymbol{L}^3$, we give an improvement of this Bernstein-type theorem. More precisely, we show that *a zero mean curvature entire graph in* $\boldsymbol{L}^3$ *which does not admit time-like points* (*namely, a graph consists of only space-like and light-like points*) *is a plane*.

1. INTRODUCTION

Consider a 2-dimensional barotropic steady flow on a simply connected domain D in the xy-plane $\boldsymbol{R}^2$ whose velocity vector field is $\boldsymbol{v} = (u, v)$, with density ρ and pressure p. We assume there are no external forces. Then

- the flow is a foliation of the integral curve of $\boldsymbol{v}$,
- ρ is a scalar field on D,
- $p\colon \boldsymbol{R} \to \boldsymbol{R}$ is a monotone function of ρ,
- $c := \sqrt{p'(\rho)}$ ($p' := dp/d\rho$) is called the *local speed of sound*.
- The following Euler's equation of motion holds:

$$dp + \frac{\rho}{2}\, d(|\boldsymbol{v}|^2) = 0. \tag{1.1}$$

We also assume the flow is *irrotational*; that is,

$$0 = \mathrm{rot}(\boldsymbol{v}) = v_x - u_y, \tag{1.2}$$

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where $v_x := \partial v/\partial x$, $u_y := \partial u/\partial y$. Here, 'the equation of continuity' is equivalent to the fact that

$$0 = \operatorname{div}(\rho \boldsymbol{v}) = (\rho u)_x + (\rho v)_y. \tag{1.3}$$

By (1.2), there exists a function $\Phi\colon D \to \boldsymbol{R}$, called the *potential* of the flow, such that $\nabla\Phi = \boldsymbol{v}$, where $\nabla\Phi := (\Phi_x, \Phi_y)$. Since p is a function of ρ, the fact $c^2 = p'(\rho)$ and (1.1) yield that

$$\rho_x = -\frac{\rho(uu_x + vv_x)}{c^2}, \quad \rho_y = -\frac{\rho(uu_y + vv_y)}{c^2}. \tag{1.4}$$

By (1.3), one can easily check that

$$0 = (c^2 - \Phi_x^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (c^2 - \Phi_y^2)\Phi_{yy}. \tag{1.5}$$

On the other hand, by (1.3), there exists a function $\Psi\colon D \to \boldsymbol{R}$, called the *stream function* of the flow, such that

$$\Psi_x = -\rho v, \qquad \Psi_y = \rho u. \tag{1.6}$$

If we set $\xi := \rho u$ and $\eta := \rho v$, (1.4) can be written as

$$(\rho^2c^2 - \xi^2 - \eta^2)(\rho_x, \rho_y) = -\rho(\xi\xi_x + \eta\eta_x, \xi\xi_y + \eta\eta_y).$$

Since

$$0 = v_x - u_y = \frac{\eta_x}{\rho} - \frac{\xi_y}{\rho} - \frac{\eta\rho_x}{\rho^2} + \frac{\xi\rho_y}{\rho^2},$$

the identity $0 = \rho(\xi^2 + \eta^2 - \rho^2c^2)(v_x - u_y)$ yields that

$$0 = (\rho^2c^2 - \Psi_y^2)\Psi_{xx} + 2\Psi_x\Psi_y\Psi_{xy} + (\rho^2c^2 - \Psi_x^2)\Psi_{yy}. \tag{1.7}$$

A flow satisfying

$$\rho c = 1 \tag{1.8}$$

is called a *Chaplygin gas flow* (see [4, p. 24] and also [11, Section 4]). For a given stream function $\Psi\colon D \to \boldsymbol{R}$ of the Chaplygin gas flow, we set

$$B_\Psi := 1 - \Psi_x^2 - \Psi_y^2. \tag{1.9}$$

Let D be a domain in the uv-plane $\boldsymbol{R}^2$. Let $f\colon D \to \boldsymbol{L}^3$ be an immersion into the Lorentz-Minkowski 3-space $\boldsymbol{L}^3$ of signature $(++-)$. We set

$$P := \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix}$$

and

$$B_f := \det(P),$$

where $\cdot$ denotes the canonical Lorentzian inner product of $\boldsymbol{L}^3$ and $\det(P)$ denotes the determinant of the 2×2 matrix P. A point $p \in U$ where $B_f(p) > 0$ (resp., $B_f(p) < 0$, $B_f(p) = 0$) is said to be *space-like* (resp., *time-like*, *light-like*). We set

$$Q := \begin{pmatrix} f_{uu} \cdot \tilde\nu & f_{uv} \cdot \tilde\nu \\ f_{vu} \cdot \tilde\nu & f_{vv} \cdot \tilde\nu \end{pmatrix},$$

where $\tilde\nu := f_u \times_L f_v$ and $\times_L$ is the canonical Lorentzian vector product of $\boldsymbol{L}^3$. Consider the matrix $W := \tilde{P}Q$ and set

$$A_f := \operatorname{trace}(W),$$

where $\tilde{P}$ is the cofactor matrix of P. We call f a *zero mean curvature surface* if A_f vanishes identically. In this paper, for the sake of simplicity, we abbreviate '**zero mean curvature**' by '**ZMC**'. A ZMC-surface consisting only of space-like points is called a *maximal surface*. On the other hand, a surface in $\boldsymbol{L}^3$ consisting only of light-like points is called a *light-like surface*. It is known that the identity $B_f = 0$ implies that $A_f = 0$ (see [21, Proposition 2.1]). In particular, any light-like surfaces are ZMC-surfaces in our sense. Moreover, at a point where $B_f \neq 0$, the mean curvature function H of f is well-defined, and $A_f = 0$ is equivalent to the condition that $H = 0$.

We now assume that f is written in the form $f(x, y) = (x, y, \Psi(x, y))$. Then it can be easily checked that $B_f = B_\Psi$ (cf. (1.9)) and

$$A_f(x, y) = (1 - \Psi_y^2)\Psi_{xx} + 2\Psi_x\Psi_y\Psi_{xy} + (1 - \Psi_x^2)\Psi_{yy}.$$

Under the condition (1.8), the equation (1.7) for the stream function Ψ reduces to

$$(1 - \Psi_y^2)\Psi_{xx} + 2\Psi_x\Psi_y\Psi_{xy} + (1 - \Psi_x^2)\Psi_{yy} = 0, \tag{1.10}$$

which implies that A_f vanishes identically. So we call this the *ZMC-equation* in $\boldsymbol{L}^3$. If $\rho c = 1$, then we have $1/\rho^2 = c^2 = dp/d\rho$; that is, $dp = d\rho/\rho^2$ is obtained. Substituting this into (1.1), we get $d(|\boldsymbol{v}|^2 - 1/\rho^2) = 0$, and so there exists a constant μ such that

$$|\boldsymbol{v}|^2 + \mu = \frac{1}{\rho^2}(= c^2). \tag{1.11}$$

By (1.6), we can rewrite this as

$$B_\Psi = \mu\rho^2. \tag{1.12}$$

By (1.11) and (1.12), the sign change of B_Ψ corresponds to the type change of the Chaplygin gas flow from sub-sonic ($|\boldsymbol{v}| < c$) to super-sonic ($|\boldsymbol{v}| > c$); that is, the sub-sonic part satisfies $B_\Psi > 0$. If $\mu = 0$, then B_Ψ vanishes identically, and the graph of Ψ gives a light-like surface. Such surfaces are discussed in the appendix, and we now consider the case $\mu \neq 0$. Since B_Ψ and μ have the same sign (cf. (1.12)), we can write

$$\rho = \frac{1}{\sqrt{|\boldsymbol{v}|^2 + \mu}} = \sqrt{\frac{1 - \Psi_x^2 - \Psi_y^2}{\mu}}. \tag{1.13}$$

By (1.11) and the fact that $|\boldsymbol{v}|^2 = \Phi_x^2 + \Phi_y^2$, (1.5) can be written as

$$(\mu + \Phi_y^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (\mu + \Phi_x^2)\ \Phi_{yy} = 0. \tag{1.14}$$

We set

$$\varphi(x, y) := \tilde{\mu}\Phi(\tilde{\mu}x, \tilde{\mu}y) \qquad (\tilde{\mu} := 1/\sqrt[4]{|\mu|}). \tag{1.15}$$

If $\mu > 0$, then (1.14) reduces to

$$(1 + \varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1 + \varphi_x^2)\varphi_{yy} = 0, \tag{1.16}$$

which is known as the condition that the graph of $\varphi(x, y)$ gives a minimal surface in the Euclidean 3-space $\boldsymbol{E}^3$. On the other hand, if $\mu < 0$, then (1.14) reduces to

$$(1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy} = 0, \tag{1.17}$$

which is the ZMC-equation (cf. (1.10)). It can be easily checked that the graph of φ is a time-like ZMC-surface in $\boldsymbol{L}^3$. In both of the two cases, it can be easily checked that ($\epsilon := \mathrm{sign}(\mu) \in \{1, -1\}$)

$$\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \frac{1}{\sqrt{\varphi_x^2 + \varphi_y^2 + \epsilon}} \begin{pmatrix} -\varphi_y \\ \varphi_x \end{pmatrix}$$

holds, where $\psi := \Psi(\tilde{\mu}x, \tilde{\mu}y)/\tilde{\mu}$. Note that Ψ satisfies (1.10) if and only if ψ satisfies (1.10). Moreover, one can easily check that

$$(\hat{\rho} :=) \frac{1}{\sqrt{\varphi_x^2 + \varphi_y^2 + \epsilon}} = \sqrt{\epsilon(1 - \psi_x^2 - \psi_y^2)} \tag{1.18}$$

and

$$\begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix} = \frac{1}{\sqrt{\epsilon(1 - \psi_x^2 - \psi_y^2)}} \begin{pmatrix} \psi_y \\ -\psi_x \end{pmatrix}.$$

This means that $\varphi \longleftrightarrow \psi$ corresponds to the duality between potentials and stream functions of Chaplygin gas flows such that

- $\mu = \pm 1 (= \epsilon)$,
- the density $\hat{\rho}$ is given as (1.18), and
- $p = p_0 - 1/\hat{\rho}$ for some constant p_0.

When $\epsilon = 1$ (resp., $\epsilon = -1$), this gives a correspondence between graphs of minimal surfaces $(x, y) \mapsto \varphi(x, y)$ in $\boldsymbol{E}^3$ and graphs of maximal surfaces $(x, y) \mapsto \psi(x, y)$ in $\boldsymbol{L}^3$ (resp., an involution on the set of graphs of time-like ZMC-surfaces in $\boldsymbol{L}^3$) which we call the *fluid mechanical duality.*

A part of the above dualities is suggested in the classical book [4]. Calabi [5] also recognized this duality for $\mu > 0$ and pointed out the following:

Fact 1.1 (Calabi's Bernstein-type theorem)**.** Suppose that the graph of a function $\psi\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ gives a maximal surface (that is, a surface consisting only of space-like points whose mean curvature function vanishes identically). Then $\psi - \psi(0, 0)$ is linear.

This is an analogue of the classical Bernstein theorem for minimal surfaces in $\boldsymbol{E}^3$. Moreover, Calabi [5] obtained the same conclusion for entire space-like ZMC-graphs in $\boldsymbol{L}^{n+1}$ ($n \leq 4$), and Cheng and Yau [6] extended this result for complete maximal hypersurfaces in $\boldsymbol{L}^{n+1}$ for $n \geq 5$. The assumption that the graph consists only of space-like points is crucial. Entire ZMC-graphs which are not planar actually exist. Typical such examples are of the form

$$\psi_0(x, y) := y + g(x), \tag{1.19}$$

where $g\colon \boldsymbol{R} \to \boldsymbol{R}$ is any C^∞-function of one variable. A point $p = (x_0, y_0) \in \boldsymbol{R}^2$ is a light-like point of ψ_0 if and only if $g'(x_0) = 0$. Moreover, if the graph of ψ_0 does not contain any light-like points, the potential function φ_0 corresponding to ψ_0 is given by

$$\varphi_0(x, y) = \pm \left(-y + \int_0^x \frac{du}{g'(u)} \right)$$

up to a constant, where the sign "$\pm$" coincides with that of g'. On the other hand, Osamu Kobayashi [18] pointed out the existence of entire graphs of ZMC-surfaces with space-like points, light-like points, and time-like points all appearing. Such a surface is called of *mixed type*. Recently, many such examples were constructed in [9].

By definition, any entire ZMC-graph of mixed type has at least one light-like point. So we give the following definition.

Definition 1.2. A light-like point p of the function ψ (i.e., $B_\psi(p) = 0$) is said to be *non-degenerate* (resp., *degenerate*) if ∇B_ψ does not vanish (resp., vanishes) at p.

At each non-degenerate light-like point, the graph of ψ changes its causal type from space-like to time-like. This case is now well understood. In fact, under the assumption that the surface is real analytic, it can be reconstructed from a real analytic null regular curve in $\boldsymbol{L}^3$ (cf. Gu [12] and also [11, 16, 17]).

On the other hand, there are several examples of ZMC-surfaces with degenerate light-like points (cf. [1, 2, 10, 14]). Moreover, a local general existence theorem for maximal surfaces with degenerate light-like points is given in [21]. For such degenerate light-like points, we need a new approach to analyze the behavior of ψ and φ. The following fact was proved by Klyachin [17] (see also [21]).

Fact 1.3 (The line theorem for ZMC-surfaces)**.** Let D be a domain of $\boldsymbol{R}^2$ and let $F\colon D \to \boldsymbol{L}^3$ be a C^3-differentiable ZMC-immersion such that $o \in D$ is a degenerate light-like point. Then, there exists a light-like line segment $\hat\sigma\,(\subset \boldsymbol{L}^3)$ passing through $F(o)$ of $\boldsymbol{L}^3$ such that $F(o)$ does not coincide with one of the two end points of $\hat\sigma$ and $F(\Sigma)$ contains $\hat\sigma$, where Σ is the set of degenerate light-like points of F.

Recently, Fact 1.3 was generalized to a much wider class of surfaces, including constant mean curvature surfaces in $\boldsymbol{L}^3$; see [21, 22]. (In [21], the general local existence theorem of surfaces which changes their causal types along degenerate light-like lines was also shown.) The asymptotic behavior of ψ along the line l consisting of degenerate light-like points is discussed in [21].

The purpose of this paper is to prove the following assertion:

Theorem A. *An entire C^3-differentiable ZMC-graph which is not a plane admits a non-degenerate light-like point if its space-like part is non-empty.*

This assertion is proved in Section 2 using the fluid mechanical duality and the half-space theorem for minimal surfaces in $\boldsymbol{E}^3$ given by Hoffman-Meeks [15]. It should be remarked that the half-space theorem does not hold for time-like ZMC-surfaces. In fact, the graph of $\varphi(x, y) := y + \log(\tan x)$ $(x \in (0, \pi/2))$ gives a properly embedded time-like ZMC-surface lying between two parallel vertical planes. In Section 2, we give further examples and provide a few questions related to Theorem A. As an application, we give the following improvement of Calabi's Bernstein-type theorem:

Corollary B. *An entire C^3-differentiable ZMC-graph which does not admit any time-like points is a plane.*

In fact, if the ZMC-graph admits a space-like point, then the assertion immediately follows from Theorem A. So it remains to show the case that the graph consists only of light-like points. However, such a graph must be a plane, as shown in the appendix (see Theorem A.1).

2. Proof of Theorem A

In this section, we prove Theorem A in the introduction. We let $\psi\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ be a C^3-function satisfying the ZMC-equation (1.10). We assume ψ admits a space-like point $q_0 \in \boldsymbol{R}^2$ but admits no non-degenerate light-like points. By Calabi's Bernstein-type theorem (cf. Fact 1.1), ψ has at least one degenerate light-like point. We set

$$F_\psi(x, y) := (x, y, \psi(x, y)),$$

which gives the ZMC-graph of ψ. We denote by ds^2 the positive semi-definite metric which is the pull-back of the canonical Lorentzian metric of $\boldsymbol{L}^3$ by F_ψ. The line theorem (cf. Fact 1.3) yields that the image of F_ψ contains a light-like line segment $\hat{\sigma}$. Then the projection of $\hat{\sigma}$ is a line segment σ on the xy-plane $\boldsymbol{R}^2$. Then σ lies on a line l on $\boldsymbol{R}^2$. If $\sigma \neq l$, then there exists an end point p of σ on l. Since p is the limit point of degenerate light-like points, p itself is also a degenerate light-like point. By applying the line theorem again, there exists a light-like line segment $\hat{\sigma}'$ containing $F_\psi(p)$ as its interior point. We denote by σ' the projection of $\hat{\sigma}'$ to the xy-plane. Since the null direction at p with respect to the metric ds^2 is uniquely determined, σ' also lies on the line l. Thus, the entire graph contains a whole light-like line containing $\hat{\sigma}$. In particular, degenerate light-like points on the graph consist of a family of straight lines in $\boldsymbol{R}^2$.

Let l and l' be two such straight lines. Then l' never meets l. In fact, if not, then there is a unique intersection point $q \in l \cap l'$. By Fact 1.3, two lines l, l' can be lifted to two light-like lines $\tilde{l}$ and $\tilde{l}'$ in $\boldsymbol{L}^3$ passing through $F_\psi(q)$. The tangential directions of $\tilde{l}$ and $\tilde{l}'$ are linearly independent light-like vectors at $F_\psi(q)$. Then by [19, Lemma 27 in Section 5], q is a time-like point, a contradiction.

Thus, the set of degenerate light-like points of F_ψ consists of a family of parallel lines in the xy-plane. Without loss of generality, we may assume that these lines are vertical and one of them is the y-axis. Then we can find a domain ($\Delta \in (0, \infty]$)

$$\Omega := \{(x, y)\,;\, 0 < x < 2\Delta\}$$

such that $q_0 \in \Omega$ and F_ψ has no light-like points on Ω and both of the lines $l = \{x = 0\}$ and $l' = \{x = 2\Delta\}$ consist of light-like points unless $\Delta = \infty$. Since there are no light-like points on Ω, the potential function $\varphi\colon \Omega \to \boldsymbol{R}$ is induced by ψ as the fluid mechanical dual. The graph of φ is a minimal surface in $\boldsymbol{E}^3$. In particular, φ is real analytic. If we succeed in proving that the map $F_\varphi(x, y) := (x, y, \varphi(x, y))$ is proper, then Theorem A follows. In fact, by the half-space theorem given in [15] the image $F_\varphi(\Omega)$ lies in a plane in $\boldsymbol{E}^3$. Then the map $F_\psi(x, y)$ also lies in a plane Π in $\boldsymbol{L}^3$ on $\overline{\Omega}$. Since $F_\psi(l)$ is light-like, the plane Π must be light-like, contradicting the fact that $q_0 \in \Omega$.

To prove the properness of F_φ, it is sufficient to show the following:

Lemma 2.1. *Let $\{p_n\}_{n=1}^\infty$ be a sequence of points in Ω accumulating to a point on l or l'. Then $\{|\varphi(p_n)|\}_{n=1}^\infty$ diverges.*

Proof. By switching the roles of l and l' if necessary, it is sufficient to consider the case that $\{p_n\}_{n=1}^\infty$ accumulates to a point on l. Taking a subsequence and using a suitable translation of the xy-plane, we may assume that $\{p_n\}_{n=1}^\infty$ converges to the origin $(0, 0) \in l$ and $p_n = (x_n, y_n)$ ($n = 1, 2, 3, \ldots$) satisfies the following properties:

- there exists $\epsilon > 0$ such that $|y_n| < \epsilon$ for each $n = 1, 2, \ldots$, and

- there exists $(\delta, 0) \in \Omega$ $(\delta > 0)$ such that

$$\delta > x_1 > x_2 > \cdots > x_n > x_{n+1} > \cdots .$$

Since l consists of degenerate light-like points, there exists a neighborhood U of $(0,0)$ such that (see [10] or [21, (6.1)])

$$\psi(x,y) = y + x^2 h(x,y) \qquad ((x,y) \in U),$$

where $h(x,y)$ is a C^1-differentiable function defined on U (see [21, Appendix A]). Taking ϵ, δ to be sufficiently small, we may assume that

$$V := \{(x,y) \in \Omega\,;\, |x| \le \delta,\ |y| < \epsilon\} \subset U.$$

Since $B_\psi > 0$, the potential function φ associated to ψ satisfies (cf. (1.18))

$$\varphi_x = \frac{\psi_y}{\rho}, \qquad \rho = \sqrt{1 - \psi_x^2 - \psi_y^2}.$$

Since

$$1 - \psi_x^2 - \psi_y^2 = -x^2\Big((2h + xh_x)^2 + 2h_y + x^2 h_y^2\Big)$$

is non-negative on the closure $\overline{V}$ of V, we can write

$$\sqrt{\rho} = |x| k(x,y), \tag{2.1}$$

where $k(x,y)$ is a non-negative continuous function defined on $\overline{V}$ such that k is positive-valued on V. We set $p_0 := (\delta, 0)$ and consider the path $\gamma_n \colon [0,1] \to V$ defined by $\gamma_n(s) := (\delta, 2sy_n)$ if $0 \le s \le 1/2$ and

$$\gamma_n(s) := (2(x_n - \delta)s - x_n + 2\delta, y_n)$$

if $1/2 \le s \le 1$, which starts at p_0 and terminates at p_n. This curve γ_n is the union of the vertical subarc $\gamma_{n,1}$ and the horizontal subarc $\gamma_{n,2}$. So we can write

$$\begin{aligned}\varphi(p_n) - \varphi(p_0) &= \int_{\gamma_n} \varphi_x dx + \varphi_y dy \\ &= \int_{\gamma_{n,2}} \varphi_x dx + \int_{\gamma_{n,1}} \varphi_y dy.\end{aligned}$$

Since $[-\epsilon, \epsilon] \ni y \mapsto \varphi_y(\delta, y) \in \boldsymbol{R}$ is a continuous function, we have that

$$\begin{aligned}\left|\int_{\gamma_{n,1}} \varphi_y dy\right| &\le \int_{\gamma_{n,1}} \Big|\varphi_y(\delta, 2ty_n)\Big|\, |dy| \\ &\le \epsilon \max_{|y| \le \epsilon} \Big|\varphi_y(\delta, y)\Big| < \infty.\end{aligned}$$

So to prove the lemma, it is sufficient to show that $\int_{\gamma_{n,2}} \varphi_x dx$ diverges as $n \to \infty$. We set

$$m := \max_{x \in [0,\delta],\ |y| \le \epsilon} k(x,y) \ (\ge 0),$$

where k is the continuous function given in (2.1). On the other hand, we can take a constant $m' (> 0)$ such that

$$\psi_y = 1 + x^2 h_y(x,y) > m' \qquad (x \in [0,\delta],\ |y| \le \epsilon),$$

since ϵ, δ can be chosen to be sufficiently small. Since $\varphi_x = \psi_y/\rho$, we have

$$\left|\int_{\gamma_{n,2}} \varphi_x dx\right| = \int_{x_n}^{\delta} \frac{1 + x^2 h_y(x,y)}{x^2 k^2(x,y)} dx$$
$$> \frac{m'}{m^2} \int_{x_n}^{\delta} \frac{dx}{x^2} = \frac{m'}{m^2}\left(\frac{1}{x_n} - \frac{1}{\delta}\right) \to \infty,$$

proving the assertion. □

Remark 2.2. In the above proof, we showed that $F_\psi(\Omega)$ lies in a plane using the fluid mechanical duality. We remark here that this can be proved by a different method. In fact, ψ satisfies the assumption of Ecker [7, Theorem G] or is a PS-graph on the convex domain Ω in the sense of Fernandez and Lopez [8]. Thus, we can conclude that $\psi(\Omega)$ lies in a light-like plane.

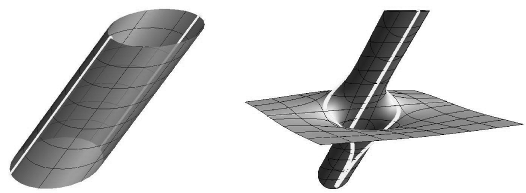

FIGURE 1. The ZMC-surfaces in Example 2.3 (left) and in Example 2.4 (right), where the white lines indicate light-like points.

In [1], the first author constructed several ZMC-surfaces foliated by circles and at most countably many straight lines. At the end of this paper, we pick up two important examples of them which contain degenerate light-like points. (In [1], these two examples are not precisely indicated. Here we show their explicit parametrization and embeddedness.)

Example 2.3 ([1, Figure 5]). We set

$$F(u,v) := (u + a\cos v, a\sin v, u),$$

where $a > 0$ and $(u,v) \in \boldsymbol{R} \times [0, 2\pi)$. Then the image of F contains two parallel degenerate light-like lines which correspond to the special values $\theta = \pm\pi/2$ (see Figure 1, left). The image of F can be characterized by the implicit function $(x-t)^2 + y^2 = a^2$. This ZMC-surface is properly embedded and is not simply connected.

Example 2.4 ([1, Figure 2]). We set

$$F(r,\theta) := \left(r + \frac{1}{2a}\log\left(\frac{ar-1}{ar+1}\right) + r\cos\theta, r\sin\theta, \frac{1}{2a}\log\left(\frac{ar-1}{ar+1}\right)\right),$$

where $a > 0$ and $\theta \in [0, 2\pi)$. This map is defined for $r > 1/a$ or $r < -1/a$, and the closure of the image of $F = (x, y, t)$ can be expressed as

$$(\Psi :=) a\sinh(at)\left((x-t)^2 + y^2\right) + 2(x-t)\cosh(at) = 0.$$

It can be checked that (Ψ_x, Ψ_y, Ψ_t) never vanishes along $\Psi = 0$. So the closure of F gives a properly embedded ZMC-surface in $\boldsymbol{L}^3$ (see Figure 1, right).

Regarding our main result, we state a few open problems:

Question 1. Does a properly embedded ZMC-surface which consists only of space-like or light-like points coincide with a plane?

If this question is affirmative, then Corollary B follows as a corollary. Suppose that we can find such a non-planar ZMC-surface S; it must contain a light-like line. In fact, if S consists only of space-like points, then S is complete, and such a surface must be a plane (see [20, Remark 1.2]). So S has a light-like point p. If p is non-degenerate, then S has a time-like point near p, so p must be degenerate. By the line theorem (Fact 1.3), S must contain a light-like line consisting of degenerate light-like points.

Question 2. Are there entire ZMC-graphs of mixed type containing degenerate light-like points?

This question needs to consider ZMC-graphs of mixed type. In fact, if we choose a function $g(x)$ satisfying $g'(0) = 0$ as in (1.19), then the y-axis consists of the degenerate light-like points. If we weaken 'entire ZMC-graphs' to 'properly embedded ZMC-surfaces of mixed type' the answer is 'yes'. In fact, Example 2.4 gives a properly embedded ZMC-surface of mixed type which contains a degenerate light-like line L. Although the space-like points never accumulate to L in the case of this example, one can show the existence of a function $\psi : U \to \boldsymbol{R}$ defined on a domain U in $\boldsymbol{R}^2$ containing the y-axis such that

- the y-axis corresponds to a degenerate light-like line,
- ψ is of mixed type or consists only of space-like points except along the y-axis.

See [3] for details. Also, the following question arises:

Question 3. Are there entire ZMC-graphs of mixed type which are not obtained as analytic extensions of Kobayashi surfaces given as in [9]?

In fact, all known examples of entire ZMC-graphs of mixed type are obtained as analytic extensions of Kobayashi surfaces (cf. [9]), and they admit only non-degenerate light-like points.

Appendix A. A property of light-like surfaces in $\boldsymbol{L}^3$

It can be easily checked that an embedded surface $S(\subset \boldsymbol{L}^3)$ is light-like if and only if the restriction of the canonical Lorentzian metric on $\boldsymbol{L}^3$ to the tangent space T_pS of each $p \in S$ is positive semi-definite but not positive definite. The purpose of this appendix is to prove the following:

Theorem A.1. *If an entire C^2-differentiable graph of $\psi\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ gives a light-like surface in $\boldsymbol{L}^3$, then $\psi - \psi(0,0)$ is a linear function.*

Proof. We set $F(x,y) = (x, y, \psi(x,y))$. Since F is a light-like surface, $\psi_x^2 + \psi_y^2 = 1$ holds on $\boldsymbol{R}^2$. Differentiating this with respect to x and y, we get two equations. Since F is light-like, $(\psi_x, \psi_y) \neq (0,0)$. By thinking ψ_x, ψ_y are unknown variables of these two equations, the determinant $\psi_{xx}\psi_{yy} - \psi_{xy}^2$ vanishes identically. So the Gaussian curvature of F with respect to the Euclidean metric of $\boldsymbol{R}^3$ vanishes identically. Then, by the Hartman-Nirenberg cylinder theorem, F must be a cylinder. (The proof of the cylinder theorem in [13] needs only C^2-differentiability.) That is, there exist a non-zero vector $\boldsymbol{a}$, a plane Π which is not parallel to $\boldsymbol{a}$, and a regular curve $\gamma\colon \boldsymbol{R} \to \Pi$ such that $F(u,v) := \gamma(u) + v\boldsymbol{a}$ gives a new parametrization of F. If F is not a plane, there exists $u_0 \in \boldsymbol{R}$ such that $\gamma'(u_0)$ and $\gamma''(u_0)$ are linearly independent. Then the point $(u,v) = (u_0, 0)$ is not an umbilical point of F. Since the asymptotic direction is uniquely determined at each non-umbilical point on a flat surface, the line theorem (cf. Fact 1.3) yields that $\boldsymbol{a}$ is a light-like vector. By a suitable homothetic transformation and an isometric motion in $\boldsymbol{L}^3$, we may set $\boldsymbol{a} := (1, 0, 1)$. Then it holds that

$$0 = \gamma' \cdot \boldsymbol{a} = x' - t'. \tag{A.1}$$

Since $\gamma' \cdot \gamma' = 0$, we have $y' = 0$. So, without loss of generality, we may assume that $y(u) = 0$. Differentiating (A.1), we have $x'' - t'' = 0$, contradicting the fact that $\gamma'(u_0)$ and $\gamma''(u_0)$ are linearly independent. Thus F is a plane. □

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