C^* -ALGEBRAS WITH WEAK* ANGELIC DUAL BALLS

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(Communicated by Adrian Ioana)

ABSTRACT. Some characterizations of hereditary C^* -subalgebras of weakly Rickart C^* -algebras with weak* angelic dual balls are given. In particular, it turns out that weak* angelicity, weak* sequentiality and weak* sequential compactness of dual balls are the same on this class, and that they are characterized by using ideals of compact elements in those algebras. As an application, it is shown that a weakly Rickart C^* -algebra has weak* angelic dual ball if and only if it coincides with the ideal of compact elements, which happens if and only if the algebra is finite dimensional.

1. INTRODUCTION

This paper is concerned with C^* -algebras that have weak* angelic dual balls. Let K be a topological space. Then K is said to be *angelic* if relatively countably compact subsets of K are relatively compact and the closure of a relatively compact subset of K coincides with the set of limits of its sequences, *sequential* if the sequential closure of a subset of K is closed, and *sequentially compact* if its sequences have convergent subsequences. If K is, further, compact and Hausdorff, then all subsets of K are automatically relatively compact. Hence, in this case, K is angelic if and only if the closure of a subset of K coincides with the set of limits of its sequences. After a moment's thought, we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) for a Banach space X in the following collection of statements.

- (a) The dual ball of X is weak* angelic.
- (b) The dual ball of X is weak* sequential.
- (c) The dual ball of X is weak* sequentially compact.

Moreover, both of the converse implications do not hold in general. Indeed, it has been shown by Martínez-Cervantes [22] that the Johnson-Lindenstrauss space JL_2 has weak* sequential but not weak* angelic dual ball; that is, (b) $\not\Rightarrow$ (a), which settled negatively a question of Plichko [25, Question 10]. See also [5] for further arguments on weak* sequential properties of JL_2 . As was also pointed out in [22], the space $C([0, \omega_1])$ has weak* sequentially compact but not weak* sequential dual ball; and hence, (c) $\not\Rightarrow$ (b).

Received by the editors June 30, 2019, and, in revised form, April 3, 2020, April 21, 2020, and April 23, 2020.

2020 *Mathematics Subject Classification.* Primary 46A50; Secondary 46B20, 46L05, 54D55, 57N17.

Key words and phrases. Angelicity, sequentiality, sequential compactness, weakly Rickart C^* -algebra, hereditary C^* -subalgebra.

This work was supported in part by Grants-in-Aid for Scientific Research Grant Number 19K14561, Japan Society for the Promotion of Science.

A rather wide class of Banach spaces with weak* angelic dual balls is that of spaces with weak* Corson compact dual balls (which includes weakly compactly generated spaces); see [16, Theorems 11.12 and 12.50, and Exercise 12.55], and [19] for some important results on weakly compactly generated spaces. However, there exists a Banach space with weak* angelic but not weak* Corson compact dual ball. An example is given by the dual space of the James tree space $\mathcal{JT}$. Indeed, $\mathcal{JT}$ is a separable Banach space with non-separable dual that contains no isomorphic copies of ℓ_1 ; see, for example, [1, Theorem 13.4.9]. By the Odell-Rosenthal theorem ([12, Theorem XIII.10]) and Bourgain-Fremlin-Talagrand theorem ([12, p.216, Theorem]), the dual space of $\mathcal{JT}$ has weak* angelic dual ball. On the other hand, a separable Banach space with weak* Corson compact second dual ball has separable dual ([16, Exercise 12.57]); and thus, the second dual ball of $\mathcal{JT}$ is not weak* Corson compact. Consequently, the class of Banach spaces with weak* angelic dual ball is strictly wider than that of spaces with weak* Corson compact dual balls.

For general Banach spaces, it is not easy to find (simple) structural properties determining weak* angelicity of dual balls. However, if we restrict our argument to C^* -algebras, a special class of (complex) Banach spaces, the results are expected to be more detailed and sharp; and, such results may provide new hints for the general case. Hence, in this paper, we attack the following problem.

Problem A. Identify the class of C^* -algebras having weak* angelic dual balls.

On the other hand, as was mentioned above, weak* angelicity and weak* sequentiality of dual balls are different to each other in general. However, there remains a chance that they are equivalent on (some classes of) C^* -algebras. We have the following problem related to Problem A.

Problem B. Are weak* angelicity and weak* sequentiality of dual balls equivalent on (some classes of) C^* -algebras?

Practically, in this paper, we focus on hereditary C^* -subalgebras of weakly Rickart C^* -algebras, and provide some conditions determining weak* angelicity of dual balls of such algebras. In particular, we see that weak* angelicity, weak* sequentiality and weak* sequential compactness of dual balls are the same on this class, and that they are characterized by using ideals of compact elements in those algebras. This provides a partial answer to both of Problems A and B. Moreover, as a consequence of the main theorem (Theorem 3.3), it turns out that a weakly Rickart C^* -algebra has weak* angelic dual ball if and only if it coincides with the ideal of compact elements, which happens if and only if the algebra is finite dimensional.

2. PRELIMINARIES AND BASIC RESULTS

A C^* -algebra $\mathfrak{A}$ is a complex Banach algebra with an involution $A \mapsto A^*$ that satisfies the Gelfand-Naimark axiom $\|A^*A\| = \|A\|^2$ for each $A \in \mathfrak{A}$. Typical examples of C^* -algebras are the algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ of all bounded linear and compact operators on a Hilbert space $\mathcal{H}$, and the space $C(K)$ of all continuous (complex-valued) functions on a compact Hausdorff space K . A C^* -algebra $\mathfrak{A}$ is said to be *unital* if it contains a (multiplicative) unit I . If $\mathfrak{A}$ is non-unital, $\mathfrak{A}$ can be embedded as a norm closed ideal of its unitization $\mathfrak{A} \oplus \mathbb{C}$, which is again a C^* -algebra. For each element A of a unital C^* -algebra $\mathfrak{A}$, let $\text{sp}(A)$ be its *spectrum*;

that is, $\text{sp}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible in } \mathfrak{A}\}$. In the case that $\mathfrak{A}$ is non-unital, $\text{sp}(A)$ is defined to be the spectrum of A considered in the unitization of $\mathfrak{A}$. It should be noted that if $\mathfrak{B}$ is a C^* -subalgebra of $\mathfrak{A}$ containing $I_{\mathfrak{A}}$ then $\text{sp}_{\mathfrak{A}}(B) = \text{sp}_{\mathfrak{B}}(B)$ for each $B \in \mathfrak{B}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^* -algebras, and let $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ be linear. Then φ is called a **-homomorphism* if $\varphi(A^*) = \varphi(A)^*$ and $\varphi(AB) = \varphi(A)\varphi(B)$ for each $A, B \in \mathfrak{A}$. We note that if $\mathfrak{A}$ is unital and $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a *-homomorphism, then $\varphi(\mathfrak{A})$ is also unital with $I_{\varphi(\mathfrak{A})} = \varphi(I_{\mathfrak{A}})$. A **-isomorphism* is a bijective *-homomorphism, and if there is a *-isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ then $\mathfrak{A}$ and $\mathfrak{B}$ are **-isomorphic*. A pair $(\mathcal{H}, \pi)$ of a (complex) Hilbert space and a *-homomorphism $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ (with $\pi(I_{\mathfrak{A}}) = I$, the identity operator on $\mathcal{H}$, if $\mathfrak{A}$ is unital) is called a *representation* of $\mathfrak{A}$ on $\mathcal{H}$; and a representation $(\mathcal{H}, \pi)$ is said to be *faithful* if π is injective. The celebrated Gelfand-Naimark theorem asserts that each C^* -algebra admits a faithful representation on a Hilbert space. In other words, all C^* -algebras can be regarded as C^* -subalgebras of $\mathcal{B}(\mathcal{H})$. The commutative version of the Gelfand-Naimark theorem proves that each unital abelian C^* -algebra $\mathfrak{A}$ is *-isomorphic to $C(K)$ for some compact Hausdorff space K .

A von Neumann algebra (acting on a Hilbert space $\mathcal{H}$) is a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed under the weak-operator topology, that is, the weak topology induced by the family of functionals on $\mathcal{B}(\mathcal{H})$ of the form $A \mapsto \langle Ax, y \rangle$. Each von Neumann algebra is unital and *monotone complete*. Namely, each bounded increasing net in a von Neumann algebra has a least upper bound in the algebra.

Let $\mathfrak{A}$ be a C^* -algebra, and let $A \in \mathfrak{A}$. Then A is said to be *self-adjoint* if $A^* = A$, and *positive*, denoted by $A \geq 0$, if A is self-adjoint and $\text{sp}(A)$ consists of non-negative real numbers. If $A \in \mathcal{B}(\mathcal{H})$ then $A \geq 0$ if and only if $\langle Ax, x \rangle \geq 0$ for each $x \in \mathcal{H}$. A self-adjoint idempotent E (that is, $E^* = E = E^2$) in $\mathfrak{A}$ is called a *projection*. If E, F are projections in $\mathfrak{A}$ then $E \leq F$ if and only if $E = EF$. The symbols $\mathfrak{A}_{\text{sa}}$ and $\mathfrak{A}_+$ stand for the sets of all self-adjoint and positive elements of $\mathfrak{A}$, respectively. We note that $\mathfrak{A}_{\text{sa}}$ forms a real Banach space while $\mathfrak{A}_+$ is a norm closed pointed cone in $\mathfrak{A}$. Thanks to the continuous functional calculus, each element of $\mathfrak{A}$ is represented as a finite linear combination of positive elements. These elementary facts about C^* -algebras are found in a nice textbook of Kadison and Ringrose [20] for operator algebras.

A C^* -subalgebra $\mathfrak{B}$ of a C^* -algebra $\mathfrak{A}$ is said to be *hereditary* in $\mathfrak{A}$ if $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ and $0 \leq A \leq B$ imply that $A \in \mathfrak{B}$. Each closed two-sided ideal of $\mathfrak{A}$ is hereditary in $\mathfrak{A}$ ([7, Proposition II.5.3.2]). Another typical example of a hereditary C^* -subalgebra is a corner, that is, a subalgebra of the form $E\mathfrak{A}E$, where E is a projection in $\mathfrak{A}$. Indeed, if $0 \leq A \leq EBE$ for some $A, B \in \mathfrak{A}$ then $0 \leq (I - E)A(I - E) \leq (I - E)EBE(I - E) = 0$ (in the unitization of $\mathfrak{A}$). This shows that $A^{1/2}(I - E) = 0$ since $(A^{1/2}(I - E))^*(A^{1/2}(I - E)) = (I - E)A(I - E) = 0$; which implies that $A(I - E) = A^{1/2}A^{1/2}(I - E) = 0$. Hence $A = AE = EA = AEA \in E\mathfrak{A}E$.

We begin with a simple lemma.

Lemma 2.1. *Let $\mathfrak{B}$ be a hereditary C^* -subalgebra of a non-unital C^* -algebra $\mathfrak{A}$. Then $\mathfrak{B}$ is hereditary in the unitization $\mathfrak{A} \oplus \mathbb{C}$ of $\mathfrak{A}$.*

Proof. Suppose that $A \in \mathfrak{A}_{\text{sa}}$, that $B \in \mathfrak{B}_+$, and that $0 \leq A + \alpha I \leq B$. Since $0 \in \text{sp}(A)$, we have $\alpha \geq 0$. On the other hand, if $\alpha > 0$ then $B - A \in \mathfrak{A}$ becomes invertible; which is impossible. Hence we have $\alpha = 0$ and $A + \alpha I = A \in \mathfrak{B}$. $\square$

Let $\mathfrak{A}$ be a C^* -algebra, and let $\mathcal{F}$ be a commutative family of elements of $\mathfrak{A}$. It is routine to show that there exists a maximal abelian $*$ -subalgebra containing $\mathcal{F}$ (by using Zorn's lemma). We shall introduce (weakly) Rickart C^* -algebras by using maximal abelian $*$ -subalgebras; see [6, 26] for the original definitions and their characterizations, and [27] for exposition of results on monotone (σ -)completeness.

Definition 2.2. Let $\mathfrak{A}$ be a C^* -algebra. Then $\mathfrak{A}$ is said to be *monotone σ -complete* if each bounded increasing sequence in $\mathfrak{A}_{\text{sa}}$ has a least upper bound in $\mathfrak{A}_{\text{sa}}$, and is called a *weakly Rickart C^* -algebra* if each maximal abelian $*$ -subalgebra of $\mathfrak{A}$ is monotone σ -complete. A *Rickart C^* -algebra* is a unital weakly Rickart C^* -algebra.

Obviously, all von Neumann algebras (or, more generally, all AW^* -algebras) are Rickart C^* -algebras. An example that distinguishes weakly Rickart C^* -algebras from those algebras is $\ell_{\infty, c}(\Gamma)$ with Γ uncountable that consists of all countably supported elements of $\ell_{\infty}(\Gamma)$.

A nice property of weakly Rickart C^* -algebra is the richness of projections. We recall that each element of a von Neumann algebra is approximated by a finite linear combination of projections. In fact, this property also holds in the setting of hereditary C^* -subalgebras of weakly Rickart C^* -algebras. The next lemma is probably known. It is included here for completeness.

Lemma 2.3. *Let $\mathfrak{B}$ be a hereditary C^* -subalgebra of a weakly Rickart C^* -algebra $\mathfrak{A}$. Then $\mathfrak{B}$ is the closed linear span of its projections.*

Proof. First we note that the unitization of a weakly Rickart C^* -algebra is a Rickart C^* -algebra; see [26, Lemma 3.6]. From this and Lemma 2.1, considering the unitization of $\mathfrak{A}$ if necessary, we may assume that $\mathfrak{A}$ is a Rickart C^* -algebra. Let A be a positive element of $\mathfrak{B}$. Then a maximal abelian $*$ -subalgebra $\mathcal{A}$ containing A (and I) is $*$ -isomorphic to $C(K)$ for some compact Hausdorff space K . Since the C^* -subalgebra $\mathcal{A} \cong C(K)$ is monotone complete, the element A can be approximated in norm by a (positive) finite linear combination of projections $p_1, \dots, p_m$ satisfying, additionally, that $p_j \leq \lambda_j A$ for a positive $\lambda_j \in \mathbb{R}$ for each j (cf. the basic theory on clopen sets in K [6, Proposition 8.1]). Since $\mathfrak{B}$ is hereditary and $A \in \mathfrak{B}$, it follows that $p_j \in \mathfrak{B}$ for all j , which finishes the proof. $\square$

3. WEAK* ANGELICITY OF DUAL BALLS

We now proceed to consider weak* angelicity of dual balls of hereditary C^* -subalgebras of weakly Rickart C^* -algebras. In this setting, the ideal of compact elements in a C^* -algebra plays a crucial role; see [2, 29] for basic references on compact elements in C^* -algebras.

Definition 3.1. Let $\mathfrak{A}$ be a C^* -algebra, and let $A \in \mathfrak{A}$ be nonzero. Then A is said to be *rank one* with respect to $\mathfrak{A}$ if $A\mathfrak{A}A \subset \mathbb{C}A$ holds. Let $R_1(\mathfrak{A})$ be the set of all rank one elements with respect to $\mathfrak{A}$, and let $\mathcal{K}(\mathfrak{A})$ be the closed linear span of $R_1(\mathfrak{A})$. By a *compact* element of $\mathfrak{A}$, we mean an element of $\mathcal{K}(\mathfrak{A})$.

We note that if A is a rank one element of $\mathfrak{A}$ then so is aA for each $a \in \mathbb{C}$. Moreover, for each $B \in \mathfrak{A}$, we have $AB\mathfrak{A}AB \subset \mathbb{C}AB$ and $BA\mathfrak{A}BA \subset \mathbb{C}BA$; that is, both AB and BA are also at most rank one. Hence $\mathcal{K}(\mathfrak{A})$ forms a norm closed two-sided ideal in $\mathfrak{A}$. It should be noted that, in a von Neumann algebra $\mathcal{R}$, rank one projections with respect to $\mathcal{R}$ coincide with minimal projections in it.

Each C^* -algebra $\mathfrak{A}$ has a faithful representation on a Hilbert space $\mathcal{H}$ that maps $\mathcal{K}(\mathfrak{A})$ into $\mathcal{K}(\mathcal{H})$ (see [30]). This result is based on some techniques in Erdős' paper [14]. Moreover, $\mathcal{K}(\mathfrak{A})$ is closely related to some algebraic notions such as socle, minimal left ideal, annihilator C^* -algebra and so on; see, for example, Almus, Blecher and Sharma [4] and Almus [3].

The following lemma is known. The proof is included here for completeness.

Lemma 3.2. *Let $\mathfrak{A}$ be a C^* -algebra. Then $\mathcal{K}(\mathfrak{A})$ is Asplund and weakly compactly generated.*

Proof. Since $\mathcal{K}(\mathfrak{A})$ is a compact C^* -algebra, it follows from [2, Theorem 8.2] that $\mathcal{K}(\mathfrak{A})$ is isometrically isomorphic to $\bigoplus_{\lambda}^{c_0} \mathcal{K}(\mathcal{H}_{\lambda})$. Smith and Ward proved in [28] that the M -ideals in a C^* -algebra are exactly the closed two-sided ideals. Consequently, $\mathcal{K}(\mathfrak{A}) \cong \bigoplus_{\lambda}^{c_0} \mathcal{K}(\mathcal{H}_{\lambda})$ is an M -ideal in the space $\bigoplus_{\lambda}^{\ell_{\infty}} \mathcal{B}(\mathcal{H}_{\lambda})$. A result by Fabián and Godefroy asserts that if a non-reflexive Banach space is an M -ideal in its bidual, then it is Asplund and weakly compactly generated (see [15, Theorem 3]). Thus $\mathcal{K}(\mathfrak{A})$ is Asplund and weakly compactly generated. $\square$

We are now ready to prove the main theorem in this paper. We note that the term “orthogonal family (E_{λ}) of projections” means a family of projections satisfying $E_{\lambda}E_{\mu} = 0$ whenever $\lambda \neq \mu$.

Theorem 3.3. *Let $\mathfrak{B}$ be a hereditary C^* -subalgebra of a weakly Rickart C^* -algebra $\mathfrak{A}$. Then the following are equivalent:*

- (i) $\mathfrak{B}$ has weak* angelic dual ball.
- (ii) $\mathfrak{B}$ has weak* sequential dual ball.
- (iii) $\mathfrak{B}$ has weak* sequentially compact dual ball.
- (iv) $\mathfrak{B}$ does not contain an isomorphic copy of ℓ_{∞} .
- (v) $\mathfrak{B}$ does not contain a C^* -subalgebra $*$ -isomorphic to ℓ_{∞} .
- (vi) $\mathfrak{B}$ does not contain any projection that is an upper bound of an infinite orthogonal family of nonzero projections in $\mathfrak{B}$.
- (vii) $\mathfrak{B} = \mathcal{K}(\mathfrak{B})$.
- (viii) $\mathfrak{B}$ is weakly compactly generated.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are obvious. If (iii) holds, then, by the Hahn-Banach theorem, each closed subspace of $\mathfrak{B}$ has weak* sequentially compact dual ball. Moreover, the property that the dual ball is weak* sequentially compact is preserved under bounded linear isomorphisms. Hence $\mathfrak{B}$ cannot contain an isomorphic copy of ℓ_{∞} because, as it is well known, the dual ball of the latter space is not weak* sequentially compact; that is, (iii) $\Rightarrow$ (iv) holds.

Suppose that (v) holds. To see (v) $\Rightarrow$ (vi), suppose on the contrary that $\mathfrak{B}$ contains a projection E_0 that is an upper bound for an infinite orthogonal family (E_{λ}) of nonzero projections in $\mathfrak{B}$. As in the proof of Lemma 2.3, we may assume that $\mathfrak{A}$ is a Rickart C^* -algebra. Let (E_n) be a countable subfamily of (E_{λ}) . Let $\mathcal{A}$ denote a maximal abelian $*$ -subalgebra of $\mathfrak{A}$ containing (E_n) , E_0 and I . Then $\mathcal{A}$ is $*$ -isomorphic to a monotone σ -complete $C(K)$ under a $*$ -isomorphism φ . Let $e_n = \varphi(E_n)$ for each $n \in \mathbb{N} \cup \{0\}$, let N be a nonempty (finite or infinite) subset of $\mathbb{N}$, and let (n_k) be a representation of N as a strictly increasing sequence of natural numbers. Since $(\sum_{k=1}^m e_{n_k})_m$ is an increasing sequence in $C(K)$, there exists its least upper bound $e_N \in C(K)$. We note that $0 \leq e_N \leq e_0$ since e_0 is an

upper bound for $(\sum_{k=1}^m e_{n_k})_m$. Now let $K_N = \{t \in K : \sup_m \sum_{k=1}^m e_{n_k}(t) = 1\} = \bigcup_{n \in \mathbb{N}} e_n^{-1}(\{1\})$. Then K_N is open in K ; and, by Urysohn's lemma, e_N coincides with the characteristic function of $\overline{K_N}$. Here we note that $\varphi^{-1}(e_N) \in \mathfrak{B}$ because $0 \leq \varphi^{-1}(e_N) \leq E_0$ and $E_0 \in \mathfrak{B}$. Put $e_\emptyset = 0$.

Let M, N be mutually disjoint subsets of $\mathbb{N}$. Then $K_M \cap K_N = \emptyset$; in which case, $K_M \subset K \setminus K_N$ and $K \setminus K_N$ is closed in K . It follows that $\overline{K_M} \subset K \setminus K_N$; that is, $K_N \subset K \setminus \overline{K_M}$. Since $\overline{K_M} = e_M^{-1}(\{1\})$ is also open in K , we derive that $\overline{K_N} \subset K \setminus \overline{K_M}$. Therefore $\overline{K_M} \cap \overline{K_N} = \emptyset$, and $e_M e_N = 0$. We also note that $e_{M \cup N} = e_M + e_N$ by $\overline{K_{M \cup N}} = \overline{K_M} \cup \overline{K_N}$.

Next, let M, N be arbitrary subsets of $\mathbb{N}$. Since $\overline{K_M} = \overline{K_{M \cap N}} \cup \overline{K_{M \setminus N}}$ and $\overline{K_{M \cap N}} \cap \overline{K_{M \setminus N}} = \emptyset$, it follows that $e_M = e_{M \cap N} + e_{M \setminus N}$. In particular, we have $e_M e_N = e_{M \cap N}$.

Let $\mathcal{M}_0$ be the linear span of $\{e_N : N \subset \mathbb{N}\}$, where $e_\emptyset = 0$ and $e_{\{n\}} = e_n$ for each $n \in \mathbb{N}$. Remark that $\mathcal{M}_0$ is a $*$ -subalgebra of $C(K)$ satisfying $\varphi^{-1}(\mathcal{M}_0) \subset \mathfrak{B}$. Define a mapping $T : \mathcal{M}_0 \rightarrow \ell_\infty$ by

$$[T(e_N)](n) = \begin{cases} 1 & (n \in N) \\ 0 & (n \notin N) \end{cases}$$

for each $N \subset \mathbb{N}$ and

$$T\left(\sum_{i=1}^m a_i e_{M_i}\right) = \sum_{i=1}^m a_i T(e_{M_i})$$

for each finite linear combination $\sum_{i=1}^m a_i e_{M_i} \in \mathcal{M}_0$. To see the well-definedness of T , suppose that $f = \sum_{i=1}^m a_i e_{M_i} = \sum_{j=1}^n b_j e_{N_j}$. For each $k \in \mathbb{N}$, let $M(k) = \{i \in \{1, 2, \dots, m\} : k \in M_i\}$ and $N(k) = \{j \in \{1, 2, \dots, n\} : k \in N_j\}$. Then it follows from

$$\left(\sum_{i \in M(k)} a_i\right) e_k = \left(\sum_{i=1}^m a_i e_{M_i}\right) e_k = \left(\sum_{j=1}^n b_j e_{N_j}\right) e_k = \left(\sum_{j \in N(k)} b_j\right) e_k,$$

that $\sum_{i \in M(k)} a_i = \sum_{j \in N(k)} b_j$. On the other hand, from the definition, we have

$$\left[\sum_{i=1}^m a_i T(e_{M_i})\right](k) = \sum_{i \in M(k)} a_i = \sum_{j \in N(k)} b_j = \left[\sum_{j=1}^n b_j T(e_{N_j})\right](k)$$

Hence T is well-defined. Moreover, according to the above, one obtains that

$$\left|\sum_{i \in M(k)} a_i\right| = \left\|\left(\sum_{i \in M(k)} a_i\right) e_k\right\|_\infty \leq \left\|\sum_{i=1}^m a_i e_{M_i}\right\|_\infty.$$

This shows that $\|T(f)\|_\infty \leq \|f\|_\infty$ for each $f \in \mathcal{M}_0$. Conversely, if $t \in \overline{K_{\mathbb{N}}}$ then there exists a net $(t_\lambda) \subset K_{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} e_n^{-1}(\{1\})$ converging to t . For each λ , there exists a unique k with $t_\lambda \in e_k^{-1}(\{1\})$. We obtain

$$|f(t_\lambda)| = |(fe_k)(t_\lambda)| = \left|\sum_{i \in M(k)} a_i\right| \leq \|Tf\|_\infty.$$

Hence it follows that $|f(t)| = \lim_\lambda |f(t_\lambda)| \leq \|Tf\|_\infty$. Since $f(t) = 0$ for each $t \notin \overline{K_{\mathbb{N}}}$, the equality $\|f\|_\infty = \|Tf\|_\infty$ holds for all $f \in \mathcal{M}_0$.

Here we remark that each element of $\mathcal{M}_0$ has the form $\sum_{i=1}^m a_i e_{M_i}$ with mutually disjoint subsets $(M_i)_{i=1}^m$ of $\mathbb{N}$ satisfying $\bigcup_{i=1}^m M_i = \mathbb{N}$. Indeed, we have $ae_N = ae_N + 0 \cdot e_{\mathbb{N} \setminus N}$ for each $a \in \mathbb{C}$ and $N \subset \mathbb{N}$. Fix an n . Suppose that each element of the form $\sum_{j=1}^n b_j e_{N_j}$ has another representation $\sum_{i=1}^m a_i e_{M_i}$ with mutually disjoint subsets $(M_i)_{i=1}^m$ of $\mathbb{N}$ satisfying $\bigcup_{i=1}^m M_i = \mathbb{N}$, then

$$\begin{aligned} \sum_{j=1}^{n+1} b_j e_{N_j} &= b_{n+1} e_{N_{n+1}} + \sum_{i=1}^m a_i e_{M_i} \\ &= b_{n+1} \sum_{i=1}^m e_{N_{n+1} \cap M_i} + \sum_{i=1}^m a_i (e_{N_{n+1} \cap M_i} + e_{M_i \setminus N_{n+1}}) \\ &= \sum_{i=1}^m (b_{n+1} + a_i) e_{N_{n+1} \cap M_i} + \sum_{i=1}^m a_i e_{M_i \setminus N_{n+1}}. \end{aligned}$$

Since $(N_{n+1} \cap M_i)_{i=1}^m \cup (M_i \setminus N_{n+1})_{i=1}^m$ is a family of mutually disjoint subsets of $\mathbb{N}$ with union $\mathbb{N}$, we have that each element of the form $\sum_{j=1}^{n+1} b_j e_{N_j}$ has a desired representation. Thus an induction argument works.

Let $f = \sum_{i=1}^m a_i e_{M_i}$ and $g = \sum_{j=1}^n b_j e_{N_j}$ be arbitrary elements of $\mathcal{M}_0$. From the preceding paragraph, we may assume that $(M_i)_{i=1}^m$ and $(N_j)_{j=1}^n$ are sequences of mutually disjoint subsets of $\mathbb{N}$ satisfying $\bigcup_{i=1}^m M_i = \bigcup_{j=1}^n N_j = \mathbb{N}$. It is obvious that $T(af) = aT(f)$ and $T(f^*) = T(f)^*$ for each $a \in \mathbb{C}$. For the addition, since

$$\begin{aligned} f + g &= \sum_{i=1}^m a_i \sum_{j=1}^n e_{M_i \cap N_j} + \sum_{j=1}^n b_j \sum_{i=1}^m e_{M_i \cap N_j} \\ &= \sum_{i=1}^m \sum_{j=1}^n (a_i + b_j) e_{M_i \cap N_j}, \end{aligned}$$

it follows that

$$\begin{aligned} T(f + g) &= \sum_{i=1}^m \sum_{j=1}^n (a_i + b_j) T(e_{M_i \cap N_j}) \\ &= \sum_{i=1}^m a_i \sum_{j=1}^n T(e_{M_i \cap N_j}) + \sum_{j=1}^n b_j \sum_{i=1}^m T(e_{M_i \cap N_j}) \\ &= \sum_{i=1}^m a_i T(e_{M_i}) + \sum_{j=1}^n b_j T(e_{N_j}) \\ &= T(f) + T(g). \end{aligned}$$

Moreover, since

$$fg = \left(\sum_{i=1}^m a_i e_{M_i} \right) \left(\sum_{j=1}^n b_j e_{N_j} \right) = \sum_{i=1}^m \sum_{j=1}^n a_i b_j e_{M_i \cap N_j},$$

we derive that

$$\begin{aligned} T(fg) &= \sum_{i=1}^m \sum_{j=1}^n a_i b_j T(e_{M_i \cap N_j}) = \left(\sum_{i=1}^m a_i T(e_{M_i}) \right) \left(\sum_{j=1}^n b_j T(e_{N_j}) \right) \\ &= T(f)T(g). \end{aligned}$$

Finally, let $\overline{T}$ be a unique bounded linear extension of T to the C^* -subalgebra $\overline{\mathcal{M}_0}$ of $C(K)$. Then $\overline{T}$ is an isometric $*$ -isomorphism from $\mathcal{M}_0$ to $\overline{T(\mathcal{M}_0)} \subset \ell_\infty$. Since $\overline{T(\mathcal{M}_0)}$ contains every sequence of scalars whose terms all come from $\{0, 1\}$ (i.e. all projections in ℓ_∞), by [24, Lemma 4.2.6], we have $\overline{T(\mathcal{M}_0)} = \ell_\infty$. Thus $\varphi^{-1}(\overline{\mathcal{M}_0})$ is a C^* -subalgebra of $\mathfrak{B}$ $*$ -isomorphic to ℓ_∞ ; a contradiction. This shows (v) $\Rightarrow$ (vi).

Assume next that (vi) holds. Let E be a projection in $\mathfrak{B}$. Suppose that E is not a finite sum of rank one projections with respect to $\mathfrak{B}$. Since $E\mathfrak{B}E \not\subset \mathbb{C}E$ and $E\mathfrak{B}E$ is a hereditary C^* -subalgebra of $\mathfrak{B}$ (and hence, of $\mathfrak{A}$), by Lemma 2.3, there exists a projection $E_1 \in E\mathfrak{B}E$ with $0 < E_1 < E$. We note that either E_1 or $E - E_1$ is not a finite sum of rank one projections with respect to $\mathfrak{B}$. It may be assumed that $E - E_1$ is not a finite sum of rank one projections with respect to $\mathfrak{B}$. An argument similar to the above generates a nonzero projection E_2 with $0 < E_2 < E - E_1$. Again, either E_2 or $E - E_1 - E_2$ is not a finite sum of rank one projections with respect to $\mathfrak{B}$. Continuing this process, one obtains an infinite orthogonal family (E_n) of proper subprojections of E ; and, E is an upper bound for (E_n) . This contradicts (vi). Therefore E must be a finite sum of rank one projections with respect to $\mathfrak{B}$, that is, $E \in \mathcal{K}(\mathfrak{B})$. Again, by Lemma 2.3, $\mathfrak{B}$ coincides with the closed linear span of projections in $\mathfrak{B}$. It follows that $\mathfrak{B} = \mathcal{K}(\mathfrak{B})$; and (vi) $\Rightarrow$ (vii) holds.

The implication (vii) $\Rightarrow$ (viii) follows from Lemma 3.2. If (viii) holds, then the real version $\mathfrak{A}_r$ (i.e. the same space considered as a real space) is also weakly compactly generated by [8, Proposition 2.2 (ii)]. It follows from [16, Theorems 11.12 and 12.50, and Exercise 12.55] that $(\mathfrak{A}_r)^*$ has weak* angelic dual ball. Moreover, the correspondence $\rho \mapsto \operatorname{Re} \rho$ is a real-linear isometry from $\mathfrak{A}^*$ onto $(\mathfrak{A}_r)^*$, and $\rho(x) = \operatorname{Re} \rho(x) - i \operatorname{Re} \rho(ix)$ holds for each $\rho \in \mathfrak{A}^*$ and each $x \in \mathfrak{A}_r$. From this, we can see that the dual ball of $\mathfrak{A}$ is also weak* angelic. This proves that (viii) $\Rightarrow$ (i). The proof is complete. $\square$

Here we recall that, by some well-known results, every weakly compactly generated von Neumann algebra must be finite dimensional. Indeed, W. B. Johnson and C. Stegall observed that weakly compactly generated duals have the Radon-Nikodym property (see [13, Page 87]). Actually, by a result due to T. Kuo, if X is a Banach space whose dual, X^* , is a subspace of a weakly compactly generated Banach space Y , then X^* has the Radon-Nikodym property (see [13, Corollary 3.7]). Therefore, every weakly compactly generated von Neumann algebra satisfies the Radon-Nikodym property. It is known that a C^* -algebra satisfies the Radon-Nikodym property if and only if it is reflexive (see [11, Theorem 6]). Furthermore, reflexive C^* -algebras are all finite dimensional (cf. [23], [17] or [18]). Therefore every weakly compactly generated von Neumann algebra must be finite dimensional.

In fact, a bit more work in the preceding theorem provides a sharper consequence for weakly Rickart C^* -algebras. Indeed, putting $\mathfrak{B} = \mathfrak{A}$ in Theorem 3.3, we obtain the following result.

Corollary 3.4. *Let $\mathfrak{A}$ be a weakly Rickart C^* -algebra. Then the following are equivalent:*

- (i) $\mathfrak{A}$ is finite dimensional.
- (ii) $\mathfrak{A}$ has weak* angelic dual ball.
- (iii) $\mathfrak{A}$ does not contain a C^* -subalgebra $*$ -isomorphic to ℓ_∞ .
- (iv) $\mathfrak{A} = \mathcal{K}(\mathfrak{A})$.
- (v) $\mathfrak{A}$ is weakly compactly generated.

Proof. The equivalence of (ii) to (v) is due to Theorem 3.3. It is clear that (i) implies the other conditions. Now suppose that $\mathfrak{A}$ is infinite dimensional. We first assume that $\mathfrak{A}$ is unital; in which case, $\mathfrak{A}$ is a Rickart C^* -algebra. Since $\mathfrak{A}$ is infinite dimensional, each maximal abelian $*$ -subalgebra $\mathcal{A}$ of $\mathfrak{A}$ is infinite dimensional; see, for example, [20, Exercise 4.6.12]. Let $\varphi : \mathcal{A} \rightarrow C(K)$ be a $*$ -isomorphism. We note that $C(K)$ is monotone σ -complete and K contains infinitely many points. Suppose that $t_1, t_2 \in K$ with $t_1 \neq t_2$. Since K is Hausdorff, we have a closed neighborhood L_1 of t_1 which does not contain t_2 . Then, by Urysohn's lemma, there exists a continuous function $f_1 : K \rightarrow [0, 1]$ with $f_1(L_1) = 0$ and $f_1(t_2) = 1$. Let $e_1 \in C(K)$ be the least upper bound for the increasing sequence $(f_1^{1/n})$. Again, by Urysohn's lemma, it turns out that e_1 is the characteristic function of the closure K_1 of $\{t \in K : \sup_n f_1^{1/n}(t) = 1\} = \{t \in K : f_1(t) > 0\}$, which does not contain t_1 . In particular, $0 < e_1 < \mathbf{1}$. Moreover, either K_1 or $K \setminus K_1$ contains infinitely many points. Replacing e_1 with $\mathbf{1} - e_1$ if necessary, we may assume that $K \setminus K_1$ is an infinite (clopen) subset of K . Then, again, an argument similar to the above generates a projection e_2 with $0 < e_2 < \mathbf{1} - e_1$ which is the characteristic function of $K_2 \subset K \setminus K_1$. It may be assumed that $K \setminus (K_1 \cup K_2)$ is infinite. Continuing this process yields an infinite orthogonal family (e_n) of projections in $C(K)$. Hence $(\varphi^{-1}(e_n))$ is an infinite orthogonal family of projections in $\mathfrak{A}$ with an upper bound $I \in \mathfrak{A}$. Thus, by Theorem 3.3, $\mathfrak{A}$ does not satisfy the properties stated in (ii) to (v). Finally, if $\mathfrak{A}$ is non-unital (and infinite dimensional), then its unitization $\mathfrak{A} \oplus \mathbb{C}$ is, in particular, not weakly compactly generated. Therefore $\mathfrak{A}$ is itself not weakly compactly generated; which proves that (v) $\Rightarrow$ (i). This completes the proof. $\square$

Remark 3.5. We note that Theorem 3.3 does not reduce to Corollary 3.4. Let $\mathcal{K}(\mathcal{H})$ be the C^* -algebra of all compact operators on an infinite dimensional Hilbert space $\mathcal{H}$. Then $\mathcal{K}(\mathcal{H})$ provides an example of hereditary C^* -subalgebra of a weakly Rickart (actually, a von Neumann) algebra $\mathcal{B}(\mathcal{H})$ which is not weakly Rickart C^* -algebra. Indeed, each two-sided closed ideal in a C^* -algebra is hereditary by [7, Proposition II.5.3.2 (i)]. On the other hand, since $\mathcal{K}(\mathcal{H}) = \mathcal{K}(\mathcal{B}(\mathcal{H}))$ is infinite dimensional and weakly compactly generated by Lemma 3.2, it cannot be a weakly Rickart C^* -algebra by Corollary 3.4.

We finally consider our results in the setting of von Neumann algebras. A von Neumann algebra $\mathcal{R}$ is said to be of *type I* if it has an abelian projection with central carrier I . If $\mathcal{R}$ has no nonzero abelian projection but has a finite projection with central carrier I , then $\mathcal{R}$ is said to be of *type II*. If $\mathcal{R}$ has no nonzero finite projections, $\mathcal{R}$ is said to be of *type III*. It is known that every von Neumann algebra is decomposed into its type I, II and III portions by a family of central projections; see [21, Section 6.5].

As a consequence of Theorem 3.3, we have the following result.

Corollary 3.6. *A von Neumann algebra containing a nontrivial hereditary C^* -subalgebra with weak* angelic dual ball has nonzero type I part.*

Proof. Suppose that $\mathcal{R}$ is a von Neumann algebra that contains a hereditary C^* -subalgebra $\mathfrak{A} (\neq \{0\})$ with weak* angelic dual ball. Since each minimal projection in $\mathcal{R}$ is abelian ([21, Proposition 6.4.3]), it is enough to show that $\mathcal{R}$ contains a minimal projection. By Theorem 3.3, we have $\mathfrak{A} = \mathcal{K}(\mathfrak{A})$. In particular, $\mathfrak{A}$ contains a rank one projection E with respect to $\mathfrak{A}$; see the proof of (vi) $\Rightarrow$ (vii) in Theorem 3.3. Now let $F \in \mathcal{R}$ be a nonzero subprojection of E . Then $F \in \mathfrak{A}$ since $\mathfrak{A}$ is hereditary in $\mathcal{R}$, which implies that $F = E$. We have thus proved that E is minimal in $\mathcal{R}$ as desired. $\square$

Recall that a *factor* is a von Neumann algebra with trivial center $\mathbb{C}I$. The following is immediate from Corollary 3.6.

Corollary 3.7. *A factor containing a nontrivial hereditary C^* -subalgebra with weak* angelic dual ball is of type I.*

Finally, we shall mention some related results. In [8–10], it is proved that preduals of von Neumann algebras, JBW*-algebras, and more generally, of JBW*-triples are 1-Plichko spaces and it is also determined when these spaces are weakly Lindelöf determined and weakly compactly generated.

ACKNOWLEDGMENT

The author would like to thank the referee for the careful reading, detailed helpful remarks and valuable comments.

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