

CONFORMAL AND CR MAPPINGS ON CARNOT GROUPS

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ABSTRACT. We consider a class of stratified groups with a CR structure and a compatible control distance. For these Lie groups we show that the space of conformal maps coincide with the space of CR and anti-CR diffeomorphisms. Furthermore, we prove that on products of such groups, all CR and anti-CR maps are product maps, up to a permutation isomorphism, and affine in each component. As examples, we consider free groups on two generators, and show that these admit very simple polynomial embeddings in \mathbb{C}^N that induce their CR structure.

1. INTRODUCTION

In this article, we consider the interplay between metric and complex geometry on some model manifolds. This is the first outcome of a larger project which aims to develop a unified theory of conformal and CR structures on the one hand, and to define explicit polynomial embeddings of certain CR manifolds into \mathbb{C}^n on the other. This kind of embedding is what permits the detailed analysis of the ∂_b operator carried out by Stein and his collaborators since the 1970s (see [6]). The analogy between CR and conformal geometry is well documented in the case of CR manifolds of hypersurface-type, see, e.g., [7, 10–14, 17]. The easiest and perhaps the most studied example in this setting is that of the Heisenberg group, taken with its sub-Riemannian structure. Here we will focus on those CR manifolds that are stratified groups and that admit a control metric compatible with the CR structure in a suitable sense.

The class of stratified groups that we consider have a particular algebraic structure, which we call *tight*. These are the indecomposable examples that mimic the Heisenberg group, in the sense that the metric and algebraic structures are very closely tied together. It turns out that the only tight stratified groups are either Heisenberg groups or groups whose Lie algebras have two generators (Corollary 3.2). Tight groups may be endowed with an abstract CR structure. We will show that

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the space of CR and anti-CR automorphisms coincides with the space of conformal maps with respect to a compatible control metric (Theorem 3.3). Hence, we consider the problem of realising our spaces as embedded manifolds. The fact that these groups embed as CR submanifolds of \mathbb{C}^n for appropriate n is a consequence of [1]; see also [9]. We will find explicit embeddings in the cases of free Lie groups with two generators and step at most 8 (Theorem 4.1). Further, we will show that, on products of tight groups, all CR maps are product maps, up to a permutation isomorphism, and are affine in each component, that is, the composition of a translation with a group automorphism (Theorem 5.5). In order to achieve this, we first show that differentiable quasiconformal mappings on product stratified groups are product mappings, up to an automorphic permutation (Theorem 5.1). The last result is a minor variation of [19, Theorem 1.1].

Here is what follows. In Section 2, we establish the definitions and properties of Carnot groups and conformal mappings that will be used throughout this paper. In Section 3, we introduce CR structures on stratified groups and define a compatible metric when the groups are tight. In this section we prove one of our main results, Theorem 3.3, and we ask whether we can see these CR groups as boundaries of domains in some \mathbb{C}^n . This is equivalent to constructing explicit embeddings in some \mathbb{C}^n , which is in turn equivalent to solving a system of differential equations. In Section 4 we find the explicit embeddings for the case of free nilpotent groups with two generators up to step 8, by solving the differential equations of the previous section. Finally, in Section 5.5, we prove our result about product groups, Theorem 5.5, which is a consequence of Theorem 3.3 and Corollary 5.3.

2. PRELIMINARIES

In this section, we define stratified Lie algebras and Lie groups, and show how to put sub-Riemannian structures on these. We also define the Pansu derivative and consider quasiconformal mappings for these structures.

2.1. Stratified Lie algebras. A Lie algebra \mathfrak{g} is said to be *stratified of step ℓ* if

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-\ell},$$

where $[\mathfrak{g}_{-j}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-j-1}$ when $1 \leq j \leq \ell$, while $\mathfrak{g}_{-\ell} \neq \{0\}$ and $\mathfrak{g}_{-\ell-1} = \{0\}$; this implies that \mathfrak{g} is nilpotent. We assume that $\dim(\mathfrak{g})$ is at least 3 to avoid degenerate cases.

We write π_j for the canonical projection of \mathfrak{g} onto \mathfrak{g}_{-j} , $\mathfrak{Z}(\mathfrak{g})$ for the centre of \mathfrak{g} , and $\text{Aut}(\mathfrak{g})$ for the group of automorphisms of \mathfrak{g} . In particular, for each $s \in \mathbb{R}^+$, the dilation $\delta_s \in \text{Aut}(\mathfrak{g})$ is defined to be $\sum_{j=1}^{\ell} s^j \pi_j$.

For a linear map of \mathfrak{g} , preserving all the subspaces \mathfrak{g}_{-j} of the stratification is equivalent to commuting with dilations, and to having a block-diagonal matrix representation. We call such maps *strata-preserving*. We write $\text{Aut}^{\delta}(\mathfrak{g})$ for the subset of $\text{Aut}(\mathfrak{g})$ of strata-preserving automorphisms; these are determined by their action on \mathfrak{g}_{-1} . A stratified Lie algebra \mathfrak{g} is said to be *totally nonabelian* if $\mathfrak{g}_{-1} \cap \mathfrak{Z}(\mathfrak{g}) = \{0\}$. If \mathfrak{g} is totally nonabelian, then \mathfrak{g} has a *finest direct sum decomposition* (see [5, Theorem 2.3]):

$$\mathfrak{g} = \bigoplus_{k=1}^K \mathfrak{g}^k,$$

where the \mathfrak{g}^k are nontrivial totally nonabelian stratified Lie algebras that commute pairwise, with the additional property that, given any direct sum decomposition $\bigoplus_{l=1}^L \tilde{\mathfrak{g}}^l$ of \mathfrak{g} into ideals, the set $\{1, \dots, K\}$ may be partitioned into disjoint subsets J_1, \dots, J_L such that

$$\tilde{\mathfrak{g}}^l = \bigoplus_{j \in J_l} \mathfrak{g}^j \quad \forall l \in \{1, \dots, L\}.$$

When $j, k \in \{1, \dots, K\}$, we write $j \sim k$ if and only if there is a strata-preserving isomorphism from \mathfrak{g}^j to \mathfrak{g}^k ; then \sim is an equivalence relation. For each equivalence class $[j]$ and each $k \in [j]$, choose a stratified Lie algebra $\mathfrak{g}^{[j]}$ isomorphic to \mathfrak{g}^j , and a strata-preserving isomorphism I^k from $\mathfrak{g}^{[j]}$ to \mathfrak{g}^k , whose inverse we write as I^{-k} .

When σ lies in S_m^\sim , the group of permutations of $\{1, \dots, m\}$ that preserve the equivalence classes of \sim , define $I^\sigma \in \text{Aut}^\delta(\mathfrak{g})$ by first setting

$$I^\sigma(X) = I^{\sigma(j)} I^{-j}(X)$$

for all $X \in \mathfrak{g}^j$ and all $j \in \{1, \dots, m\}$, and then extending this definition to \mathfrak{g} by linearity. It is well known and easy to check that the map $\sigma \mapsto I^\sigma$ embeds S_m^\sim in $\text{Aut}^\delta(\mathfrak{g})$. We denote the image by $\text{Perm}(\mathfrak{g})$.

2.2. Stratified Lie groups. Let G be a stratified Lie group of step ℓ . This means that G is connected and simply connected, and its Lie algebra \mathfrak{g} is stratified with ℓ layers. The identity of G is written e

Since G is nilpotent, connected, and simply connected, the exponential map \exp is a bijection from \mathfrak{g} to G , with inverse \log . In particular, if $\{U_j : j = 1, \dots, n\}$ is a basis of \mathfrak{g} , we may write every point of G as $\exp(u_1 U_1 + \dots + u_n U_n)$, which we denote by (u_1, \dots, u_n) and call exponential coordinates of the first kind. We also write δ_s for the automorphism of G given by $\exp \circ \delta_s \circ \log$. The differential $T \mapsto (T_*)_e$ is a one-to-one correspondence between automorphisms of G and of \mathfrak{g} , and $T = \exp \circ (T_*)_e \circ \log$. We denote by $\text{Aut}(G)$ the group of automorphisms of G , and by $\text{Aut}^\delta(G)$ the subgroup of automorphisms that commute with dilations.

A stratified connected simply connected Lie group G is called *totally nonabelian* or a *direct product* if its Lie algebra is totally nonabelian or a Lie algebra direct sum. The *finest direct product decomposition* of the group is that associated to the finest direct sum decomposition of the Lie algebra.

First, we state and prove a preliminary lemma.

Lemma 2.1. *Suppose that G is a simply connected nilpotent Lie group, with an orthonormal basis $\{U_j : j = 1, \dots, n\}$ for its Lie algebra. Let $\sum_{n=0}^\infty c_n z^n$ be the power series of the function $z/(1 - e^{-z})$ (extended to 0 by continuity), which converges in the ball with centre 0 and radius 2π . Then the left-invariant vector field X corresponding to $X \in \mathfrak{g}$, evaluated at $\exp(Y)$ in G , is given in exponential coordinates of the first kind by*

$$X_{\exp Y} = \left\langle \left(\sum_{k=0}^\infty c_k \text{ad}^k(Y) \right) X, U_j \right\rangle \partial_{u_j},$$

where the sum terminates when k is sufficiently large as $\text{ad}(Y)$ is nilpotent.

Proof. The derivative of the exponential map \exp at $Y \in \mathfrak{g}$ is given by

$$\frac{1 - \exp(-\text{ad}(Y))}{\text{ad}(Y)} = \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)!} \text{ad}^k(Y)$$

(see, for example, [15, Theorem 2.14.3]), and the series terminates because G is nilpotent. \square

The coefficients c_j may be determined inductively from the condition

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} z^k \right) \left(\sum_{k=0}^{\infty} c_k z^k \right) = 1,$$

and we find that

$$c_0 = 1, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{12}, \quad c_3 = 0, \quad c_4 = \frac{-1}{720}, \quad \dots$$

In a stratified group G , more can be said. We say that a function f on G is homogeneous of degree d if $f(\delta_s x) = s^d f(x)$, and a differential operator D on G is homogeneous of degree e (which may be negative) if Df is homogeneous of degree $d + e$ whenever f is homogeneous of degree d . We write $\deg(f)$ and $\deg(D)$ for these degrees.

If $X \in \mathfrak{g}_{-k}$, then the associated vector field X is homogeneous of degree $-k$. If we take a basis $\{U_j : j = 1, \dots, n\}$ of \mathfrak{g} , where each U_j belongs to some $\mathfrak{g}_{-d(j)}$, and use exponential coordinates of the first kind on G , that is, we write

$$(u_1, \dots, u_n) := \exp(u_1 U_1 + \dots + u_n U_n),$$

then the coordinate function u_j is homogeneous of degree $d(j)$. A vector field that is homogeneous of degree d is a linear combination of the left-invariant vector fields U_1, \dots, U_n , with coefficients c_j that are homogeneous functions, and $d = \deg(c_j) + \deg(U_j)$.

2.3. The Pansu differential. We denote by L_p the left translation by p in G , that is, $L_p q = pq$ for all $q \in G$. The subbundle HG of the tangent bundle TG given by $H_p G = (L_p)_*(\mathfrak{g}_{-1})$ is called the *horizontal distribution*. We write Ω for an arbitrary nonempty connected open subset of G . The differential of a differentiable map $f : \Omega \rightarrow G$ is written f_* . We recall that a continuous map $f : \Omega \rightarrow G$ is *Pansu differentiable* at $p \in \Omega$ if the limit

$$\lim_{s \rightarrow 0^+} \delta_s^{-1} \circ L_{f(p)}^{-1} \circ f \circ L_p \circ \delta_s(q)$$

converges, uniformly for q in compact subsets of G , to a strata-preserving homomorphism of G , written $Df_p(q)$. If f is Pansu differentiable at p , then $\log \circ Df_p \circ \exp$ is a Lie algebra homomorphism, written df_p , and

$$df_p(X) = \lim_{s \rightarrow 0^+} \log \circ \delta_s^{-1} \circ L_{f(p)}^{-1} \circ f \circ L_p \circ \delta_t \circ \exp(X)$$

exists, uniformly for X in compact subsets of \mathfrak{g} . We call Df_p the *Pansu derivative* and df_p the *Pansu differential* of f at p . By construction, both Df_p and df_p commute with dilations, and so in particular, df_p is a strata-preserving Lie algebra homomorphism.

Note that if T is a strata-preserving automorphism of G , then its Pansu derivative $DT(p)$ coincides with T at every point, and its Pansu differential $dT(p)$ coincides with the Lie differential $\log \circ T \circ \exp$ at every point. Thus our notation is a little different from the standard Lie theory notation, but is not ambiguous.

2.4. The sub-Riemannian distance. We fix a scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g}_{-1} , and we define a left-invariant Riemannian metric on the horizontal distribution by the formula

$$(2.1) \quad \langle V, W \rangle_p = \langle (L_{p^{-1}})_*(V), (L_{p^{-1}})_*(W) \rangle$$

for all $V, W \in H_p G$ and all $p \in G$. This gives rise to a left-invariant *sub-Riemannian* or *Carnot–Carathéodory* distance function ϱ on G . To define this, we first say that a smooth curve γ is *horizontal* if $\dot{\gamma}(t) \in H_{\gamma(t)} G$ for every t . Then we define the distance $\varrho(p, q)$ between points p and q by

$$\varrho(p, q) := \inf \int_0^1 ((\dot{\gamma}(t), \dot{\gamma}(t))_{\gamma(t)})^{1/2} dt,$$

where in the infimum we take all horizontal curves $\gamma : [0, 1] \rightarrow G$ such that $\gamma(0) = p$ and $\gamma(1) = q$. The distance function is homogeneous, symmetric, and left-invariant, that is,

$$s^{-1} \varrho(\delta_s p, \delta_s q) = \varrho(p, q) = \varrho(q, p) = \varrho(rp, rp) \quad \forall p, q, r \in G \quad \forall s \in \mathbb{R}^+;$$

in particular, $\varrho(p, q) = \varrho(q^{-1}p, e)$. The stratified group G , equipped with the distance ϱ , is known as a *Carnot group*.

2.5. Quasiconformal automorphisms and maps. We write $S(V)$ for the unit sphere in a normed vector space V .

Suppose that $\lambda \geq 1$. We say that $T \in \text{Aut}^\delta(\mathfrak{g})$ is λ -*quasiconformal* if and only if

$$\max \{ \|TX\| : X \in S(\mathfrak{g}_{-1}) \} \leq \lambda \min \{ \|TX\| : X \in S(\mathfrak{g}_{-1}) \}.$$

Of course, every $T \in \text{Aut}^\delta(\mathfrak{g})$ is λ -quasiconformal for sufficiently large λ .

Suppose that $s \in \mathbb{R}^+$. In a Carnot group, the distortion $H(f, p, s)$ of a map $f : \Omega \rightarrow G$ at a point $p \in \Omega$ and at scale $s \in \mathbb{R}^+$ is defined by

$$H(f, p, s) = \frac{\sup \{ \varrho(f(x), f(p)) : x \in \Omega, \varrho(x, p) \leq s \}}{\inf \{ \varrho(f(x), f(p)) : x \in \Omega, \varrho(x, p) \geq s \}}.$$

The map f is λ -*quasiconformal in Ω* if

$$\limsup_{s \rightarrow 0^+} H(f, p, s) \leq \lambda \quad \forall p \in \Omega,$$

and f is *quasiconformal* if it is λ -quasiconformal for some $\lambda \in \mathbb{R}^+$.

If the map f is C^1 , then it is λ -quasiconformal in Ω if and only if its Pansu differential df_p is λ -quasiconformal at all $p \in \Omega$. It is known that 1-quasiconformal maps on Carnot groups and on some sub-Riemannian manifolds are smooth (see [2, 3]); such maps are also known as *conformal maps*.

3. CR STRATIFIED GROUPS AND CARNOT GROUPS

In this section, we consider CR structures on stratified groups and Carnot groups; we consider an example with an illustrious history, and construct many new examples of Carnot groups as boundaries of domains.

3.1. CR stratified groups. Let G be a stratified group such that $\dim \mathfrak{g}_{-1} = 2m$ and let n be the integer such that $2m + n = \dim G$. We define an *almost complex structure* on \mathfrak{g}_{-1} to be a linear isomorphism $J : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$ such that $J^2 = -\text{Id}$ and

$$(3.1) \quad [X, Y] = [JX, JY] \quad \text{and} \quad [X, JY] = -[JX, Y]$$

for all $X, Y \in \mathfrak{g}_{-1}$. A stratified group G equipped with such a mapping J is said to be a *CR stratified group of type (m, n)* . Let $L = \text{span}\{X - iJX : X \in \mathfrak{g}_{-1}\}$. It is easy to check that (3.1) is equivalent to L being abelian in the complexification $\mathfrak{g}_{\mathbb{C}}$ of \mathfrak{g} .

We say that $T \in \text{Aut}^{\delta}(\mathfrak{g})$ is a *CR automorphism* or an *anti-CR automorphism* if

$$T|_{\mathfrak{g}_{-1}}J = JT|_{\mathfrak{g}_{-1}} \quad \text{or} \quad T|_{\mathfrak{g}_{-1}}J = -JT|_{\mathfrak{g}_{-1}};$$

equivalently, $T_{\mathbb{C}}(L) \subseteq L$ or $\overline{T_{\mathbb{C}}}(L) \subseteq L$, where $T_{\mathbb{C}}$ denotes the linear extension of T to $\mathfrak{g}_{\mathbb{C}}$. Notice that the inverse of a CR automorphism is also a CR automorphism. A diffeomorphism $f : \Omega \rightarrow \Upsilon$ between domains in G is a *CR mapping* or an *anti-CR mapping* if and only if its Pansu differential df_p is a CR automorphism or an anti-CR automorphism for every $p \in \Omega_1$. In this section we will study the structure and the CR diffeomorphisms of CR stratified groups. In particular, we will show that, for a class of these groups, the space of conformal maps with respect to a compatible metric coincides with the space of CR maps. Last but not least, we will show some explicit embeddings of CR stratified groups into \mathbb{C}^{m+n} via a CR diffeomorphism.

3.2. Tight groups. We say that a stratified group is *tight* if \mathfrak{g} is totally nonabelian, its finest direct product decomposition has only one factor, and $\dim \mathfrak{g}_{-2} = 1$. Equivalently,

$$\mathfrak{g}_{-1} = \text{span}\{X_1, \dots, X_m, Y_1, \dots, Y_m\},$$

where

$$(3.2) \quad [X_j, X_l] = [Y_j, Y_l] = 0 \quad \text{and} \quad [X_j, Y_l] = \delta_{jl}U \quad \forall j, l = 1, \dots, m.$$

It is straightforward to check that the space L , defined by

$$L := \text{span}\{X_j - iY_j : j = 1, \dots, m\},$$

satisfies $[L, L] = \{0\}$. So tight stratified groups are CR with respect to the almost complex structure determined by the requirements that $JX_j = Y_j$ and $JY_j = -X_j$ for every $j = 1, \dots, m$.

Lemma 3.1. *Let G be a tight CR stratified group. If $m > 1$, then U is central, that is, $[X_j, U] = [Y_j, U] = 0$ when $j = 1, \dots, m$.*

Proof. We argue by contradiction. Suppose that $[X_j, U] \neq 0$ for some j . Renumbering if necessary, we may assume that $j = 1$. Then $[X_1, [X_2, Y_2]] \neq 0$. However, by the Jacobi identity and (3.2),

$$[X_1, [X_2, Y_2]] = [[X_1, X_2], Y_2] + [X_2, [X_1, Y_2]] = 0,$$

which gives a contradiction. We may show that $[Y_j, U] = 0$ similarly. \square

Corollary 3.2. *Let G be a tight CR stratified group with $\dim(\mathfrak{g}_{-1}) = 2m$. Then exactly one of the following holds:*

- (i) $\mathfrak{g}_{-1} = \text{span}\{X_1, Y_1\}$ and $\mathfrak{g}_{-3} \neq \{0\}$,

- (ii) $\mathfrak{g}_{-1} = \text{span}\{X_1, \dots, X_m, Y_1, \dots, Y_m\}$, $\mathfrak{g}_{-2} = \text{span}\{U\}$ and $\mathfrak{g}_{-3} = \{0\}$. In this case \mathfrak{g} is the Heisenberg algebra of dimension $2m + 1$.

When G is tight, we consider the element $U^* \in \mathfrak{g}^*$ uniquely defined by $U^*(U) = 1$ and $U^*(X) = 0$ for all $X \in \bigoplus_{k \neq 2} \mathfrak{g}_{-k}$, and the left-invariant one-form θ such that $\theta_e = U^*$. Then the bilinear form

$$B_\theta(X, Y) = d\theta(X, JY)$$

is a scalar product on \mathfrak{g}_{-1} for which $\{X_1, \dots, X_m, Y_1, \dots, Y_m\}$ is an orthonormal basis. Moreover, B_θ is compatible with J , in the sense that

$$B_\theta(JX, JY) = B_\theta(X, Y) \quad \forall X, Y \in \mathfrak{g}_{-1}.$$

We define a Carnot group structure on G using the left-invariant metric on the horizontal subbundle that coincides with B_θ at the identity.

Theorem 3.3. *Let G be a tight stratified group with the Carnot distance determined by B_θ . Let $f : \Omega \rightarrow G$ be a homeomorphism from a connected open subset Ω of G onto its image. Then f is 1-quasiconformal if and only if f is CR or anti-CR.*

Proof. We say that $T \in \text{Aut}^\delta(\mathfrak{g})$ is conformal if

$$(3.3) \quad \|T(X)\| = \lambda \|X\| \quad \forall X \in \mathfrak{g}_{-1},$$

or, equivalently, if $T^t T = \lambda^2 \text{Id}$ for some $\lambda \in \mathbb{R}_+$. Here T^t denotes the transpose with respect to B_θ . It is well known that f is 1-quasiconformal if and only if df_p is conformal for every $p \in \Omega$ [3]. Therefore, it is enough to show that every conformal automorphism is a CR or anti-CR automorphism, and vice versa. Since G is tight, either $\dim \mathfrak{g}_{-1} = 2$ or G is the Heisenberg group of dimension $2m + 1$. It is straightforward to show in both cases that for all $T \in \text{Aut}^\delta(\mathfrak{g})$, the condition $T^t|_{\mathfrak{g}_{-1}} T|_{\mathfrak{g}_{-1}} = \lambda \text{Id}$ is equivalent to $T|_{\mathfrak{g}_{-1}} J = JT|_{\mathfrak{g}_{-1}}$ or $T|_{\mathfrak{g}_{-1}} J = -JT|_{\mathfrak{g}_{-1}}$. \square

The theorem above holds for every left-invariant metric that is compatible with the CR structure. Indeed, let B' be any scalar product on \mathfrak{g}_{-1} with the property that

$$B'(JX, JY) = B'(X, Y) \quad \forall X, Y \in \mathfrak{g}_{-1}.$$

Then there is $A \in GL(2\ell, \mathbb{R})$ such that $B'(X, Y) = B_\theta(AX, AY)$. The compatibility condition and the definition of A imply that $B_\theta(JAX, JAY) = B_\theta(AJX, AJY)$, which in turn implies that $AJ = JA$ or $AJ = -JA$. Therefore A induces a CR or anti-CR automorphism of \mathfrak{g} , and so by Theorem 3.3 the left-invariant metrics with respect to B_θ and B' are conformally equivalent.

3.3. CR embeddings of tight groups. We are now deriving a system of equations whose solutions, if they exist, yield explicit CR embeddings of tight groups G into \mathbb{C}^{m+n} that generalise those considered by Nagel and Stein; our embeddings are more closely related to work of Andreotti and Hill [1], and generalise work of the fourth-named author and her collaborators [16, 17]. As the Heisenberg groups are well understood, we concentrate on the case where $\dim \mathfrak{g}_{-1} = 2$ and \mathfrak{g}_{-3} is nontrivial. Our construction involves several steps.

First, extend an orthonormal basis $\{X, Y\}$ of \mathfrak{g}_{-1} to a basis $\{X, Y, U_1, \dots, U_n\}$ of \mathfrak{g} , where we choose the U_j from the iterated commutators of X and Y in order to first span \mathfrak{g}_{-2} , then \mathfrak{g}_{-3} , and so on. We then extend the inner product on \mathfrak{g}_{-1}

to an inner product on \mathfrak{g} so that our basis is orthonormal. We use exponential coordinates of the first kind, and take an element of G to be

$$(x, y, u_1, \dots, u_n) := \exp(xX + yY + u_1U_1 + \dots + u_nU_n).$$

By Lemma 2.1, the left-invariant vector field T corresponding to an element T of \mathfrak{g} is given in these coordinates by

$$\begin{aligned} T_{\exp Y} &= \langle T, X \rangle \partial_x + \langle T, Y \rangle \partial_y + \sum_{j=1}^n \left\langle \left(\sum_{k=0}^{\infty} c_k \operatorname{ad}^k(Y) \right) T, U_j \right\rangle \partial_{u_j} \\ &= a_T \partial_x + b_T \partial_y + \sum_{j=1}^n p_{T,j}(x, y, u_1, \dots, u_n) \partial_{u_j}, \end{aligned}$$

where $a_T = \langle T, X \rangle$, $b_T = \langle T, Y \rangle$, and $p_{T,j}(x, y, u_1, \dots, u_n)$ is equal to

$$\sum_{l=1}^n \sum_{k=0}^{\infty} c_k \left\langle \operatorname{ad}^k(xX + yY + u_1U_1 + \dots + u_nU_n)T, U_l \right\rangle \partial_{u_l}.$$

The functions $p_{T,j}$ are polynomials of bounded degree, since the series above has finitely many nonzero terms, and are homogeneous if T is homogeneous. In particular,

$$X = \partial_x + \sum_{j=1}^n p_{X,j}(x, y, u_1, \dots, u_n) \partial_{u_j}$$

and

$$Y = \partial_y + \sum_{j=1}^n p_{Y,j}(x, y, u_1, \dots, u_n) \partial_{u_j}.$$

Now we seek to map G into the surface

$$\{(x, y, u_1, \dots, u_n, v_1, \dots, v_n) \in \mathbb{R}^{2+2n} : v_j = q_j(x, y, u_1, \dots, u_n), j = 1, \dots, n\},$$

where the q_j are homogeneous polynomials of positive degree, using the map ϕ , defined by taking $\phi(x, y, u_1, \dots, u_n)$ equal to

$$(x, y, u_1, \dots, u_n, q_1(x, y, u_1, \dots, u_n), \dots, q_n(x, y, u_1, \dots, u_n)).$$

The map ϕ is evidently a C^∞ embedding, and 0 lies on the surface. In the obvious extension of our coordinate system, the differential ϕ_* of ϕ satisfies

$$\phi_*(T) = T + \sum_{j=1}^n (Tq_j) \partial_{v_j}.$$

We identify $(x, y, u_1, \dots, u_n, v_1, \dots, v_n) \in \mathbb{R}^{2+2n}$ with $(z, w_1, \dots, w_n) \in \mathbb{C}^{1+n}$, where $z = x + iy$ and $w_j = u_j + iv_j$. When we do this, our embedding is a CR embedding if and only if the complex $(1, 0)$ vector field $Z = X + iY$ on G maps to a $(1, 0)$ vector field tangent to the surface in \mathbb{C}^{1+n} . Now

$$\phi_*(Z) = \phi_*(X) + i\phi_*(Y),$$

and this is a $(1, 0)$ vector field if and only if the coefficient of ∂_{v_j} is i times the coefficient of ∂_{u_j} for all j , that is,

$$(Xq_j) + i(Yq_j) = i(p_{X,j} + ip_{Y,j}),$$

or equivalently,

$$(3.4) \quad \begin{cases} Xq_j &= -p_{Y,j}(x, y, u_1, \dots, u_n), \\ Yq_j &= p_{X,j}(x, y, u_1, \dots, u_n). \end{cases}$$

This is a nontrivial system of differential equations, and we do not know whether it can be solved in general. There are certainly many examples where this is possible, for instance, if \mathfrak{g} is filiform—see [16, 18]. In the next section, we will consider the case of free nilpotent Lie groups with two generators and use an alternative coordinate system to solve (3.4) for the case when these Lie groups have step at most 8.

If the stratified group G is not tight, there is no obvious canonical choice of a compatible Carnot structure. Hence the extent to which we can generalise our study of the interplay between conformal and CR structures in the general case is unclear.

4. FREE NILPOTENT LIE GROUPS

In this section we focus on solving the system of equations (3.4) in the tight case, that is, for free nilpotent Lie groups whose Lie algebra has two generators and step at least 2. We introduce some further notation, that in some cases will replace that of the previous sections. Denote by $\mathfrak{f}_{2,s}$ the free nilpotent Lie algebra of step s with two generators, and let $n = \dim \mathfrak{f}_{2,s}$. Recall that $\mathfrak{f}_{2,s}$ is the biggest nilpotent Lie algebra of step s generated by iterated brackets of two generators X_1 and X_2 . The elements in the linear span of

$$[X_{\alpha_1}, \dots, [X_{\alpha_{k-1}}, X_{\alpha_k}] \dots],$$

where $X_{\alpha_1}, \dots, X_{\alpha_k}$ are vectors in $\mathfrak{f}_{2,s}$, are said to have *length at most k* . We now recall the recursive definition of the Hall basis [8] for $\mathfrak{f}_{2,s}$. Each element in the basis is a monomial in the generators. The generators X_1 and X_2 are elements of the basis and of length 1. Assume that we have defined basis elements of lengths $1, \dots, \ell - 1$ and that they are simply ordered in such a way that $X < Y$ if $\text{length}(X) < \text{length}(Y)$. If $\text{length}(X) = r$ and $\text{length}(Y) = t$, and $\ell = r + t$, then $[X, Y]$ is a basis element of length ℓ if:

1. X and Y are basis elements and $X > Y$, and
2. if $X = [Z, W]$, then $Y \geq W$.

Number the basis elements using this ordering, i.e., $X_3 = [X_2, X_1]$, $X_4 = [X_3, X_1]$, $X_5 = [X_3, X_2]$, etc. Consider a basis element X_i as a bracket in the lower order basis elements, $[X_{j_1}, X_{k_1}]$, where $j_1 > k_1$. If we repeat this process with X_{j_1} , we get $X_i = [[X_{j_2}, X_{k_2}], X_{k_1}]$, where $k_2 \leq k_1$ by the Hall basis conditions. Continuing in this fashion, we end up with

$$(4.1) \quad X_i = [[\dots [[X_2, X_{i_1}], X_{i_2}], \dots, X_{i_{m-1}}], X_{i_m}],$$

where $i_1 = 1$ and $i_\ell \leq i_{\ell+1}$ when $2 \leq \ell \leq m - 1$. Since this expansion involves m brackets, we shall write $d(i) = m$ and define $d(1) = d(2) = 0$. This process naturally associates a multi-index $I(i) = (a_1, \dots, a_n)$ to each Hall basis element X_i , defined by $a_r = \#\{t : i_t = r\}$. Note that $I(i) = (0, \dots, 0)$ for $i = 1, 2$. Let $x = (x_1, \dots, x_n)$ be coordinates in \mathbb{R}^n . For every $j \geq 3$, we define the monomial p_j

by

$$p_j(x) := \frac{(-1)^{d(j)}}{I(j)!} x^{I(j)},$$

where $x^{I(j)} = x_1^{I(j)_1} \cdots x_n^{I(j)_n}$ and $I(j)! = I(j)_1! \cdots I(j)_n!$, if $I(j) = (I(j)_1, \dots, I(j)_n)$. Notice that $I(j)_1 \geq 1$. It will be convenient to represent the bracket (4.1) by the vector $(2, 1, i_2, \dots, i_m)$. We stress that any one of the formula (4.1) for X_i , the vector $(2, 1, i_2, \dots, i_m)$, or the monomial p_i , uniquely describes the other two.

The vector fields

$$X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2} + \sum_{j \geq 3} p_j \frac{\partial}{\partial x_j}$$

generate the Lie algebra $\mathfrak{f}_{2,s}$. We now rewrite (3.4) using these vector fields as generators. Thus, we look for polynomials q_j solving

$$\begin{cases} X_1 q_j &= -p_j, \\ X_2 q_j &= 0 \end{cases}$$

for every $j \geq 3$. Since p_j is a monomial, the first equation integrates to

$$q_j = c_j x_1 p_j + r_j,$$

where $c_j \in (0, 1]$ and $r_j = r_j(x_2, \dots, x_n)$. We substitute this in the second equation to obtain

$$c_j x_1 X_2 p_j + X_2 r_j = c_j x_1 \left(\frac{\partial}{\partial x_2} p_j + \sum_{k > 2} p_k \frac{\partial}{\partial x_k} p_j \right) + \frac{\partial}{\partial x_2} r_j + \sum_{\ell > 2} p_\ell \frac{\partial}{\partial x_\ell} r_j = 0.$$

Since $I(t)_1 \geq 1$ when $t \geq 3$, it follows that $\partial r_j / \partial x_2$ is the only term that does not depend on x_1 , so it is zero, and $r_j = r_j(x_3, \dots, x_n)$. Hence for a free nilpotent Lie algebra with two generators, the system (3.4) can be solved if we can find $r_j(x_3, \dots, x_n)$ such that

$$(4.2) \quad c_j x_1 \left(\frac{\partial}{\partial x_2} p_j + \sum_{k > 2} p_k \frac{\partial}{\partial x_k} p_j \right) + \sum_{\ell > 2} p_\ell \frac{\partial}{\partial x_\ell} r_j = 0$$

for all $j \geq 3$. We now solve this system of equations for free Lie algebras up to step 8.

Theorem 4.1. *Let $\mathfrak{f}_{2,s}$ be a free nilpotent Lie algebra of step s at most 8. Then the system of equations (4.2) admits a solution of the form*

$$r_j(x_3, \dots, x_n) = \sum_{k > 2} a_k^j x_k + \sum_{\ell > 2} b_\ell^j x_\ell^2,$$

for some $a_k^j, b_\ell^j \in \mathbb{R}$. In particular, if $s \leq 5$, then $b_\ell^j = 0$.

Vice versa, if $s \geq 9$, then there is $j \geq 3$ such that $\sum_{k > 2} a_k^j x_k + \sum_{\ell > 2} b_\ell^j x_\ell^2$ does not solve (4.2).

For a free nilpotent Lie algebra with two generators, we may represent the monomials p_ℓ by the vector $(2, 1, \ell_1, \dots, \ell_m)$. We stress that this vector is not the same as $I(\ell)$. Vice versa, in order for such a vector to represent a nonzero monomial p_ℓ , it must be that $(2, 1, \ell_1, \dots, \ell_{m-1})$ represents a basis vector higher than X_{ℓ_m} and with $X_{\ell_m} > X_{\ell_{m-1}}$. Using these rules, we may easily construct all vectors for

TABLE 1. Monomials up to step 8.

Step	Vectors	Monomials
2	(2, 1)	$p_3 = -x_1$
3	(2, 1, 1), (2, 1, 2)	$p_4 = \frac{1}{2}x_1^2, p_5 = x_1x_2$
4	(2, 1, 1, 1), (2, 1, 1, 2) (2, 1, 2, 2)	$p_6 = -\frac{1}{6}x_1^3, p_7 = -\frac{1}{2}x_1^2x_2$ $p_8 = -\frac{1}{2}x_1x_2^2$
5	(2, 1, 1, 1, 1), (2, 1, 1, 1, 2) (2, 1, 1, 2, 2), (2, 1, 2, 2, 2) (2, 1, 1, 3), (2, 1, 2, 3)	$p_9 = \frac{1}{24}x_1^4, p_{10} = \frac{1}{6}x_1^3x_2$ $p_{11} = \frac{1}{4}x_1^3x_2^2, p_{12} = \frac{1}{6}x_1x_2^3$ $p_{13} = -\frac{1}{5}x_1^2x_3, p_{14} = -x_1x_2x_3$
6	(2, 1, 1, 1, 1, 1), (2, 1, 1, 1, 1, 2) (2, 1, 1, 1, 2, 2), (2, 1, 1, 2, 2, 2) (2, 1, 2, 2, 2, 2), (2, 1, 1, 1, 3) (2, 1, 1, 2, 3), (2, 1, 2, 2, 3) (2, 1, 2, 4)	$p_{15} = -\frac{1}{120}x_1^5, p_{16} = -\frac{1}{24}x_1^4x_2$ $p_{17} = -\frac{1}{12}x_1^3x_2^2, p_{18} = -\frac{1}{12}x_1^2x_2^3$ $p_{19} = -\frac{1}{24}x_1x_2^4, p_{20} = \frac{1}{6}x_1^3x_3$ $p_{21} = \frac{1}{2}x_1^2x_2x_3, p_{22} = \frac{1}{2}x_1x_2^2x_3$ $p_{23} = -x_1x_2x_4$
7	(2, 1, 1, 1, 1, 1, 1), (2, 1, 1, 1, 1, 1, 2) (2, 1, 1, 1, 1, 2, 2), (2, 1, 1, 1, 2, 2, 2) (2, 1, 1, 2, 2, 2, 2), (2, 1, 2, 2, 2, 2, 2) (2, 1, 1, 1, 1, 3), (2, 1, 1, 1, 2, 3) (2, 1, 1, 2, 2, 3), (2, 1, 2, 2, 2, 3) (2, 1, 1, 3, 3), (2, 1, 2, 3, 3) (2, 1, 1, 1, 4), (2, 1, 1, 1, 5) (2, 1, 1, 2, 4), (2, 1, 1, 2, 5) (2, 1, 2, 2, 4), (2, 1, 2, 2, 5)	$p_{24} = \frac{1}{720}x_1^6, p_{25} = \frac{1}{120}x_1^5x_2$ $p_{26} = \frac{1}{48}x_1^4x_2^2, p_{27} = \frac{1}{36}x_1^3x_2^3$ $p_{28} = \frac{1}{48}x_1^2x_2^4, p_{29} = \frac{1}{120}x_1x_2^5$ $p_{30} = -\frac{1}{24}x_1^4x_3, p_{31} = -\frac{1}{6}x_1^3x_2x_3$ $p_{32} = -\frac{1}{4}x_1^2x_2^2x_3, p_{33} = -\frac{1}{6}x_1^2x_2^3x_3$ $p_{34} = \frac{1}{4}x_1^2x_2^3, p_{35} = \frac{1}{2}x_1x_2^2x_3^2$ $p_{36} = \frac{1}{6}x_1^3x_4, p_{37} = \frac{1}{6}x_1^3x_5$ $p_{38} = \frac{1}{2}x_1^2x_2x_4, p_{39} = \frac{1}{2}x_1^2x_2x_5$ $p_{40} = \frac{1}{2}x_1x_2^2x_4, p_{33} = \frac{1}{2}x_1x_2^2x_5$
8	(2, 1, 1, 1, 1, 1, 1, 1), (2, 1, 1, 1, 1, 1, 1, 2) (2, 1, 1, 1, 1, 1, 2, 2), (2, 1, 1, 1, 1, 2, 2, 2) (2, 1, 1, 1, 2, 2, 2, 2), (2, 1, 1, 2, 2, 2, 2, 2) (2, 1, 2, 2, 2, 2, 2, 2), (2, 1, 1, 1, 1, 1, 3) (2, 1, 1, 1, 1, 2, 3), (2, 1, 1, 1, 2, 2, 3) (2, 1, 1, 2, 2, 2, 3), (2, 1, 2, 2, 2, 2, 3) (2, 1, 1, 1, 3, 3), (2, 1, 1, 2, 3, 3) (2, 1, 2, 2, 3, 3), (2, 1, 1, 1, 1, 4) (2, 1, 1, 1, 1, 5), (2, 1, 1, 1, 2, 4) (2, 1, 1, 1, 2, 5), (2, 1, 1, 2, 2, 4) (2, 1, 1, 2, 2, 5), (2, 1, 2, 2, 2, 4) (2, 1, 2, 2, 2, 5), (2, 1, 1, 3, 4) (2, 1, 1, 3, 5), (2, 1, 2, 3, 4) (2, 1, 2, 3, 5), (2, 1, 1, 2, 6) (2, 1, 2, 2, 6), (2, 1, 2, 2, 7)	$p_{41} = -\frac{1}{5040}x_1^7, p_{42} = -\frac{1}{720}x_1^6x_2$ $p_{43} = -\frac{1}{240}x_1^5x_2^2, p_{44} = -\frac{1}{144}x_1^4x_2^3$ $p_{45} = -\frac{1}{144}x_1^3x_2^4, p_{46} = -\frac{1}{240}x_1^2x_2^5$ $p_{47} = -\frac{1}{720}x_1^6x_3, p_{48} = \frac{1}{120}x_1^5x_3$ $p_{49} = \frac{1}{24}x_1^4x_2x_3, p_{50} = \frac{1}{12}x_1^3x_2^2x_3$ $p_{51} = \frac{1}{12}x_1^2x_2^3x_3, p_{52} = \frac{1}{24}x_1x_2^4x_3$ $p_{53} = -\frac{1}{12}x_1^3x_2^2, p_{54} = -\frac{1}{4}x_1^2x_2^3$ $p_{55} = -\frac{1}{4}x_1x_2^2x_3^2, p_{56} = \frac{1}{24}x_1^4x_4$ $p_{57} = -\frac{1}{24}x_1^4x_5, p_{58} = -\frac{1}{6}x_1^3x_2x_4$ $p_{59} = -\frac{1}{6}x_1^3x_2x_5, p_{60} = -\frac{1}{4}x_1^2x_2^2x_4$ $p_{61} = -\frac{1}{4}x_1^2x_2^2x_5, p_{62} = -\frac{1}{6}x_1x_2^3x_4$ $p_{63} = -\frac{1}{6}x_1x_2^3x_5, p_{64} = \frac{1}{2}x_1^2x_3x_4$ $p_{65} = \frac{1}{2}x_1^2x_3x_5, p_{66} = x_1x_2x_3x_4$ $p_{67} = x_1x_2x_3x_5, p_{68} = \frac{1}{2}x_1^2x_2x_6$ $p_{69} = \frac{1}{2}x_1x_2^2x_6, p_{70} = \frac{1}{2}x_1x_2^2x_7$

a given step. For example, if the step is 2, we only have the monomial p_3 , corresponding to the vector (2, 1). In step 3, we have to add two more monomials, p_4 , and p_5 , corresponding to the vectors (2, 1, 1) and (2, 1, 2). We write all vectors and monomials up to step 8 in Table 1.

Theorem 4.1 will be a consequence of the following three lemmas.

Lemma 4.2. *Let $\mathfrak{f}_{2,s}$ be a free nilpotent Lie algebra of step s at most 8. Then for all $j, k \geq 3$,*

$$x_1 p_k \frac{\partial}{\partial x_k} p_j \in \text{span}\{p_\ell\}$$

for some $\ell = \ell(j, k)$.

Proof. We say that X_j has height $h(j)$ if $X_j \in \mathfrak{g}_{-h(j)}$. Observe that p_j is homogeneous of degree $h(j) - 1$. It follows that, if $x_1 p_k \frac{\partial}{\partial x_k} p_j \neq 0$, then it is homogeneous

of the same degree as p_j . If this is the case, for our statement to be true, it must be that $h(\ell) = h(j)$. Moreover, $\frac{\partial}{\partial x_k} p_j$ can only be nonzero if $3 \leq k \leq 7$. Using these observations, the claim readily follows by inspecting Table 1. \square

Lemma 4.3. *Let $\mathfrak{f}_{2,s}$ be a free nilpotent Lie algebra of step s at most 8. Then for every $j \geq 3$,*

$$x_1 \frac{\partial}{\partial x_2} p_j \in \begin{cases} \text{span}\{p_\ell\} & \text{if } j \neq 23, \\ \text{span}\{x_4 p_4\} & \text{if } j = 23, \\ \text{span}\{x_6 p_6\} & \text{if } j = 68, \\ \text{span}\{x_7 p_7\} & \text{if } j = 70 \end{cases}$$

for some $\ell = \ell(j)$. In particular, if $s \leq 5$, then $x_1 \frac{\partial}{\partial x_2} p_j \in \text{span}\{p_\ell\}$.

Proof. Given p_j , we study the action of $x_1 \frac{\partial}{\partial x_2}$ on the associated vector $(2, 1, j_2, \dots, j_m)$. If $j_k \neq 2$ for all k , then $x_1 \frac{\partial}{\partial x_2} p_j = 0$ and we are done. Otherwise, let k be the smallest integer such that $j_k = 2$. Then the action of $x_1 \frac{\partial}{\partial x_2}$ replaces the j_k th entry with 1, and more precisely,

$$(2, 1, j_2, \dots, j_m) \mapsto (2, 1, j_2, \dots, j_{k-1}, 1, j_{k+1}, \dots, j_m).$$

If $(2, 1, j_2, \dots, j_{k-1}, 1, j_{k+1}, \dots, j_m)$ represents a monomial p_ℓ for some ℓ , then we are done. By inspecting Table 1, this always occurs except for the vectors $(2, 1, 2, 4)$, $(2, 1, 1, 2, 6)$, and $(2, 1, 2, 2, 7)$, which represent the monomials p_{23}, p_{68} , and p_{70} . In these cases,

$$x_1 \frac{\partial}{\partial x_2} p_{23} = -x_1 \frac{\partial}{\partial x_2} x_1 x_2 x_4 = -x_1^2 x_4 = -x_4 p_4$$

and, similarly, $x_1 \frac{\partial}{\partial x_2} p_{68} = -3x_6 p_6$ and $x_1 \frac{\partial}{\partial x_2} p_{70} = -2x_7 p_7$. \square

Lemma 4.4. *Let $\mathfrak{f}_{2,s}$ be a free nilpotent Lie algebra of step 9. Then there is j for which $\sum_{k>2} a_k^j x_k + \sum_{\ell>2} b_\ell^j x_\ell^2$ is not a solution of (4.2) for every $a_k^j, b_\ell^j \in \mathbb{R}$.*

Proof. When $s = 9$, consider the monomial $p_j = x_1 x_2 x_4 x_5$, corresponding to the vector $(2, 1, 2, 4, 5)$. A direct computation shows that every solution r_j of (4.2) for this monomial contains $c_j x_4^2 x_5$ for some $c_j \neq 0$. \square

Proof of Theorem 4.1. In view of Lemmas 4.2 and 4.3, if $j \neq 23$ we may rewrite the equation (4.2) as

$$\sum_{v>2} a_v p_v + \sum_{\ell>2} p_\ell \frac{\partial}{\partial x_\ell} r_j = \sum_{\ell>2} \left(a_\ell + \frac{\partial}{\partial x_\ell} r_j \right) p_\ell = 0$$

for some constants a_v . Then $r_j = -\sum_{\ell>3} a_\ell x_\ell$ is a solution. If $j = 23$, then (4.2) becomes

$$-c_4 x_4 p_4 + \sum_{v>2, v \neq 4} b_v p_v + \sum_{\ell>2} p_\ell \frac{\partial}{\partial x_\ell} r_{23} = 0$$

for some constants b_v . Hence $r_{23} = -\sum_{v>2, v \neq 4} b_v x_v + \frac{c_4}{2} x_4^2$ is a solution. A quadratic solution can be found in the same way for $j = 68$ and $j = 70$. \square

Theorem 4.1 suggests that all free tight groups can be embedded into \mathbb{C}^{1+n} by means of polynomial functions. The pattern that these polynomials follow is at the moment unknown to the authors. For example, we find linear solutions for groups up to step 5 and then again at step 7, whereas quadratic solutions are only sufficient

up to step 8. However, our result gives an algorithm for finding these embeddings for anyone to use and that we plan to implement for MAPLE. Hopefully, this will provide us with enough examples to look again at the global picture.

5. PRODUCTS

We say that a mapping on G is *affine* if it is the composition of a left translation with an element in $\text{Aut}^\delta(G)$.

Theorem 5.1. *Suppose that G is a totally nonabelian Carnot group, with finest direct product decomposition $G^1 \times \cdots \times G^m$, where $m > 1$. Let $f : G \rightarrow G$ be a C^1 quasiconformal map. Then f is composed of a group automorphism that permutes the groups G^j and a product bi-Lipschitz map.*

Proof. We write either p or (p^1, \dots, p^m) for a typical element of G .

The Pansu differential of a C^1 global quasiconformal mapping f is continuous, and hence its $\text{Perm}(\mathfrak{g})$ component is constant. This is an automorphism of \mathfrak{g} and by conjugation with the exponential may be considered as an automorphism of G , and is therefore quasiconformal. By composing with the inverse of this automorphism if necessary, we may assume that the Pansu differential df_p of f is a product automorphism (see [5, Corollary 3.4]).

If we take a horizontal curve γ in one of the factors G^j , then $f \circ \gamma$ is again a horizontal curve, whose Pansu derivative is $df \circ \dot{\gamma}$, and so $f \circ \gamma$ moves in the factor G^j and is fixed in the other factors. The groups G^j mutually commute, and it follows immediately that f is a product map: we may find maps $f^j : G^j \rightarrow G^j$ such that

$$f(p^1, \dots, p^m) = (f^1(p^1), \dots, f^m(p^m)) \quad \forall p \in G.$$

The Pansu differential df_p is also a product map:

$$df_p = (df_{p^1}^1, \dots, df_{p^m}^m) \quad \forall p \in G.$$

If f is λ -quasiconformal, it follows immediately that when $j \neq k$,

$$\max \{ \|df_{p^j}^j(X)\| : X \in S(\mathfrak{g}_{-1}^j) \} \leq \lambda \min \{ \|df_{p^k}^k(X)\| : X \in S(\mathfrak{g}_{-1}^k) \}$$

for all $p \in G$. Define

$$c_k = \inf \left\{ \|df_{p^k}^k(X)\| : X \in S(\mathfrak{g}_{-1}^k), p^k \in G^k \right\}$$

for all k , and now fix k such that $c_k = \min\{c_j : j \in \{1, \dots, m\}\}$. Then, when $j \neq k$,

$$\sup \{ \|df_{p^j}^j(X)\| : X \in S(\mathfrak{g}_{-1}^j), p^j \in G^j \} \leq \lambda c_k.$$

Fix j different from k . Since

$$\begin{aligned} & \sup \left\{ \|df_{p^k}^k(X)\| : X \in S(\mathfrak{g}_{-1}^k), p^k \in G^k \right\} \\ & \leq \lambda \inf \{ \|df_{p^j}^j(X)\| : X \in S(\mathfrak{g}_{-1}^j), p^j \in G^j \}, \end{aligned}$$

it follows that

$$\sup \{ \|df_{p^k}^k(X)\| : X \in S(\mathfrak{g}_{-1}^k), p^k \in G^k \} \leq \lambda^2 c_k.$$

Since f is not constant, $c_k \neq 0$, and now each map f^j is bi-Lipschitz, and, by considering horizontal curves, we conclude that

$$c_k \varrho(p, q) \leq \varrho(f(p), f(q)) \leq \lambda^2 c_k \varrho(p, q) \quad \forall p, q \in G,$$

as required. \square

Remark 5.2. The argument above shows that if f is defined in a domain Ω in G , then f is locally a product mapping. Of course, this does not imply that f is a product mapping, unless Ω is a product domain. However, if f is 1-quasiconformal, then stronger conclusions do hold.

Corollary 5.3. *Suppose that G is a totally nonabelian Carnot group, with finest direct product decomposition $G^1 \times \cdots \times G^m$, where $m > 1$. Let $f : \Omega \rightarrow G$ be a 1-quasiconformal map from a domain Ω in G onto its image. Then f is the restriction to Ω of the composition of a group automorphism that permutes the groups G^j and a product affine map.*

Proof. By [3], f is smooth. By [4, Theorem 4.1], f is an affine map. In particular, f extends analytically to a conformal map on all of G . By Theorem 5.1, it follows that f is the composition of a group automorphism that permutes the groups G^j and a product map. \square

Remark 5.4. We recall that if G is the Heisenberg group H^n , then conformal maps on a domain in G are restrictions of the action of an element of $SU(1, n + 1)$ [12]. However, if G is the product of m Heisenberg groups H^m where $m \geq 2$, then most elements in $SU(1, n_1 + 1) \times \cdots \times SU(1, n_m + 1)$ do not induce conformal maps on domains in G . Indeed, from the previous corollary, conformal maps are affine in this case.

5.1. CR mappings on product groups.

Theorem 5.5. *Suppose that G is a totally nonabelian Carnot group, with finest direct product decomposition $G^1 \times \cdots \times G^m$, where $m > 1$. Suppose that G^j is tight when $j = 1, \dots, m$. Let $f : G \rightarrow G$ be a CR mapping. Then f is the composition of a group automorphism that permutes the groups G^j and a product CR mapping.*

Proof. By Theorem 3.3, for tight Carnot groups CR and anti-CR diffeomorphisms are the same as conformal mappings. The conclusion now follows from Corollary 5.3. \square

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