

## INTERMEDIATE $C^*$ -ALGEBRAS OF CARTAN EMBEDDINGS

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ABSTRACT. Let  $A$  be a  $C^*$ -algebra and let  $D$  be a Cartan subalgebra of  $A$ . We study the following question: if  $B$  is a  $C^*$ -algebra such that  $D \subseteq B \subseteq A$ , is  $D$  a Cartan subalgebra of  $B$ ? We give a positive answer in two cases: the case when there is a faithful conditional expectation from  $A$  onto  $B$ , and the case when  $A$  is nuclear and  $D$  is a  $C^*$ -diagonal of  $A$ . In both cases there is a one-to-one correspondence between the intermediate  $C^*$ -algebras  $B$ , and a class of open subgroupoids of the groupoid  $G$ , where  $\Sigma \rightarrow G$  is the twist associated with the embedding  $D \subseteq A$ .

### 1. INTRODUCTION

An interesting, and common, type of question in the study of operator algebras is the following. Suppose  $D$  is an algebra embedded in an algebra  $A$  in such a way that the inclusion has some nice properties. If  $B$  is an algebra intermediate to  $D$  and  $A$ : i.e.  $D \subseteq B \subseteq A$ , does the embedding of  $D$  in  $B$  have the same nice properties? Is there a way to classify all the intermediate algebras  $B$ ?

Perhaps the most famous of these results is the Galois correspondence for the crossed products of von Neumann algebra factors by discrete groups. This says that if  $N$  is a factor von Neumann algebra, and  $G$  is a discrete group acting freely on  $N$ , then the map  $H \mapsto N \rtimes H$  gives a one-to-one correspondence between subgroups of  $H \subseteq G$  and intermediate von Neumann algebras  $N \subseteq M \subseteq N \rtimes G$ . This result is due to Izumi, Longo and Popa [24], building on the work of Choda [11]. An alternative, elegant proof was given by Cameron and Smith [9].

Similar Galois correspondence results have been proved for reduced crossed-products of  $C^*$ -algebras. If  $A$  is a  $C^*$ -algebra and  $G$  is a discrete group acting on  $A$  by automorphisms, then Choda [12] gives a one-to-one correspondence between subgroups of  $G$  and a certain class of intermediate  $C^*$ -algebras  $A \subseteq B \subseteq A \rtimes_r G$ . An intermediate algebra  $B$  is in the desired class if, among other things, there is a faithful conditional expectation  $A \rtimes_r G \rightarrow B$ . Cameron and Smith [10], generalising an earlier result of Izumi [23] for finite groups, showed if  $A$  is a simple  $C^*$ -algebra and  $G$  is a discrete group acting on  $A$  by outer automorphisms, then the map  $H \mapsto A \rtimes H$  is a one-to-one correspondence between subgroups of  $H \subseteq G$  and

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$C^*$ -algebras  $B$  satisfying  $A \subseteq B \subseteq A \rtimes G$ . In this case there is always a faithful conditional expectation from  $A \rtimes_r G$  onto an intermediate  $C^*$ -algebra.

Beyond the rigid structure of crossed products, there are striking examples of these intermediate algebra type results. Suppose  $D$  is a Cartan subalgebra of a von Neumann algebra  $M$ . Aoi proved that if  $D \subseteq N \subseteq M$ , then  $D$  is also a Cartan subalgebra of  $N$  [2]. Cameron, Pitts and Zarikian have given an alternative proof of this theorem [8]. Feldman and Moore [19, 20] gave a one-to-one correspondence between Cartan embeddings and measured equivalence relations. Thus, analogous to the Galois correspondence results mentioned above, Aoi's result gives a one-to-one correspondence between sub-measured equivalence relations and von Neumann algebras  $D \subseteq N \subseteq M$ . Alternatively, Donsig, Fuller and Pitts [15] give a measure-free description of Cartan embeddings based on extensions of inverse semigroups. There, Aoi's theorem is used to show a one-to-one correspondence between a class of sub-inverse monoids of an inverse monoid  $S$  and intermediate von Neumann algebras  $D \subseteq N \subseteq M$ . These results are generalized beyond Cartan embeddings in [16].

In this note we address an analogous problem in the  $C^*$ -algebra setting. Suppose  $D$  is a Cartan subalgebra of a  $C^*$ -algebra  $A$ , in the sense of Renault [32]. Renault showed that there is a twist  $\Sigma \xrightarrow{q} G$  such that  $A$  can be identified with the reduced  $C^*$ -algebra of the twist  $C_r^*(\Sigma; G)$  and  $D$  is identified with  $C(G^{(0)})$ . In Theorem 3.3, we exhibit a one-to-one correspondence between open subgroupoids  $H \subseteq G$  and  $C^*$ -algebras  $B$  such that  $D \subseteq B \subseteq A$  and  $D$  is Cartan in  $B$ . We further show in Theorem 3.5 that this map gives a one-to-one correspondence between subgroupoids  $H$  which are both closed and open in  $G$  and intermediate  $C^*$ -algebras  $D \subseteq B \subseteq A$  with a faithful conditional expectation  $F: A \rightarrow B$ .

However, a direct analogue of Aoi's theorem does not hold in this context: indeed, there are Cartan embeddings  $D \subseteq A$  with intermediate subalgebras  $D \subseteq B \subseteq A$  where  $D$  is not Cartan in  $B$ . A relatively simple example is given in Example 5.1. Stronger results can be found with some additional, very natural, hypotheses.

Two decades before Renault's work on Cartan subalgebras of  $C^*$ -algebras [32], Kumjian introduced the stronger property of  $C^*$ -diagonal [25]. In Theorem 4.5 we give a Galois-correspondence type result in this context. Suppose  $D$  is a  $C^*$ -diagonal of a nuclear  $C^*$ -algebra  $A$ . Let  $\Sigma \xrightarrow{q} G$  be the twist corresponding to the embedding  $D \subseteq A$ . The map  $H \mapsto C_r^*(\Sigma_H; H)$  gives a one-to-one correspondence between open subgroupoids  $H \subseteq G$  with  $H^{(0)} = G^{(0)}$  and all  $C^*$ -algebras  $B$  such that  $D \subseteq B \subseteq A$ . In particular, if  $D \subseteq B \subseteq A$  then  $D$  is a  $C^*$ -diagonal in  $B$ . In contrast with the work of [10, 12, 23] there is not necessarily a conditional expectation onto the intermediate subalgebra.

Takeishi [34] showed nuclearity of  $A$  is equivalent to the amenability of  $G$ . Let  $G^{(0)} \subseteq H \subseteq G$  be an open subgroupoid, and let  $\Sigma_H \rightarrow H$  be the corresponding subtwist of  $\Sigma \rightarrow G$ . In Theorem 4.2, amenability of  $G$  is used to show that, viewing  $C_r^*(\Sigma_H; H)$  as a subalgebra of  $C_r^*(\Sigma; G)$ , an element  $a \in C_r^*(\Sigma; G)$  is in  $C_r^*(\Sigma_H; H)$  if and only if  $a$  (when viewed as a function on  $G$ ) vanishes off  $H$ . This is a key ingredient in the proof of Theorem 4.5. When the amenability condition is removed, we show in Theorem 4.4 that  $B$  is contained in the  $C^*$ -algebra of functions supported on  $H$  and  $C_r^*(\Sigma_H; H) \subseteq B$ .

In the final section we focus on the case of  $C^*$ -algebraic crossed products, where we strengthen our results beyond the nuclear case. Let  $\Gamma$  be a discrete group acting

on a compact Hausdorff space  $X$  by homeomorphisms. To apply Theorem 4.5 to a crossed product we would require that the algebra  $C(X)$  is a C\*-diagonal in the reduced crossed-product  $C(X) \rtimes \Gamma$ , and that  $C(X) \rtimes \Gamma$  is nuclear. This happens if and only if  $\Gamma$  is an amenable group acting freely on  $X$ . We can, however, prove a version of Theorem 4.5 which does not require that  $\Gamma$  is amenable, and instead assume  $\Gamma$  has the approximation property of Cowling and Haagerup [13]. Let  $\Gamma$  be a group which satisfies the approximation property and acts freely on a compact Hausdorff space  $X$ . Let  $\Gamma \times X$  be the corresponding transformation groupoid. We prove in Corollary 5.8 that the map  $H \mapsto C_r^*(H)$  gives a one-to-one correspondence between open subgroupoids  $H \subseteq \Gamma \times X$  with  $\{e\} \times X \subseteq H$  and all C\*-algebras  $B$  satisfying  $C(X) \subseteq B \subseteq C(X) \rtimes_r \Gamma$ . Thus, while there is not a Galois correspondence from the subgroups of  $\Gamma$ , there is a Galois-type correspondence from the open subgroupoids of the transformation groupoid  $\Gamma \times X$ . This result is further evidence of the value of the groupoid approach to C\*-algebras, even in the relatively straightforward setting of crossed products by discrete group actions.

This Galois-type correspondence is a corollary of a spectral theorem for bimodules in  $C(X) \rtimes_r \Gamma$ . In Theorem 5.7 we show that there is a one-to-one correspondence between the open subsets of  $\Gamma \times X$  and the norm-closed  $C(X)$ -bimodules in  $C(X) \rtimes \Gamma$ . A similar spectral theorem is proved for actions of groups satisfying the approximation property on simple C\*-algebras in [10]. This result is also a direct analogue of the Spectral Theorem for Bimodules for Cartan embeddings in von Neumann algebras; see [8, 15, 21].

In the von Neumann algebra setting,  $L^\infty(X, \mu)$  is Cartan in the crossed product  $L^\infty(X, \mu) \rtimes \Gamma$  if and only if the action of  $\Gamma$  on the measure space  $(X, \mu)$  is (measurably) free [35, Corollary V.7.7]. It has been argued, e.g. by Tomiyama [36], that topological freeness is the correct analogue of free actions on measure spaces. There is good reason for this viewpoint. However, Corollary 5.8 shows that there are settings when free actions, not topologically free actions, are needed in order to get desirable analogues of von Neumann algebra results.

## 2. PRELIMINARIES

We recall the key details of Cartan embeddings in C\*-algebras and their relation to twists. Recall that if  $A$  is a C\*-algebra and  $D$  is a subalgebra, then  $n \in A$  normalizes  $D$  if  $nDn^* \cup n^*Dn \subseteq D$ . We denote the normalizers of  $D$  in  $A$  by  $N(A, D)$ . The subalgebra  $D$  is *regular* in  $A$  if  $N(A, D)$  spans a dense subset of  $A$ .

**Definition 2.1.** Let  $A$  be a C\*-algebra. A maximal abelian C\*-subalgebra  $D \subseteq A$  is *Cartan* in  $A$ , or a *Cartan subalgebra* of  $A$ , if

- (i)  $D$  contains an approximate identity for  $A$ ;
- (ii) there is a faithful conditional expectation  $E: A \rightarrow D$ ;
- (iii)  $D$  is regular in  $A$ .

The subalgebra  $D$  is a *C\*-diagonal* of  $A$  if  $D$  is Cartan in  $A$  and

- (iv) every pure state on  $D$  extends uniquely to a pure state on  $A$ , i.e.  $D$  has the *unique extension property* in  $A$ .

It is worth noting that the conditional expectation  $E$  above is unique [32, Proposition 4.2]. Archbold, Bunce and Gregson [3, Corollary 2.7] classify precisely when an abelian subalgebra  $D$  of a C\*-algebra  $A$  satisfies the unique extension property of Definition 2.1(iv) in terms of properties of the conditional expectation. This

allows us to give the following alternative differentiation between  $C^*$ -diagonals and Cartan embeddings when  $A$  is unital.

**Theorem 2.2** (c.f. [3, Corollary 2.7]). *Let  $D$  be a Cartan subalgebra of a unital  $C^*$ -algebra  $A$ , with faithful conditional expectation  $E: A \rightarrow D$ . Then  $D$  is a  $C^*$ -diagonal of  $A$  if and only if*

$$\{E(a)\} = \overline{\text{conv}}\{uau^* : u \in D, u \text{ unitary}\} \cap D,$$

for all  $a \in A$ .

Suppose  $D$  is a Cartan subalgebra of  $A$ , with associated conditional expectation  $E$ . We will briefly recap how to construct the groupoid twist from the embedding  $D \subseteq A$ . As  $D$  is abelian,  $D \simeq C_0(X)$  for a locally compact Hausdorff space  $X$ . If  $n \in A$  normalizes  $D$  then  $n^*n \in D$ , and thus  $n^*n$  can be viewed as a continuous positive function on  $X$ . Let

$$s(n) = \{x \in X : n^*n(x) > 0\}.$$

By [25, Proposition 6] the normalizer  $n$  defines a partial homeomorphism  $\beta_n$  on  $X$ , with domain  $s(n)$  and range  $s(n^*)$  satisfying

$$(n^*dn)(x) = (dnn^*)(\beta_n(x))$$

for all  $d \in D$  and  $x \in s(n)$ .

For every normalizer  $n \in N(A, D)$  and  $x \in s(n)$ , define a linear functional  $[n, x]$  on  $A$  by

$$[n, x](a) = \frac{E(n^*a)(x)}{(n^*n)(x)^{1/2}}.$$

Denote by  $\Sigma$  the collection of all such linear functionals  $[n, x]$ . The set  $\Sigma$  becomes a groupoid with partial multiplication

$$[n, x][m, y] = [nm, y]$$

when  $x = \beta_m(y)$ ; and inverse

$$[n, x]^{-1} = [n^*, \beta_n(x)].$$

Further, as a collection of linear functionals,  $\Sigma$  inherits the relative weak\*-topology. Under this topology  $\Sigma$  is a topological groupoid. There is an action of the unit circle  $\mathbb{T}$  on  $\Sigma$  by

$$\lambda \cdot [n, x] = [\bar{\lambda}n, x].$$

Letting  $G = \Sigma/\mathbb{T}$ , we have the twist

$$\mathbb{T} \times X \hookrightarrow \Sigma \xrightarrow{q} G.$$

The groupoid  $G$  is given the quotient topology. Equivalently, the topology on  $G$  is generated by the basic sets

$$Z(n) = \{q([n, x]) : x \in s(n)\},$$

for  $n \in N(A, D)$ . Under this topology  $G$  is a topologically principal, étale groupoid [32]. If  $D$  is a  $C^*$ -diagonal of  $A$  then  $G$  is a principal, étale groupoid [25]. This twist will usually be abbreviated to  $\Sigma \rightarrow G$ , or  $\Sigma \xrightarrow{q} G$  when we wish to emphasize the quotient map.

To construct the associated line bundle  $L$  over  $G$ , we put the following equivalence relation on  $\mathbb{C} \times \Sigma$ . Define  $(z_1, \gamma_1) \sim (z_2, \gamma_2)$  if there is a  $\lambda \in \mathbb{T}$  such that  $(z_1, \gamma_1) = (\bar{\lambda}z_2, \lambda \cdot \gamma_2)$ . Thus  $[z\lambda, \gamma] = [z, \lambda \cdot \gamma]$ . Let  $L = (\mathbb{C} \times \Sigma)/\sim$ . The continuous

surjection  $P: L \rightarrow G$  defined by  $P([z, \gamma]) = q(\gamma)$  makes  $L$  a Fell line bundle over  $G$ . For  $[z, \gamma] \in L$  we define  $\|[z, \gamma]\| = |z|$ .

The collection of continuous compactly supported cross-sections  $C_c(\Sigma; G)$  of  $L$  becomes a  $*$ -algebra under the following operations. For  $f, g \in C_c(\Sigma; G)$  the convolution product on  $C_c(\Sigma; G)$  is defined by

$$f * g(\gamma) = \sum_{\gamma=\alpha\beta} f(\alpha)g(\beta), \quad \gamma \in G.$$

The adjoint is given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}, \quad \gamma \in G.$$

The reduced C\*-algebra  $C_r^*(\Sigma; G)$  of the twist  $\Sigma \rightarrow G$  is the unique closure of the  $*$ -algebra  $C_c(\Sigma; G)$  so that the restriction map  $E: C_c(\Sigma; G) \rightarrow C_c(G^{(0)})$  extends to a faithful conditional expectation  $E: C_r^*(\Sigma; G) \rightarrow C_0(G^{(0)})$ . A detailed description of the construction of the reduced C\*-algebra  $C_r^*(\Sigma; G)$  can be found in [5, Section 2].

We recall main theorems of [25] and [32].

**Theorem 2.3** (c.f. [25] and [32]). *Let  $D$  be a Cartan subalgebra of a C\*-algebra  $A$ . Then  $A$  is isomorphic to  $C_r^*(\Sigma; G)$  via an isomorphism which carries  $D$  to  $C_0(G^{(0)})$ .*

*Conversely if  $\Sigma \rightarrow G$  is a twist with  $G$  topologically principal and étale then  $C_0(G^{(0)})$  is a Cartan subalgebra of  $C_r^*(\Sigma; G)$ . The algebra  $C_0(G^{(0)})$  is a C\*-diagonal of  $C_r^*(\Sigma; G)$  if and only if  $G$  is a principal groupoid.*

*Remark 2.4.* If  $D$  is a Cartan subalgebra of  $A$  with associated twist  $\Sigma \rightarrow G$ , Theorem 2.3 allows us to identify  $A$  with  $C_r^*(\Sigma; G)$  and  $D$  with  $C_0(G^{(0)})$ . In the sequel we will do this without comment.

*Remark 2.5.* In both [25] and [32] only separable C\*-algebras are studied. Theorem 2.3, however, does hold in the non-separable case if effective groupoids are considered in place of topologically principal groupoids [26, Corollary 7.6].

Our main result, Theorem 4.5, applies when the groupoid is amenable. Amenability of groupoids was introduced by Renault [31, Chapter II.3]. For our purposes we will use a characterization of amenability in terms of positive-type functions. Let  $G$  be a groupoid. A function  $h: G \rightarrow \mathbb{C}$  is of *positive-type* if for all  $x \in G^{(0)}$  and finite subsets  $F \subseteq G^x$  the matrix  $[h(\eta^{-1}\gamma)]_{\eta, \gamma \in F}$  is a positive matrix.

**Theorem 2.6** (c.f. [7, Theorem 5.6.18]). *Let  $G$  be an étale locally compact groupoid. Then  $G$  is amenable if and only if there is a net of sup-bounded, positive-type functions in  $C_c(G)$  which converges uniformly to 1 on compact subsets of  $G$ .*

We note the following theorem of Takeishi [34] which classifies when  $C_r^*(\Sigma; G)$  is nuclear. The case when the twist is trivial can be found in [7, Theorem 5.6.18], and for more general groupoids in [1, Corollary 6.2.14].

**Theorem 2.7** (c.f. [34, Theorem 5.4]). *Let  $\Sigma \rightarrow G$  be a twist, with  $G$  an étale locally compact groupoid. Then  $C_r^*(\Sigma; G)$  is nuclear if and only if  $G$  is amenable.*

One can also construct the *full* C\*-algebra  $C^*(\Sigma; G)$  of the twist  $\Sigma \rightarrow G$  by completing  $C_c(\Sigma; G)$  with respect to the supremum norm over all  $*$ -representations of  $C_c(\Sigma; G)$ . If  $f \in C_c(\Sigma; G)$  is supported on an open bisection  $U \subseteq G$ , then  $f^* * f \in C_c(G^{(0)})$  and  $\|f\|_{C^*(\Sigma; G)} \leq \|f\|_\infty$ . It follows that for each compact  $K \subseteq G$  there is a  $C_K > 0$  such that if  $f \in C_c(\Sigma; G)$  and  $f$  is supported on  $K$ , then

$$(1) \quad \|f\|_{C^*(\Sigma; G)} \leq C_K \|f\|_\infty;$$

see [7, page 205]. We note the following result due to Sims and Williams [33].

**Theorem 2.8** (cf. [33, Theorem 1]). *Let  $\Sigma \rightarrow G$  be a twist, with  $G$  an étale, Hausdorff, amenable groupoid. Then the reduced  $C^*$ -algebra  $C_r^*(\Sigma; G)$  is isomorphic to the full  $C^*$ -algebra  $C^*(\Sigma; G)$ .*

### 3. INTERMEDIATE SUBALGEBRAS OF CARTAN EMBEDDINGS

Our primary goal is to study  $C^*$ -algebras  $B$  satisfying  $D \subseteq B \subseteq A$ , when  $D$  is Cartan in  $A$ . In particular, does the (twisted) groupoid  $C^*$ -algebra structure of  $A$  force  $B$  to have a similar structure? The main result of this section, Theorem 3.5, gives a positive answer in the case when there is a faithful conditional expectation from  $A$  to  $B$ .

*Notation 3.1.* If  $\Sigma \xrightarrow{q} G$  is a twist and  $H \subseteq G$ , we let  $\Sigma_H = q^{-1}(H)$ .

We recall the following result.

**Lemma 3.2** (c.f. [5, Lemma 2.19]). *Let  $\Sigma \xrightarrow{q} G$  be a twist. Let  $H$  be an open subgroupoid of  $G$ , with  $G^{(0)} \subseteq H$ . Then  $\Sigma_H \rightarrow H$  is a twist. Further, the natural map  $C_c(\Sigma_H; H) \hookrightarrow C_c(\Sigma; G)$  given by extending functions by zero induces an embedding  $\iota$  of  $C_r^*(\Sigma_H; H)$  into  $C_r^*(\Sigma; G)$ .*

Here is a sketch of the proof. That  $\Sigma_H \rightarrow H$  is a twist is straightforward. The second claim requires some care. Define  $\iota : C_c(\Sigma_H; H) \hookrightarrow C_c(\Sigma; G)$  by  $\iota(f)(\gamma) = \chi_{\Sigma_H}(\gamma)f(\gamma)$ . In [5] it is shown that  $\|h\|_{C_r^*(\Sigma_H; H)} \leq \|\iota(h)\|_{C_r^*(\Sigma; G)}$ , so  $\iota^{-1}$  extends to an epimorphism of  $\overline{\iota(C_c(\Sigma_H; H))}$  onto  $C_r^*(\Sigma_H; H)$ . That it is an isomorphism follows from the faithfulness of the conditional expectation of  $C_r^*(\Sigma; G)$  onto  $C^*(G^{(0)})$ .

We now describe the intermediate subalgebras of Cartan embeddings.

**Theorem 3.3.** *Let  $D$  be Cartan in  $A$ , with twist  $\Sigma \rightarrow G$ . Let  $D \subseteq B \subseteq A$  be an intermediate subalgebra. Then  $D$  is Cartan in  $B$  if and only if there exists an open subgroupoid  $H \subseteq G$  with  $H^{(0)} = G^{(0)}$  and  $B = \iota(C_r^*(\Sigma_H; H))$ , where  $\iota$  is the inclusion from Lemma 3.2.*

*Further, in this case  $H = \{\gamma \in G : q^{-1}(\gamma)|_B \neq 0\}$ .*

*Proof.* Lemma 3.2 shows that when  $H$  is an open subgroupoid  $D$  is a Cartan subalgebra of  $C_r^*(\Sigma_H; H)$ .

Suppose  $D \subseteq B \subseteq A$  with  $D$  Cartan in  $B$ . Take  $[n, x] \in \Sigma$  such that  $q([n, x]) \in H$ . Since we are assuming that  $D$  is regular in  $B$ , it follows that there is a normalizer  $m \in N(B, D)$  such that  $[n, x](m) \neq 0$ . That is,

$$\frac{E(n^*m)(x)}{(n^*n)(x)^{1/2}} \neq 0.$$

As  $E(n^*m)(x) \neq 0$  it follows that  $[n, x] = [m, x]$  [30, Lemma 8.7]. As  $N(B, D)$  is a  $*$ -semigroup, it follows that  $\Sigma_H$  and  $H$  are groupoids. That  $H$  is open follows from the definition of the topology on  $G$ .

Finally, that  $B = C_r^*(\Sigma_H; H)$  follows from Theorem 2.3.  $\square$

Given  $C^*$ -algebras  $D \subseteq B \subseteq A$  with  $D$  Cartan in  $A$ , it is not, in general, easy to tell whether  $D$  is Cartan in  $B$ . Indeed it may be not be. See Example 5.1. We will now give a class of intermediate  $C^*$ -algebras  $B$  where we are guaranteed that  $D$  is Cartan in  $B$ . These will be in one-to-one correspondence with the subgroupoids  $G^{(0)} \subseteq H \subseteq G$  with  $H$  both open and closed in  $G$ .

In the Galois correspondence results in [10, 12, 23] there is always a conditional expectation onto the intermediate algebras  $A \subseteq B \subseteq A \rtimes_r \Gamma$ . In the following theorem we classify the intermediate subalgebras  $B$  in Cartan embeddings for which there is a conditional expectation onto  $B$ .

**Lemma 3.4.** *Let  $\Sigma \rightarrow G$  and let  $H$  an open subgroupoid of  $G$ . Then there is a conditional expectation  $F: C_r^*(\Sigma; G) \rightarrow C_r^*(\Sigma_H; H)$  if and only if  $H$  is closed.*

*Proof.* We will follow a similar line of proof as in [6, Proposition 4.1]. Assume  $H$  is not closed. Then there is a net  $(\gamma_n)$  in  $H$  which converges to some  $\gamma \in G \setminus H$ . Let  $U \subseteq G$  be an open bisection containing  $\gamma$  and choose  $f \in C_c(\Sigma; G)$  supported on  $U$  such that  $|f(\gamma)| = 1$ . As  $\gamma_n \rightarrow \gamma$ , we may assume that each  $\gamma_n \in U$ . Since  $H$  is open, there are open sets  $V_n \subseteq H \cap U$  such that  $\gamma_n \in V_n$ . For each  $n$  we choose  $g_n \in C_c(G^{(0)})$  with  $g_n$  supported on  $r(V_n)$  and  $g_n(r(\gamma_n)) = 1$ .

Since for each  $n$ ,  $g_n \in C_c(G^{(0)}) \subseteq C_c(\Sigma_H; H)$ , it follows that

$$F(g_n * f) = g_n * F(f).$$

Further, note  $g_n * f$  is supported on  $V_n \subseteq H$ . Thus  $g_n * f \in C_c(\Sigma_H; H)$  and  $F(g_n * f) = g_n * f$ . Finally, note that for any  $h \in C_r^*(\Sigma; G)$ ,  $g_n * h(\gamma_n) = h(\gamma_n)$ , since  $g_n(r(\gamma_n)) = 1$ . Hence

$$F(f)(\gamma_n) = g_n * F(f)(\gamma_n) = F(g_n * f)(\gamma_n) = g_n * f(\gamma_n) = f(\gamma_n).$$

Since  $|f(\gamma)| = 1$ , it follows that  $|F(f)(\gamma_n)| \rightarrow 1$ . Thus  $|F(f)(\gamma)| = 1$ . However, since  $\gamma \notin H$ , and  $F(f) \in C_c(\Sigma_H; H)$ , we must have  $F(f)(\gamma) = 0$ . This is a contradiction, and hence  $H$  is closed.

Conversely, if  $H$  is both open and closed in  $G$  then the restriction map induces a faithful conditional expectation from  $C_r^*(\Sigma; G)$  onto  $C_r^*(\Sigma_H; H)$ .  $\square$

**Theorem 3.5.** *Let  $A$  be a  $C^*$ -algebra and let  $D \subseteq A$  be a Cartan subalgebra. Let  $\Sigma \rightarrow G$  be the corresponding twist. The following statements hold.*

- (1) *Let  $D \subseteq B \subseteq A$  be an intermediate subalgebra. If there is a conditional expectation  $F: A \rightarrow B$  then  $D$  is Cartan in  $B$ .*
- (2) *The map  $H \mapsto C_r^*(\Sigma_H; H)$  gives a one-to-one correspondence between subgroupoids  $G^{(0)} \subseteq H \subseteq G$  which are both open and closed in  $G$ , and the intermediate  $C^*$ -algebras  $D \subseteq B \subseteq A$  for which there is a conditional expectation  $F: A \rightarrow B$ .*

*Proof.* Let  $B$  be a  $C^*$ -algebra such that  $D \subseteq B \subseteq A$  and there is a conditional expectation  $F: A \rightarrow B$ . Let  $E: A \rightarrow D$  be the unique faithful conditional expectation. Then  $(E|_B) \circ F = E$ , from which it follows that  $F$  is necessarily faithful.

Let  $n \in A$  be an intertwiner of  $D$ . That is, for each  $d \in D$  there is a  $d' \in D$  such that  $nd = d'n$ . It follows that for  $d \in D$

$$F(n)d = F(nd) = F(d'n) = d'F(n).$$

Hence  $F(n)$  is also an intertwiner of  $D$ . By [17, Proposition 3.3] the normalizers  $N(A, D)$  are the closure of all intertwiners. Since  $F$  is a continuous map, it follows that if  $n \in N(A, D)$  is a normalizer then  $F(n) \in N(B, D)$ . Since  $D$  is regular in  $A$ , if  $b \in B$  there is a sequence  $(a_n)_n \in \text{span } N(A, D)$  such that  $a_n \rightarrow b$ . Hence  $F(a_n) \rightarrow F(b) = b$ . It follows that  $\text{span } N(B, D)$  is dense in  $B$ , so  $D$  is Cartan in  $B$ .

By Theorem 3.3, since  $D$  is Cartan in  $B$ , there is an open subgroupoid  $H \subseteq G$  with  $H^{(0)} = G^{(0)}$  such that  $C_r^*(\Sigma_H; H)$ . By Lemma 3.4,  $H$  is also closed.  $\square$

#### 4. NUCLEAR $C^*$ -ALGEBRAS AND INTERMEDIATE ALGEBRAS OF $C^*$ -DIAGONALS

In this section we consider  $C^*$ -algebras  $A$  containing a  $C^*$ -diagonal  $D$ , with associated twist  $\Sigma \rightarrow G$ . We will see that if  $D \subseteq B \subseteq A$  and  $A$  is nuclear, then  $D$  is necessarily a  $C^*$ -diagonal in  $B$ . This will give a one-to-one correspondence between open subgroupoids of  $G$  and the intermediate  $C^*$ -algebras  $B$ , Theorem 4.5.

The following theorem tells us that, in the amenable case, determining whether or not an element  $a \in C_r^*(\Sigma; G)$  lies in  $C_r^*(\Sigma_H; H)$  or not, depends solely on the support of  $a$  as a function in  $C_0(\Sigma; G)$ .

*Notation 4.1.* Let  $\Sigma \rightarrow G$  be a twist. If  $H \subseteq G$  is an open subgroupoid we let

$$A_H = \{f \in C_r^*(\Sigma; G) : \text{for all } \gamma \in \Sigma \setminus \Sigma_H \text{ } \gamma(f) = 0\}.$$

In general  $C_r^*(\Sigma_H; H) \subseteq A_H$ . We suspect this containment may be strict in some circumstances.

**Theorem 4.2.** *Let  $\Sigma \rightarrow G$  be a twist with  $G$  an étale, Hausdorff, amenable groupoid. Let  $H$  be an open subgroupoid of  $G$ . Then  $A_H$  is equal to the canonical copy (as in Lemma 3.2) of  $C_r^*(\Sigma_H; H)$  within  $C_r^*(\Sigma; G)$ .*

*Proof.* By Theorem 2.6, there is a net of positive-type functions  $h_i \in C_c(G)$  converging uniformly to 1 on compact subsets of  $G$ , with  $\sup_{\gamma \in G} |h_i(\gamma)| \leq 1$ . View  $C_c(\Sigma; G)$  as continuous cross-sections. For each  $i$ , define a map

$$\begin{aligned} m_{h_i} : C_c(\Sigma; G) &\rightarrow C_c(\Sigma; G) \\ f &\mapsto h_i f, \end{aligned}$$

where for  $\gamma \in G$  and  $f(\gamma) = [\lambda_\gamma, \gamma]$  we have  $(h_i f)(\gamma) = [\lambda_\gamma h_i(\gamma), \gamma]$ . By [34, Lemma 4.2],  $m_{h_i}$  extends to a contractive completely positive map on  $C_r^*(\Sigma; G)$ . As  $G$  is amenable,  $C_r^*(\Sigma; G) = C^*(\Sigma; G)$ , by Theorem 2.8. Note that  $m_{h_i}(f)$  converges to  $f$  in  $C^*(\Sigma; G) = C_r^*(\Sigma; G)$ , by Equation (1). We further note that, for any  $f \in C_r^*(\Sigma; G)$ ,  $m_{h_i}(f) \in C_c(\Sigma; G)$ .

Suppose  $f \in A_H$ . To show that  $f \in C_r^*(\Sigma_H; H)$  it suffices to show that  $m_{h_i}(f) \in C_r^*(\Sigma_H; H)$  for every  $i$ . As  $m_{h_i}(f)$  lies in  $C_c(\Sigma; G)$ , and vanishes off  $H$ ,  $m_{h_i}(f) \in C_c(\Sigma_H; H)$ . It follows that  $f \in C_r^*(\Sigma_H; H)$ , concluding the proof.  $\square$

We recall the following corollary to Theorem 2.2. It was observed in the unital case by Muhly, Qiu and Solel in the proof of Proposition 4.4 of [29].

**Proposition 4.3.** *Suppose  $D \subseteq A$  is a  $C^*$ -diagonal. Then, for any  $a \in A$  and normalizer  $n \in N(A, D)$ ,  $nE(n^*a)$  is a normalizer of  $D$  in the norm-closed  $D$ -bimodule generated by  $a$ .*

*Proof.* If  $A$  is unital the result follows from Theorem 2.2. Details can be found in [17, Proposition 3.10].

Suppose now that  $A$  is not unital. Let  $\tilde{A}$  be the unitization of  $A$ . It can be shown that  $D + \mathbb{C}I$  is a maximal abelian subalgebra of  $\tilde{A}$  and  $D + \mathbb{C}I$  has the unique extension property in  $\tilde{A}$ . By [3, Corollary 2.7] there is a conditional expectation  $\tilde{E} : \tilde{A} \rightarrow D + \mathbb{C}I$  determined by

$$\tilde{E}(x) = \overline{\text{conv}}\{uxu^* : u \in D + \mathbb{C}I \text{ unitary}\} \cap (D + \mathbb{C}I).$$

By [3, Remarks 2.8(iii)], the restriction of  $\tilde{E}$  to  $A$  maps  $A$  onto  $D$ . As the conditional expectation  $E : A \rightarrow D$  is unique [32, Proposition 4.3],  $\tilde{E}|_A = E$ .



Note that if  $n \in N(A, D)$  then  $n \in N(\tilde{A}, D + CI)$ . It follows now as in the unital case, that for any  $n \in N(A, D)$ ,  $nE(n^*a)$  normalizes  $D$  and is in the norm-closed  $D$ -bimodule generated by  $A$ .  $\square$

We now have all the ingredients for the main theorems of this section.

**Theorem 4.4.** *Let  $D$  be a  $C^*$ -diagonal of  $A$  with associated twist  $\Sigma \rightarrow G$ . Let  $D \subseteq B \subseteq A$  be an intermediate subalgebra. Let  $H = \{q(\sigma) : \sigma \in \Sigma, \sigma|_B \neq 0\}$ . Then  $H$  is an open subgroupoid of  $G$  and*

$$\overline{\text{span}}N(B, D) = C_r^*(\Sigma_H; H) \subseteq B \subseteq A_H.$$

*Proof.* Take  $\sigma \in \Sigma$  and  $b \in B$  such that  $\sigma(b) \neq 0$ . By definition of  $\Sigma$ , there is an  $n \in N(A, D)$  and  $x$  in  $\hat{D}$  such that  $\sigma = [n, x]$ . Thus we have

$$0 \neq [n, x](b) = \frac{E(n^*b)(x)}{(n^*n)(x)^{1/2}}.$$

Thus  $E(n^*b) \neq 0$ , and hence  $nE(n^*b) \neq 0$ . By Proposition 4.3,  $nE(n^*b) \in B \cap N(A, D)$ . Let  $m = nE(n^*b)$ . For any  $a \in A$

$$[m, x](a) = \frac{E(m^*a)(x)}{(m^*m)(x)^{1/2}} = \frac{E(n^*a)(x)}{|E(n^*a)(x)|} \sigma(a) = \sigma(a).$$

Thus  $q([m, x]) = q(\sigma)$ . Thus

$$(2) \quad H = \{q([m, x]) : m \in N(B, D)\},$$

and hence  $H \subseteq G$  is an open subgroupoid. Let  $B_0 = \overline{\text{span}}\{n \in N(B, D)\}$ . Then  $B_0$  is a  $C^*$ -algebra satisfying  $B_0 \subseteq B$ . By definition,  $D$  is regular in  $B_0$ , and thus  $B_0 = C_r^*(\Sigma_H; H)$ , by Equation (2), Theorem 3.3 and Theorem 2.3.  $\square$

**Theorem 4.5.** *Let  $A$  be a nuclear  $C^*$ -algebra, and let  $D$  be a  $C^*$ -diagonal in  $A$ . Let  $\Sigma \rightarrow G$  be the corresponding twist for the embedding  $D \subseteq A$ .*

*Then there is a one-to-one correspondence between  $C^*$ -algebras  $B$  satisfying  $D \subseteq B \subseteq A$  and open subgroupoids  $H$  satisfying  $G^{(0)} \subseteq H \subseteq G$ , given by an isomorphism  $B \simeq C_r^*(\Sigma_H; H)$ .*

*Proof.* The result follows from Theorem 4.2 and Theorem 4.4.  $\square$

**Corollary 4.6.** *Let  $A$  be a nuclear  $C^*$ -algebra, and let  $D$  be a  $C^*$ -diagonal of  $A$ . If  $B$  is a  $C^*$ -algebra satisfying  $D \subseteq B \subseteq A$ , then  $D$  is a  $C^*$ -diagonal of  $B$ .*

## 5. CROSSED PRODUCTS

Let  $\Gamma$  be a discrete group acting on a compact Hausdorff space  $X$  by homeomorphisms. The set  $\Gamma \times X$  becomes the *transformation groupoid* under the partial multiplication

$$(g_1, g_2 \cdot x)(g_2, x) = ((g_1g_2), x)$$

and inversion

$$(g, x)^{-1} = (g^{-1}, g \cdot x).$$

With the product topology the transformation groupoid  $\Gamma \times X$  is étale, and clearly Hausdorff. Let  $C(X) \rtimes_r \Gamma$  be the reduced crossed product. One can show that  $C(X) \rtimes_r \Gamma = C_r^*(\Gamma \times X)$ .

We view  $C(X)$  as a subalgebra of  $C(X) \rtimes_r \Gamma$  and denote the canonical unitary representation of  $\Gamma$  in  $C(X) \rtimes_r \Gamma$  by  $\{u_g\}_{g \in \Gamma}$ . Denote by  $E$  the usual faithful

conditional expectation from  $C(X) \rtimes_r \Gamma$  to  $C(X)$ ; see [7, Proposition 4.1.9]. Recall that the action of  $\Gamma$  on  $X$  is *free* if for each  $g \neq e$  in  $\Gamma$  the set  $\{g \cdot x = x : x \in X\}$  is empty. The action is *topologically free* if for each  $g \neq e$  in  $\Gamma$  the set  $\{g \cdot x = x : x \in X\}$  has empty interior. Zeller-Meier [37] showed that  $C(X)$  is maximal abelian in  $C(X) \rtimes_r \Gamma$  if and only if the action of  $\Gamma$  on  $X$  is topologically free. Thus,  $C(X)$  is Cartan in  $C(X) \rtimes_r \Gamma$  if and only if the action of  $\Gamma$  on  $X$  is topologically free. Further  $C(X)$  is a  $C^*$ -diagonal in  $C(X) \rtimes_r \Gamma$  if and only if the action of  $\Gamma$  on  $X$  is free. This can be seen by the fact that the transformation groupoid  $\Gamma \times X$  is principal if and only if the action of  $\Gamma$  is free. One can alternatively show that freeness of the action of  $\Gamma$  implies that

$$E(a) \in \overline{\text{conv}}\{uau^* : u \in C(X), u \text{ unitary}\},$$

see [22, Proposition 11.1.19]. Thus  $C(X)$  has the unique extension property in  $C(X) \rtimes_r \Gamma$  by Theorem 2.2 if and only if the action of  $\Gamma$  on  $X$  is free. It follows that if  $\Gamma$  acts by a topologically free action on  $X$  then  $C(X)$  is Cartan in  $C(X) \rtimes \Gamma$  with associated twist being simply the trivial twist on the transformation groupoid  $\Gamma \times X$ .

Consider a reduced crossed product  $C(X) \rtimes_r \Gamma$ , where  $\Gamma$  is a discrete group acting on a compact Hausdorff space  $X$ . Recall that every element  $a \in C(X) \rtimes_r \Gamma$  is uniquely determined by its Fourier series. We write

$$a \sim \sum_{g \in \Gamma} a_g u_g$$

where  $a_g = E(au_g^*) \in C(X)$ . To be wholly consistent with the general theory of  $C^*$ -diagonals discussed above one would consider the elements  $u_g E(u_g^* a)$ , as in Proposition 4.3, but we will stick to the usual convention of setting  $a_g = E(au_g^*)$  when working with crossed products. Note that, if the action of  $\Gamma$  is free then, as in Proposition 4.3,  $a_g u_g$  is in the norm-closed  $C(X)$ -bimodule generated by  $a$ .

The elements of the transformation groupoid  $\Gamma \times X$  act as linear functionals on  $C(X) \rtimes_r \Gamma$ . As our Fourier coefficients are of the form  $E(au_g^*)$  and *not*  $u_g E(u_g^* a)$ , we define the linear functionals slightly differently than in the general Cartan/ $C^*$ -diagonal case, with the normalizers acting on the other side. Thus, for  $a \in C(X) \rtimes_r \Gamma$  and  $(g, x) \in \Gamma \times X$  we have

$$(g, x)(a) = E(au_g^*)(x) = a_g(x),$$

when  $a$  has Fourier series  $a \sim \sum_{g \in \Gamma} a_g u_g$ .

The following example shows that Theorem 4.5 need not hold for Cartan embeddings even when  $C(X) \rtimes_r \Gamma$  is nuclear.

**Example 5.1.** Let  $\overline{\mathbb{D}}$  be the closed unit disk in  $\mathbb{C}$ . Let  $\mathbb{Z}$  act on  $\overline{\mathbb{D}}$  by an irrational rotation. That is, we fix  $\alpha \in \mathbb{T}$  such that  $\{\alpha^n : n \in \mathbb{N}\}$  is infinite and for  $n \in \mathbb{Z}$  and  $z \in \overline{\mathbb{D}}$ , we define

$$n \cdot z = \alpha^n z.$$

Then the action of  $\mathbb{Z}$  on  $\overline{\mathbb{D}}$  is topologically free. It is not a free action since 0 is fixed. Hence  $C(\overline{\mathbb{D}})$  is Cartan in  $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}$ , but it is not a  $C^*$ -diagonal. Let  $u \in C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}$  be the unitary generating the representation of  $\mathbb{Z}$ .

Consider the ideal

$$J = \{f \in C(\overline{\mathbb{D}}) : f(0) = 0\}$$

in  $C(\overline{\mathbb{D}})$ . Since  $\{0\}$  is fixed under our action, the set

$$K = \overline{\text{span}}\{fu^n : f \in J\}$$

is an ideal in  $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}$ , [14, Proposition VIII.3.3]. Let  $q$  denote the quotient map

$$q: C(\overline{\mathbb{D}}) \rtimes \mathbb{Z} \rightarrow (C(\overline{\mathbb{D}}) \rtimes \mathbb{Z})/K \simeq C(\mathbb{T});$$

indeed, for  $N \in \mathbb{N}$ ,

$$q\left(\sum_{n=-N}^N f_n u^n\right) = \sum_{n=-N}^N f_n(0)z^n \in C(\mathbb{T}).$$

Let

$$Y = \{f \in C(\mathbb{T}) : f(1) = f(-1)\}$$

and let  $B = q^{-1}(Y)$ . Then  $B$  is a  $C^*$ -algebra satisfying

$$C(\overline{\mathbb{D}}) \subseteq B \subseteq C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}.$$

For any  $x \in \mathbb{D}$  and  $r \in \mathbb{Z}$ , a calculation shows that for  $f \in C(\overline{\mathbb{D}})$  and  $n \in \mathbb{Z}$ ,

$$(r, x)(fu^n) = \begin{cases} 0 & \text{if } r \neq n; \\ f(r \cdot x) & \text{when } r = n. \end{cases}$$

Suppose  $r \in \mathbb{Z}$  is odd. Let  $b = u^r - u^{-r} \in B$ . Then for any  $x \in \overline{\mathbb{D}}$ ,  $(r, x)(b) \neq 0$ . On the other hand, when  $r$  is even,  $u^r \in B$ , so for  $x \in \overline{\mathbb{D}}$ ,  $(x, r)(u^r) \neq 0$ . Hence,  $\{\sigma \in \mathbb{Z} \times \overline{\mathbb{D}} : \sigma|_B \neq 0\} = \mathbb{Z} \times \overline{\mathbb{D}}$ . Theorem 3.3 thus implies that  $C(\overline{\mathbb{D}})$  is not Cartan in  $B$ .

Example 5.1 shows that in general there is not a nice correspondence between subgroupoids and intermediate  $C^*$ -algebras of a Cartan embedding. It may be that Theorem 3.5 is the best we can hope for in that generality. The following corollary to Theorem 3.5 recovers a special case of Choda's results [12].

**Corollary 5.2.** *Let  $\Gamma$  be a discrete group acting topologically freely on a connected Hausdorff space  $X$ . Then the map  $H \mapsto C(X) \rtimes_r H$  gives a one-to-one correspondence between subgroups  $H \subseteq \Gamma$  and intermediate  $C^*$ -algebras  $C(X) \subseteq B \subseteq C(X) \rtimes_r \Gamma$  for which there is a faithful conditional expectation  $C(X) \rtimes_r \Gamma \rightarrow B$ .*

*Proof.* Since  $X$  is connected the only subgroupoids of  $\Gamma \times X$  which are both closed and open are of the form  $H \times X$  where  $H \subseteq \Gamma$  is a subgroup. Thus, the result follows from Theorem 3.5.  $\square$

We now turn our attention to free actions. To apply Theorem 4.5 directly we also need to know when  $C(X) \rtimes_r \Gamma$  is nuclear. This happens precisely when the action of  $\Gamma$  on  $X$  is amenable, which in turn happens precisely when the transformation groupoid  $\Gamma \times X$  is amenable. Thus, for  $C(X)$  to be a  $C^*$ -diagonal in a nuclear  $C(X) \rtimes_r \Gamma$  we need the action of  $\Gamma$  on  $X$  to be free and amenable. However, if the action of  $\Gamma$  on  $X$  is both free and amenable, then  $\Gamma$  itself must be amenable [28, Corollary 4.3]. Hence Theorem 4.5, when applied to crossed products, applies only to free actions of amenable groups. We can loosen this condition.

If  $a \in C(X) \rtimes \Gamma$  with Fourier series  $a \sim \sum_{g \in \Gamma} a_g u_g$ . One cannot expect the series  $\sum_{g \in \Gamma} a_g u_g$  to converge in norm (though it does converge in the Bures-topology in the enveloping von Neumann algebra [27]). With the right class of groups  $\Gamma$  we can, however, recover any element  $a \in C(X) \rtimes_r \Gamma$  from its Fourier series.

**Definition 5.3.** A discrete group  $\Gamma$  has the *approximation property* if 1 is in the weak\*-closure of the space of finitely supported functions on  $\Gamma$ .<sup>1</sup>

The approximation property was introduced by Cowling and Haagerup in [13], where they showed that both amenable and weakly amenable groups satisfy the approximation property. The following result was proved by Bédos and Conti [4, Section 5], generalizing earlier work of Exel [18].

**Theorem 5.4.** [4] *Let  $\Gamma$  be a discrete group satisfying the approximation property and let  $C(X) \rtimes_r \Gamma$  be a reduced crossed product. If  $a \in C(X) \rtimes_r \Gamma$  has Fourier series  $a \sim \sum_{g \in \Gamma} a_g u_g$ , then*

$$a \in \overline{\text{span}}\{a_g u_g : g \in \Gamma\}.$$

*Remark 5.5.* Theorem 5.4 applies to twisted crossed products. Thus, the results which follow will also apply in that generality.

**Corollary 5.6.** *Let  $\Gamma$  be a discrete group satisfying the approximation property which acts freely on a compact Hausdorff space  $X$ . If  $M \subseteq C(X) \rtimes_r \Gamma$  is a norm-closed  $C(X)$ -bimodule then*

$$M = \overline{\text{span}}\{n \in N(M, C(X))\} = \overline{\text{span}}\{f u_g : f \in C(X), f u_g \in M\}.$$

*Proof.* Since  $\Gamma$  acts freely on  $X$ , if  $a \in M$  with  $a \sim \sum_{g \in \Gamma} a_g u_g$ , then  $a_g u_g \in M$ , by Proposition 4.3. The result follows now from Theorem 5.4.  $\square$

In [10], Cameron and Smith prove a spectral theorem for  $A$ -bimodules in  $A \rtimes_r \Gamma$ , where  $\Gamma$  is a discrete group satisfying the approximation property, and  $A$  is a simple  $C^*$ -algebra. There, the bimodules are determined by subsets of  $\Gamma$ . When  $\Gamma$  acts freely on a compact Hausdorff space  $X$ , we show that the  $C(X)$ -bimodules are determined by open subsets of the transformation groupoid  $\Gamma \times X$ .

For  $g \in \Gamma$  we denote by  $\pi_g$  the projection map on subsets  $U \subseteq \Gamma \times X$  defined by

$$\pi_g(U) = \{x \in X : (g, x) \in U\}.$$

**Theorem 5.7** (Spectral Theorem for Bimodules). *Let  $\Gamma$  be a discrete group satisfying the approximation property, which acts freely on a compact Hausdorff space  $X$ . The map*

$$U \mapsto \overline{\text{span}}\{f u_g : f \in C(X), \text{supp}(f) \subseteq \pi_g(U)\},$$

*defines a one-to-one correspondence between open subsets  $U \subseteq \Gamma \times X$  and closed  $C(X)$ -bimodules in  $C(X) \rtimes_r \Gamma$ .*

*Proof.* Let  $U \subseteq \Gamma \times X$  be open. Let

$$N_U = \overline{\text{span}}\{f u_g : f \in C(X), \text{supp}(f) \subseteq \pi_g(U)\} \subseteq C(X) \rtimes_r \Gamma.$$

Then  $N_U$  is a norm-closed  $C(X)$ -bimodule. To show that the map  $\Phi : U \mapsto N_U$  is a surjective map from open subsets of  $\Gamma \times X$  and closed  $C(X)$ -bimodules in  $C(X) \rtimes_r \Gamma$  we construct the inverse map.

Let  $N \subseteq C(X) \rtimes_r \Gamma$  be a norm-closed  $C(X)$ -bimodule. For each  $g \in \Gamma$  define the set

$$U_g = \bigcup \{\text{supp}(f) : f \in C(X), f u_g \in N\}.$$

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<sup>1</sup>To be explicit,  $\Gamma$  has the approximation property if there exists a net  $(f_\alpha)$  in  $c_c(\Gamma)$  such that for every  $\varphi \in \ell^1(\Gamma)$ ,  $\sum_{\gamma \in \Gamma} \varphi(\gamma) f_\alpha(\gamma) \rightarrow \sum_{\gamma \in \Gamma} \varphi(\gamma)$ .

Note that since  $N$  is a  $C(X)$ -bimodule the set

$$J_g = \{f \in C(X) : fu_g \in N\}$$

is an ideal in  $C(X)$ . Indeed  $J_g = C_0(U_g)$ . Let

$$U_N = \bigcup_{g \in \Gamma} \{g\} \times U_g.$$

Since each  $U_g$  is open,  $U_N$  is an open subset of  $\Gamma \times X$ . Denote by  $\Psi$  the map  $\Psi: N \mapsto U_N$ .

That  $\Psi$  is the inverse of  $\Phi$  follows from Corollary 5.6.  $\square$

Since, if  $B$  is a  $C^*$ -algebra satisfying  $C(X) \subseteq B \subseteq C(X) \rtimes_r \Gamma$ , then  $B$  is a  $C(X)$ -bimodule, we get the following Corollary, which extends Theorem 4.5.

**Corollary 5.8.** *Let  $\Gamma$  be a discrete group satisfying the approximation property which acts freely on a compact Hausdorff space  $X$ . Then the map*

$$H \mapsto \overline{\text{span}}\{fu_g : f \in C(X), \text{supp}(f) \subseteq \pi_g(U)\}$$

*defines a one-to-one correspondence between open subgroupoids  $H \subseteq \Gamma \times X$  with  $\{e\} \times X \subseteq H$  and intermediate  $C^*$ -algebras  $B$  with  $C(X) \subseteq B \subseteq C(X) \rtimes_r \Gamma$ .*

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