PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 8, Pages 1–10 (January 8, 2021)
https://doi.org/10.1090/bproc/67

HALF-SPACE TYPE THEOREM FOR TRANSLATING SOLITONS OF THE MEAN CURVATURE FLOW IN EUCLIDEAN SPACE

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(Communicated by Guofang Wei)

Abstract. In this paper, we determine which half-space contains a complete translating soliton of the mean curvature flow and it is related to the well-known half-space theorem for minimal surfaces. We prove that a complete translating soliton does not exist with respect to the velocity v in a closed half-space $\mathcal{H}_{\widetilde{\mathrm{v}}} = \{x \in \mathbb{R}^{n+1} \mid \langle x, \widetilde{\mathrm{v}}\rangle \leq 0\}$ for $\langle \mathrm{v}, \widetilde{\mathrm{v}}\rangle > 0$, whereas in a half-space $\mathcal{H}_{\widetilde{\mathrm{v}}}$, $\langle \mathrm{v}, \widetilde{\mathrm{v}}\rangle \leq 0$, a complete translating soliton can be found. In addition, we extend this property to cones: there are no complete translating solitons with respect to v in a right circular cone $C_{\mathrm{v},a} = \{x \in \mathbb{R}^{n+1} \mid \langle \frac{x}{\|x\|}, \mathrm{v}\rangle \leq a < 1\}$.

1. Introduction

In minimal surface theory, the half-space theorem proved by Hoffman and Meeks [18] is a well-known global property: *A connected, proper, possibly branched, non-planar minimal surface in* $\mathbb{R}^3$ *is not contained in a half-space.* They used two main elements, namely, the scaling invariance of minimality of the minimal surfaces in Euclidean space and the maximum principle. Specifically, the facts that a half-catenoid converges to the punctured plane through a scale-down and the maximum principle were used to prove the half-space theorem. On the one hand, the generalization of the half-space theorem fails in a higher dimensional Euclidean space. In this case, a higher dimensional catenoid is contained in a domain bounded by two parallel hyperplanes. On the other hand, generalizations of the half-space theorem to other geometric surfaces, such as constant mean curvature surfaces, in various ambient spaces has been successful (see, [3, 10, 12, 28–30, 33, 39]).

In this paper, we consider translating solitons of the mean curvature flow (MCF). A smooth family of immersions $F : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$ is a solution of the MCF if F satisfies the following parabolic equation:

$$\frac{\partial}{\partial t} F(p, t) = \overrightarrow{H}(p, t), \tag{1.1}$$

for all $(p, t) \in \Sigma \times [0, T)$, where $\overrightarrow{H}$ is the mean curvature vector. The MCF is a negative gradient flow of the area functional. It is also generally known that any closed hypersurface develops singularities within a finite time under the MCF. Therefore,

Received by the editors June 24, 2020, and, in revised form, November 12, 2020.
2020 *Mathematics Subject Classification.* Primary 53C42; Secondary 53E10.
The first author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2019R1C1C1004819) and a KIAS Individual Grant (MG070801) at Korea Institute for Advanced Study.
The second author was supported by NRF (NRF-2017R1E1A1A03070495 and NRF-2020R1A2C1A01005698).

it is important to study the singularities of the MCF. Huisken [19] and Huisken and Sinestrari [20] demonstrated that there are two types of singularities, type-I and type-II, represented by a self-shrinker and a translating soliton, respectively. The translating soliton of the MCF is not only a blow-up limit flow of a type-II singularity but also a special solution that moves only in a constant direction v without deforming its shape under the MCF; the solution form is as follows:

$$F(p,t) = F(p) + \mathrm{v}t,$$

where $F(p) = F(0,p)$.

For a self-shrinker, Cavalcante and Espinar [4] proved that the only properly immersed self-shrinker Σ is contained in one of the closed half-spaces determined by a hyperplane P is $\Sigma = P$. Some of the known translating solitons such as grim reaper cylinders, translating bowls and Δ-wings are contained in a half-space. In [40], Shahriyari proved the non-existence of complete graphical translating solitons in a cylindrical domain and Møller [31] extended it to both a higher dimensional case and complete proper embedded translating solitons. Recently, Chini and Møller [8] demonstrated that there are no properly immersed n-dimensional translating solitons contained in the intersection domain of two transverse half-spaces that are parallel to v in $\mathbb{R}^{n+1}$. According to the various examples in Section 2, there are complete translating solitons in a half-space that is containing v. However, such translating solitons are not contained in a half-space that does not contain v. This is generally true, as proved through the following theorem:

Theorem 1.1. *There are no complete translating solitons for the velocity* v *under the MCF in a closed half-space* $\mathcal{H}_{\widetilde{\mathrm{v}}} = \{x \in \mathbb{R}^{n+1} \mid \langle x, \widetilde{\mathrm{v}}\rangle \leq 0\}$ *with* $\langle \mathrm{v}, \widetilde{\mathrm{v}}\rangle > 0$.

Corollary 1.2. *There are no complete translating solitons in any bounded domain of* $\mathbb{R}^{n+1}$.

Because translating solitons are minimal hypersurfaces in Euclidean space endowed with a conformally flat metric, they share an analogical property with minimal surfaces in Euclidean space. However, the Corollary 1.2 shows that translating solitons have a different property compared to that in the minimal surface theory. More precisely, a complete minimal surface in a ball in $\mathbb{R}^3$ exists, which was proved by Nadirashvili [32]. We note that Chini and Møller [8] proved the non-existence of complete proper translating solitons in a bounded domain.

As another result in the minimal surface theory, Omori [37] considered the non-existence of a minimal isometric immersion with the bounded below sectional curvature in the cones of Euclidean space using the existence of a sequence of points, which is now popularly known as the Omori–Yau maximum principle. The Omori–Yau maximum principle was introduced in [6, 37, 42] and can be applied to non-compact cases: Omori [37] proved the Omori–Yau maximum principle for the Hessian of a function bounded from above on a complete Riemannian manifold with the sectional curvature bounded from below. Yau [42] and Cheng and Yau [6] established for the Laplacian of a function bounded from above on a complete Riemannian manifold with the Ricci curvature bounded from below. Various versions of the Omori–Yau maximum principle have been used to prove the geometric problems in [5, 7, 8, 25, 43]. A detailed introduction to various Omori–Yau maximum principles and their applications can be found in [1]. Based on the Omori–Yau maximum principle, we extend the results of Omori [37] for minimal surfaces to translating solitons of the MCF:

Theorem 1.3. *There are no complete translating solitons for the velocity* v *under the MCF in a right circular cone* $C_{\mathrm{v},a} = \{x \in \mathbb{R}^{n+1} \mid \langle \frac{x}{\|x\|}, \mathrm{v}\rangle \leq a < 1\}$.

As previous study, Impera and Rimoldi [21] demonstrated that an n-dimensional f-stochastically complete translating soliton cannot be contained in a lower cone containing the direction of the translation under the MCF. In [41], Xin established a version of the Omori–Yau maximum principle for complete proper translating solitons in Euclidean space, which is valid for a higher codimensional case.

2. Translating solitons of the MCF

Definition 2.1. A *translating soliton* M with the velocity v is a hypersurface in $\mathbb{R}^{n+1}$ if it is a solution to the MCF such that the solution form is as follows:

$$F(p,t) = F(p) + \mathrm{v}t,$$

where $F(p) = F(p,0)$. As an equivalent condition, it satisfies the following equation:

$$\overrightarrow{H} = \mathrm{v}^{\perp},$$

where $\overrightarrow{H}$ is the mean curvature vector field of M.

The velocity v indicates the direction of the translation of the translating soliton under the MCF. Up to scaling and rotation in $\mathbb{R}^{n+1}$, it is possible to assume $\mathrm{v} = e_{n+1}$. There are various translating solitons, some of which are listed herein. We can find more translating solitons for examples, see [8, 15, 27] and the references therein.

Example 2.2 (Product of minimal submanifold)**.** The simplest translating soliton is a plane parallel to v in $\mathbb{R}^3$ as a product of a line and $\mathbb{R}$ parallel to the direction v. From this perspective, the translating solitons in $\mathbb{R}^{n+1}$ can be constructed as the product of an $(n-1)$-dimensional minimal submanifold M and $\mathbb{R}$ parallel to the direction v, *i.e.*, $M \times \mathbb{R}$. There are numerous of translating solitons arising from minimal submanifolds. In [32], Nadirashvili constructed a complete, non-proper, minimal disk in the unit ball. Specifically, a complete non-proper translating soliton can be obtained from Nadirashvili's minimal surface.

Example 2.3 (Grim reaper cylinders)**.** The grim reaper $y = -\log\cos(x)$ is a translating soliton on $\mathbb{R}^2$, *i.e.*, the only eternal solution of the MCF in $\mathbb{R}^2$, which is also known as the curve-shortening flow. Its product surface, which is a cylindrical surface of the grim reaper, is called a *canonical grim reaper cylinder* whose suitable combination of rotation and dilation is called a *grim reaper cylinder.* The following parametrization is for a family of grim reaper cylinders:

$$X_\theta(u,v) = \left(s, t, -\frac{1}{\cos^2(\theta)}\log\cos(s\cos(\theta)) + t\tan(\theta)\right).$$

In particular, the grim reaper cylinder is a one-parameter family of cylindrical surfaces from the canonical grim reaper cylinder to the plane parallel to $\mathrm{v} = e_3$.

Example 2.4 (Translating bowl and winglike translator)**.** Altschuler and Wu [2] and Clutterbuck, Schnürer and Schulze [9] showed the existence of the translating bowl and the winglike translator. These are rotationally symmetric translating solitons and can thus be represented as an immersion $X : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ parametrized by

$$X(s, \phi_1, \cdots, \phi_{n-1}) = (x(s)\Phi(\phi_1, \cdots, \phi_{n-1}), y(s)),$$

where Φ is an orthogonal parametrization of the $(n-1)$-dimensional unit sphere. The profile curve $\gamma(s) = (x(s), y(s))$ parametrized by arc-length satisfies the following differential equation:

$$x'(1-n) + nyy' + y(x'y'' - x''y') = 0.$$

In particular, the translating bowl and winglike translator have an asymptotic behavior of $y = x^2$. The authors [24] rediscovered their asymptotic behaviors of the profile curve using the phase-plane method to the above differential equation.

Example 2.5 (Generalized winglike translator)**.** As a generalization of the winglike translator, Kunikawa [26] constructed an m-dimensional translating soliton in $\mathbb{R}^n$. Let N be any minimal submanifold in $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$ and $r : [0,\infty] \to \mathbb{R}$ be a function satisfying

$$r'' = (1 + r'^2)\left(1 - \frac{(m-1)r'}{t}\right),$$

which is an m-dimensional winglike translator equation. The immersion $F : M \to \mathbb{R}^n$ defined by $F(t,p) = (tp, r(t))$ where $p \in N$ and $t \in [0,\infty]$ is an m-dimensional translating soliton with the velocity $e_n \in \mathbb{R}^n$.

Example 2.6 (Helicoidal translating solitons)**.** Halldorsson [13] proved the existence of the helicoidal rotating solitons under the MCF, which are also known as the helicoidal translating solitons. The authors [24] completely classified the profile curves and analyzed their asymptotic behaviors in the same way as those of the translating bowl or winglike translator. Consider a helicoidal translating soliton Σ with the pitch h whose helicoidal axis is the z-axis. We can parametrize Σ as $X : \Sigma \to \mathbb{R}^3$ by

$$X(s,t) = (x(s)\cos(t), x(s)\sin(t), y(s) + ht),$$

such that the profile curve (x, y) parametrized by arc-length satisfies the following differential equation:

$$(x^2 + 2h^2x'^2)y' + x(h^2 + x^2)(x'y'' - y'x'') - 2xx'(x^2 + h^2x'^2) = 0.$$

Example 2.7 (Δ-wing)**.** The existence of Δ-wings, which are graphs over the strips $\mathbb{R} \times (-b, b)$ and where $b > \frac{\pi}{2}$, was proved by Hoffman, Ilmanen, Martín and White [14]. In particular, Δ-wing has the asymptotic behaviors of grim reaper cylinders with the angle $\theta = \pm\arccos(2b/\pi)$. They [14] also proved the following classification theorem: For every $b > \frac{\pi}{2}$, there is (up to the translation) a unique, complete and strictly convex translating soliton $u : \mathbb{R} \times (-b, b) \to \mathbb{R}$, *i.e.*, the only other complete translating graphs are the grim reaper cylinders and the translating bowl.

Example 2.8 (Semi-graph translating soliton)**.** Hoffman, Martín and White [16] proved the existence and uniqueness of a two-parameter family of translating solitons of the MCF. The family has several types of translating solitons, namely, Scherkenoids, Scherk translators, pithchfork translators and helicoid-like translators. The constructions of graphical translating solitons are over parallelograms and strips with infinite boundary values, which are derived from the minimal surface theory because the translating soliton is a minimal surface in a metric measure space that is conformally changed from Euclidean metric.

Example 2.9 (Translating solitons using gluing technique)**.** Using the grim reaper cylinder and the plane as building blocks, Nguyen [34] constructed the new examples of self-translating solitons with four ends using a gluing technique, two of which are asymptotic to planes and the other two ends are exponentially asymptotic to a grim reaper cylinder, which are called the *translating tridents*. Specifically, Hoffman, Martín and White [17] rediscovered the existence of a one-parametric family of the translating solitons, *i.e.*, Nguyen's tridents. Nguyen [36] also constructed doubly periodic translating solitons with numerous but finite grim reaper cylinders during each period, which is an extension of the result of Nguyen [35] on the desingularization of a finite family of grim reaper cylinders. Dávila, del Pino and Nguyen [11] constructed embedded, complete translating solitons homeomorphic to the Costa-Hoffman-Meeks minimal surfaces, which are a desingularization of the union of a winglike translator and a translating bowl.

The proofs of Theorems 1.1 and 1.3 are provided in Sections 3 and 4, respectively and are based on the following the Omori–Yau maximum principle:

Theorem 2.10 (Omori–Yau maximum principle)**.** *Let M be a connected and complete Riemannian manifold with Ricci curvature bounded below. Then, there exists a sequence $\{x_k\}$ on M such that*

$$\lim_{k\to\infty} f(x_k) = f^*, \quad \|\nabla f(x_k)\| < \frac{1}{k}, \quad \triangle f(x_k) < \frac{1}{k},$$

for any $f \in C^2(M)$ with $f^* = \sup_M f < \infty$.

3. Proof of Theorem 1.1

Let M be an n-dimensional complete translating soliton with the velocity v under the MCF in $\mathbb{R}^{n+1}$. For the unit normal vector $\widetilde{\mathrm{v}}$ of the closed half-space $\mathcal{H}_{\widetilde{\mathrm{v}}}$, we denote $c = \langle \mathrm{v}, \widetilde{\mathrm{v}} \rangle > 0$. Let us assume that $M \subset \mathcal{H}_{\widetilde{\mathrm{v}}} = \{x \in \mathbb{R}^{n+1} \mid \langle x, \widetilde{\mathrm{v}} \rangle \le 0\}$.

To prove that the Ricci curvature of M is bounded below, we take an orthonormal basis $\{e_k\}_{k=1}^n$ of T_pM in which the second fundamental form $A(p)$ is diagonal at the considered point p on M. The Ricci curvature with respect to $\{e_k\}_{k=1}^n$ at p is then

$$\begin{aligned} \mathrm{Ric}_p(e_i, e_i) &= -\sum_{j=1, j\neq i}^{n} \kappa_j(p)\kappa_i(p) \\ &= \kappa_i^2(p) - \sum_{j=1}^{n} \kappa_j(p)\kappa_i(p) \\ &= \kappa_i^2(p) - H(p)\kappa_i(p). \end{aligned} \tag{3.1}$$

The following inequality holds from $-1 \le H \le 1$ and equation (3.1):

$$\mathrm{Ric}_p(e_i, e_i) = -H(p)\kappa_i(p) + \kappa_i^2(p) = \left(\kappa_i(p) - \frac{H(p)}{2}\right)^2 - \frac{H^2(p)}{4} \ge -\frac{1}{4}.$$

Therefore, the Omori–Yau maximum principle holds on M.

We assume $\widetilde{\mathrm{v}} = \mathrm{v} + a$, where a is a constant vector and $\widetilde{\mathrm{v}}$ and v are unit constant vectors such that $\langle \widetilde{\mathrm{v}}, \mathrm{v} \rangle = c$, where c is a positive constant. We define the height function $\phi = \langle x, \widetilde{\mathrm{v}} \rangle$ of M with respect to $\widetilde{\mathrm{v}}$. By direct computation, we obtain the following:

$$\begin{aligned} \|\nabla\phi\|^2 &= \|\widetilde{\mathrm{v}}^\top\|^2 = 1 - \|\widetilde{\mathrm{v}}^\perp\|^2, \\ \triangle\phi &= H\langle \nu, \widetilde{\mathrm{v}} \rangle. \end{aligned}$$

By the Omori–Yau maximum principle, there is a sequence $\{x_k\}$ on M such that

$$\lim_{k\to\infty} \phi(x_k) = \sup_M \phi < 0,$$

$$\frac{1}{k^2} > \|\nabla\phi(x_k)\|^2 = 1 - \|\widetilde{\mathrm{v}}^\perp(x_k)\|^2, \tag{3.2}$$

$$\frac{1}{k} > \triangle\phi(x_k) = H(x_k)\langle \nu(x_k), \widetilde{\mathrm{v}} \rangle. \tag{3.3}$$

We observe the following from the equation (3.2):

$$\lim_{k\to\infty} \|\widetilde{\mathrm{v}}^\perp(x_k)\|^2 = 1,$$

which indicates that ν becomes parallel to $\widetilde{\mathrm{v}}$ as k tends toward infinity. By the inequality (3.3), we obtain the following equation after passing to the sequence $\{x_k\}$:

$$0 \geq \lim_{k\to\infty} \triangle\phi(x_k) = \lim_{k\to\infty} \langle \nu(x_k), \mathrm{v} \rangle \langle \nu(x_k), \widetilde{\mathrm{v}} \rangle = \langle \widetilde{\mathrm{v}}, \mathrm{v} \rangle = c > 0.$$

Therefore, we arrive at a contradiction. As a consequence, there are no complete translating solitons in $\mathcal{H}_{\widetilde{\mathrm{v}}}$.

Remark 3.1. To prove Theorem 1.1, we use the condition $\langle \widetilde{\mathrm{v}}, \mathrm{v} \rangle = c > 0$. If we consider $\langle \widetilde{\mathrm{v}}, \mathrm{v} \rangle \leq 0$, then there are several counter examples, namely, hyperplanes parallel to v if $\langle \widetilde{\mathrm{v}}, \mathrm{v} \rangle = 0$ and grim reaper cylinders or translating bowls if $\langle \widetilde{\mathrm{v}}, \mathrm{v} \rangle < 0$. The novelty of Theorem 1.1 is twofold. Firstly, the theorem holds for any dimension. Secondly, for the half-space theorem for minimal surfaces in Euclidean space, the properness is a necessary condition. The theorem holds regardless of properness on the translating solitons.

Remark 3.2. Chini and Møller [8] proved a half-space type theorem of a proper translating soliton in a bi-half-space, which is the intersection domain of two transverse half-spaces that are parallel to v in $\mathbb{R}^{n+1}$. It directly follows that there are no proper translating solitons in any bounded domain of $\mathbb{R}^{n+1}$. However, as mentioned in Remark 3.1, Theorem 1.1 and Corollary 1.2 hold regardless of whether the translating soliton is proper or not.

4. Proof of Theorem 1.3

Let M be an n-dimensional complete translating soliton with v in a right circular cone $C_{\mathrm{v},a} = \{x \in \mathbb{R}^{n+1} \mid \langle \frac{x}{\|x\|}, \mathrm{v} \rangle \leq a < 1\}$, where a is a constant. For $C_{\mathrm{v},a} \subset \mathcal{H}_{\mathrm{v}}$ with $a \leq 0$, it is sufficient to consider only $0 < a < 1$ based on Theorem 1.1. We already know that the Ricci curvature of M is bounded from below by $-\frac{1}{4}$. Thus, the Omori–Yau maximum principle holds on M.

By direct computation, we can calculate the following equations:

$$\begin{aligned}
\nabla\|x\| &= \frac{x^\top}{\|x\|},\\
\|\nabla\|x\|\|^2 &= \frac{\|x^\top\|^2}{\|x\|^2} \leq 1,\\
\triangle\|x\| &= \frac{1}{\|x\|}\left(\frac{1}{2}\triangle\|x\|^2 - \|\nabla\|x\|\|^2\right) = \frac{1}{\|x\|}\left(\langle x, \overrightarrow{H}\rangle + n - \|\nabla\|x\|\|^2\right).
\end{aligned}$$

We define the non-positive function $\psi = \langle x, \mathrm{v}\rangle - a\|x\|$ that satisfies the following:

$$\|\nabla\psi\| = \|\mathrm{v}^\top - a\nabla\|x\|\| = \left\|\mathrm{v}^\top - a\frac{x^\top}{\|x\|}\right\| \geq \left|\|\mathrm{v}^\top\| - a\frac{\|x^\top\|}{\|x\|}\right|, \tag{4.1}$$

$$\triangle\psi = H^2 - a\triangle\|x\|, \tag{4.2}$$

By applying the Omori–Yau maximum principle to ψ, there is a sequence $\{x_k\}$ such that

$$\begin{aligned}
\lim_{k\to\infty}\psi(x_k) &= \sup_M \psi \leq 0,\\
\frac{1}{k} &> \|\nabla\psi(x_k)\|,\\
\frac{1}{k} &> \triangle\psi(x_k).
\end{aligned}$$

From the equation (4.1), we first consider the following:

$$\|\mathrm{v}^\top(x_k)\| - \frac{1}{k} < a\frac{\|x_k^\top\|}{\|x_k\|} < \frac{1}{k} + \|\mathrm{v}^\top(x_k)\|. \tag{4.3}$$

If $H(x_k)$ converges to zero as $k\to\infty$, then $\|\mathrm{v}^\top(x_k)\| = 1$ and we have a direct contradiction. Because H is bounded, we can consider a subsequence $\{x_{k_l}\}$ such that $H(x_{k_l})$ converges as $l\to\infty$. Thus, the subsequence guarantees the property of the Omori–Yau maximum principle and the convergence of $H(x_{k_l})$. We replace the sequence of the Omori–Yau maximum principle with the subsequence $\{x_{k_l}\}$, which indicates that both $\|\mathrm{v}^\top(x_{k_l})\|^2$ and $\frac{\|x_{k_l}^\perp\|^2}{\|x_{k_l}\|^2}$ converge as $l\to\infty$. From the equation (4.3), we have the following equation:

$$1 - \lim_{l\to\infty} H^2(x_{k_l}) = a^2\left(1 - \lim_{l\to\infty}\frac{\|x_{k_l}^\perp\|^2}{\|x_{k_l}\|^2}\right),$$

Thus, by multiplying with $H^2(x_{k_l})$, we obtain the following equation:

$$\lim_{l\to\infty} a^2\left\langle \frac{x_{k_l}}{\|x_{k_l}\|}, \overrightarrow{H}(x_{k_l})\right\rangle^2 = \lim_{l\to\infty}\left(H^4(x_{k_l}) - (1-a^2)H^2(x_{k_l})\right). \tag{4.4}$$

In particular, the following inequality is obtained:

$$\lim_{l\to\infty}\left(H^2(x_{k_l}) + a^2 - 1\right) \geq 0.$$

From the equation (4.2), we then have

$$\begin{aligned}\frac{1}{k} > \triangle\psi(x_k) &= H^2(x_k) - a\left(\left\langle \frac{x_k}{\|x_k\|}, \vec{H}(x_k)\right\rangle + \frac{n}{\|x_k\|} - \frac{\|\nabla\|x_k\|\|^2}{\|x_k\|}\right) \\ &\geq H^2(x_k) - a\left|\left\langle \frac{x_k}{\|x_k\|},\ \vec{H}(x_k)\right\rangle\right| - \frac{an}{\|x_k\|}.\end{aligned}$$

Thus, the following inequality holds:

$$\frac{1}{k} + \frac{an}{\|x_k\|} > H^2(x_k) - a\left|\left\langle \frac{x_k}{\|x_k\|},\ \vec{H}(x_k)\right\rangle\right|. \tag{4.5}$$

We replace the sequence of the Omori–Yau maximum principle with the subsequence $\{x_{k_l}\}$. Inserting the equation (4.4) into the inequality (4.5) and passing to the subsequence $\{x_{k_l}\}$, the following inequality is obtained:

$$\lim_{l\to\infty}\left(\frac{1}{k_l} + \frac{an}{\|x_{k_l}\|}\right) \geq \lim_{l\to\infty}\left(H^2(x_{k_l}) - \sqrt{H^4(x_{k_l}) - (1-a^2)H^2(x_{k_l})}\right). \tag{4.6}$$

In addition, we define a positive function defined on $[1-a^2, 1]$ as

$$\alpha(t) = t - \sqrt{t^2 - (1-a^2)t}.$$

Bacause $\alpha'(t) < 0$, $\alpha(t)$ is a decreasing and positive function such that $\alpha(1) = 1-a$ is the minimum value in $[1-a^2, 1]$. It is possible to assume $\inf_M \|x\| \geq \frac{2an}{1-a}$ through a translation along the direction $-\mathrm{v}$. We then arrive at a contradiction using the inequality (4.6) as follows:

$$\begin{aligned} 0 &\geq \lim_{l\to\infty}\alpha(H^2(x_{k_l})) - \lim_{l\to\infty}\left(\frac{1}{k_l} + \frac{an}{\|x_{k_l}\|}\right) \\ &\geq 1 - a - \lim_{l\to\infty}\left(\frac{1}{k_l} + \frac{1-a}{2}\right) \\ &\geq \frac{1-a}{2} > 0.\end{aligned}$$

Therefore, there are no complete translating solitons with the direction v of the translation under the MCF in $\mathcal{C}_{\mathrm{v},a} \subset \mathbb{R}^{n+1}$.

Remark 4.1. Because we consider $-1 \leq a < 1$ in Theorem 1.3, Theorem 1.1 is induced by Theorem 1.3. There are several counter examples of complete translating solitons if the right circular cone contains v, namely, the translating bowls and winglike translators are contained in $\{x \in \mathbb{R}^{n+1} \mid \langle \frac{x}{\|x\|}, \mathrm{v}\rangle \geq a\}$ for a constant a.

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