

## THE PRODUCT FORMULA FOR REGULARIZED FREDHOLM DETERMINANTS

THOMAS BRITZ, ALAN CAREY, FRITZ GESZTESY, ROGER NICHOLS,  
 FEDOR SUKOCHEV, AND DMITRIY ZANIN

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ABSTRACT. For trace class operators  $A, B \in \mathcal{B}_1(\mathcal{H})$  ( $\mathcal{H}$  a complex, separable Hilbert space), the product formula for Fredholm determinants holds in the familiar form

$$\det_{\mathcal{H}}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) = \det_{\mathcal{H}}(I_{\mathcal{H}} - A)\det_{\mathcal{H}}(I_{\mathcal{H}} - B).$$

When trace class operators are replaced by Hilbert–Schmidt operators  $A, B \in \mathcal{B}_2(\mathcal{H})$  and the Fredholm determinant  $\det_{\mathcal{H}}(I_{\mathcal{H}} - A)$ ,  $A \in \mathcal{B}_1(\mathcal{H})$ , by the 2nd regularized Fredholm determinant  $\det_{\mathcal{H},2}(I_{\mathcal{H}} - A) = \det_{\mathcal{H}}((I_{\mathcal{H}} - A)\exp(A))$ ,  $A \in \mathcal{B}_2(\mathcal{H})$ , the product formula must be replaced by

$$\det_{\mathcal{H},2}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) = \det_{\mathcal{H},2}(I_{\mathcal{H}} - A)\det_{\mathcal{H},2}(I_{\mathcal{H}} - B) \\
 \times \exp(-\operatorname{tr}_{\mathcal{H}}(AB)).$$

The product formula for the case of higher regularized Fredholm determinants  $\det_{\mathcal{H},k}(I_{\mathcal{H}} - A)$ ,  $A \in \mathcal{B}_k(\mathcal{H})$ ,  $k \in \mathbb{N}$ ,  $k \geq 2$ , does not seem to be easily accessible and hence this note aims at filling this gap in the literature.

### 1. INTRODUCTION

The purpose of this note is to prove a product formula for regularized (modified) Fredholm determinants extending the well-known Hilbert–Schmidt case.

To set the stage, we recall that if  $A \in \mathcal{B}_1(\mathcal{H})$  is a trace class operator on the complex, separable Hilbert space  $\mathcal{H}$ , that is, the sequence of (necessarily nonnegative) eigenvalues  $\lambda_j((A^*A)^{1/2})$ ,  $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , of  $|A| = (A^*A)^{1/2}$  (the singular values of  $A$ ), ordered in nonincreasing magnitude and counted according to their multiplicity, lies in  $\ell^1(\mathbb{N}_0)$ , the Fredholm determinant  $\det_{\mathcal{H}}(I_{\mathcal{H}} - A)$  associated with  $I_{\mathcal{H}} - A$ ,  $A \in \mathcal{B}_1(\mathcal{H})$ , is given by the absolutely convergent infinite product

$$(1.1) \quad \det_{\mathcal{H}}(I_{\mathcal{H}} - A) = \prod_{j \in \mathbb{N}_0} [1 - \lambda_j(A)],$$

where  $\lambda_j(A)$ ,  $j \in \mathbb{N}_0$ , are the (generally, complex) eigenvalues of  $A$  ordered again with respect to nonincreasing absolute value and now counted according to their algebraic multiplicity.

A celebrated property of  $\det_{\mathcal{H}}(I_{\mathcal{H}} - \cdot)$  that (like the analog of (1.1)) is shared with the case where  $\mathcal{H}$  is finite-dimensional, is the product formula

$$(1.2) \quad \det_{\mathcal{H}}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) = \det_{\mathcal{H}}(I_{\mathcal{H}} - A)\det_{\mathcal{H}}(I_{\mathcal{H}} - B), \quad A, B \in \mathcal{B}_1(\mathcal{H})$$

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(see, e.g., [4, pp. 162–163], [8, Theorem XIII.105 (a)], [9, Theorem 3.8], [10, Theorem 3.5 (a)], [11, Theorem 3.4.10 (f)], [12, p. 44]).

When extending these considerations to operators  $A \in \mathcal{B}_p(\mathcal{H})$ , with  $\mathcal{B}_p(\mathcal{H})$ ,  $p \in [1, \infty)$ , the  $\ell^p(\mathbb{N}_0)$ -based trace ideals (i.e., the eigenvalues  $\lambda_j((A^*A)^{1/2})$ ,  $j \in \mathbb{N}_0$ , of  $(A^*A)^{1/2}$  now lie in  $\ell^p(\mathbb{N}_0)$ , see, e.g., [4, Sect. III.7]), the  $k$ th regularized Fredholm determinant  $\det_{\mathcal{H},k}(I_{\mathcal{H}} - A)$ ,  $k \in \mathbb{N}$ ,  $k \geq p$ , associated with  $I_{\mathcal{H}} - A$ ,  $A \in \mathcal{B}_k(\mathcal{H})$ , is given by

$$(1.3) \quad \begin{aligned} \det_{\mathcal{H},k}(I_{\mathcal{H}} - A) &= \prod_{j \in \mathbb{N}_0} \left( [1 - \lambda_j(A)] \exp \left( \sum_{\ell=1}^{k-1} \ell^{-1} \lambda_j(A)^\ell \right) \right) \\ &= \det_{\mathcal{H}} \left( (I_{\mathcal{H}} - A) \exp \left( \sum_{\ell=1}^{k-1} \ell^{-1} A^\ell \right) \right), \quad k \geq p \end{aligned}$$

(see, e.g., [2, pp. 1106–1116], [4, pp. 166–169], [9], [10, pp. 75–76], [11, pp. 187–191], [12, p. 44]). In particular, the first line in (1.3) resembles the structure of canonical infinite product representations of entire functions according to Weierstrass, Hadamard, and Borel (see, e.g., [7, Vol. 2, Ch. II.10]).

We note that  $\det_{\mathcal{H},k}(I_{\mathcal{H}} - \cdot)$  is continuous on  $\mathcal{B}_\ell(\mathcal{H})$  for  $1 \leq \ell \leq k$ , and

$$(1.4) \quad \det_{\mathcal{H},k}(I_{\mathcal{H}} - AB) = \det_{\mathcal{H},k}(I_{\mathcal{H}} - BA), \quad A, B \in \mathcal{B}(\mathcal{H}), \quad AB, BA \in \mathcal{B}_k(\mathcal{H})$$

(this extends to the case where  $A$  maps between different Hilbert spaces  $\mathcal{H}_2$  and  $\mathcal{H}_1$  and  $B$  from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ , etc.).

The analog of the simple product formula (1.2) no longer holds for  $k \geq 2$  and it is well-known in the special Hilbert–Schmidt case  $k = 2$  that (1.2) must be replaced by

$$(1.5) \quad \begin{aligned} \det_{\mathcal{H},2}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) &= \det_{\mathcal{H},2}(I_{\mathcal{H}} - A) \det_{\mathcal{H},2}(I_{\mathcal{H}} - B) \\ &\quad \times \exp(-\operatorname{tr}_{\mathcal{H}}(AB)), \quad A, B \in \mathcal{B}_2(\mathcal{H}) \end{aligned}$$

(see, e.g., [4, p. 169], [10, p. 76], [11, p. 190], [12, p. 44]). Recently, some of us needed the extension of (1.5) to general  $k \in \mathbb{N}$ ,  $k \geq 3$ , in [1], but were not able to find it in the literature; hence, this note aims at closing this gap.

More precisely, we were interested in a product formula for  $\det_{\mathcal{H},k}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B))$  for  $A, B \in \mathcal{B}_k(\mathcal{H})$  in terms of  $\det_{\mathcal{H},k}(I_{\mathcal{H}} - A)$  and  $\det_{\mathcal{H},k}(I_{\mathcal{H}} - B)$ ,  $k \in \mathbb{N}$ ,  $k \geq 3$ . As kindly pointed out to us by Rupert Frank, the particular case where  $A$  is a finite rank operator, denoted by  $F$ , and  $B \in \mathcal{B}_k(\mathcal{H})$  was considered in [5, Lemma 1.5.10] (see also, [6, Proposition 4.8 (ii)]), and the result

$$(1.6) \quad \det_{\mathcal{H},k}((I_{\mathcal{H}} - F)(I_{\mathcal{H}} - B)) = \det_{\mathcal{H}}(I_{\mathcal{H}} - F) \det_{\mathcal{H},k}(I_{\mathcal{H}} - B) \exp(\operatorname{tr}_{\mathcal{H}}(p_n(F, B))),$$

with  $p_n(\cdot, \cdot)$  a polynomial in two variables and of finite rank, was derived. An extension of this formula to three factors, that is,

$$(1.7) \quad \begin{aligned} \det_{\mathcal{H},k}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - F)(I_{\mathcal{H}} - B)) &= \det_{\mathcal{H}}(I_{\mathcal{H}} - F) \det_{\mathcal{H},k}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) \\ &\quad \times \exp(\operatorname{tr}_{\mathcal{H}}(p_n(A, F, B))), \end{aligned}$$

with  $p_n(\cdot, \cdot, \cdot)$  a polynomial in three variables and of finite rank, was derived in [3, Lemma C.1].

The result we have in mind is somewhat different from (1.6) in that we are interested in a quantitative version of the following fact:

**Theorem 1.1.** *Let  $k \in \mathbb{N}$ , and suppose  $A, B \in \mathcal{B}_k(\mathcal{H})$ . Then*

$$(1.8) \quad \det_{\mathcal{H},k}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) = \det_{\mathcal{H},k}(I_{\mathcal{H}} - A)\det_{\mathcal{H},k}(I_{\mathcal{H}} - B) \\ \times \exp(\operatorname{tr}_{\mathcal{H}}(X_k(A, B))),$$

where  $X_k(\cdot, \cdot) \in \mathcal{B}_1(\mathcal{H})$  is of the form

$$(1.9) \quad X_1(A, B) = 0, \\ X_k(A, B) = \sum_{j_1, \dots, j_{2k-2}=0}^{k-1} c_{j_1, \dots, j_{2k-2}} C_1^{j_1} \dots C_{2k-2}^{j_{2k-2}}, \quad k \geq 2,$$

with

$$(1.10) \quad c_{j_1, \dots, j_{2k-2}} \in \mathbb{Q}, \\ C_\ell = A \text{ or } B, \quad 1 \leq \ell \leq 2k-2, \\ k \leq \sum_{\ell=1}^{2k-2} j_\ell \leq 2k-2, \quad k \geq 2.$$

Explicitly, one obtains:

$$(1.11) \quad X_1(A, B) = 0, \\ X_2(A, B) = -AB, \\ X_3(A, B) = 2^{-1}[(AB)^2 - AB(A+B) - (A+B)AB], \\ X_4(A, B) = 2^{-1}(AB)^2 - 3^{-1}[AB(A+B)^2 + (A+B)^2AB \\ + (A+B)AB(A+B)] \\ + 3^{-1}[(AB)^2(A+B) + (A+B)(AB)^2 + AB(A+B)AB] \\ - 3^{-1}(AB)^3, \\ \text{etc.}$$

When taking traces (what is actually needed in (1.8)), this simplifies to

$$(1.12) \quad \operatorname{tr}_{\mathcal{H}}(X_1(A, B)) = 0, \\ \operatorname{tr}_{\mathcal{H}}(X_2(A, B)) = -\operatorname{tr}_{\mathcal{H}}(AB), \\ \operatorname{tr}_{\mathcal{H}}(X_3(A, B)) = -\operatorname{tr}_{\mathcal{H}}(ABA + BAB - 2^{-1}(AB)^2), \\ \operatorname{tr}_{\mathcal{H}}(X_4(A, B)) = -\operatorname{tr}_{\mathcal{H}}(A^3B + A^2B^2 + AB^3 + 2^{-1}(AB)^2 \\ - (AB)^2A - B(AB)^2 + 3^{-1}(AB)^3), \\ \text{etc.}$$

We present the proof of a quantitative version of Theorem 1.1 in two parts. In the next section we prove an algebraic result, Lemma 2.4, that is the key to the analytic part of the argument appearing in the final section on regularized determinants.

## 2. THE COMMUTATOR SUBSPACE IN THE ALGEBRA OF NONCOMMUTATIVE POLYNOMIALS

To prove a quantitative version of Theorem 1.1 and hence derive a formula for  $X_k(A, B)$ , we first need to recall some facts on the commutator subspace of an algebra of noncommutative polynomials.

Let  $\text{Pol}_2$  be the free polynomial algebra in 2 (noncommuting) variables,  $A$  and  $B$ . Let  $W$  be the set of noncommutative monomials (words in the alphabet  $\{A, B\}$ ). (We recall that the set  $W$  is a semigroup with respect to concatenation, 1 is the neutral element of this semigroup, that is, 1 is an empty word in this alphabet.) Every  $x \in \text{Pol}_2$  can be written as a sum

$$(2.1) \quad x = \sum_{w \in W} \hat{x}(w)w.$$

Here the coefficients  $\hat{x}(w)$  vanish for all but finitely many  $w \in W$ .

Let  $[\text{Pol}_2, \text{Pol}_2]$  be the commutator subspace of  $\text{Pol}_2$ , that is, the linear span of commutators  $[x_1, x_2]$ ,  $x_1, x_2 \in \text{Pol}_2$ .

**Lemma 2.1.** *One has  $x \in [\text{Pol}_2, \text{Pol}_2]$  provided that*

$$(2.2) \quad \sum_{m=1}^{L(w)} \hat{x}(\sigma^m(w)) = 0, \quad w \in W.$$

Here,  $L(w)$  is the length of each word  $w = w_1 w_2 \cdots w_{L(w)}$ ,  $\sigma$  is the cyclic shift given by  $\sigma(w) = w_2 \cdots w_{L(w)} w_1$ .

*Proof.* One notes that

$$(2.3) \quad x = \sum_{w \in W} \hat{x}(w)w = \hat{x}(1) + \sum_{w \neq 1} L(w)^{-1} \sum_{m=1}^{L(w)} \hat{x}(\sigma^m(w))\sigma^m(w).$$

Obviously,  $(\sigma^m(w) - w) \in [\text{Pol}_2, \text{Pol}_2]$  for each positive integer  $m$  and thus,

$$(2.4) \quad x \in \left( \hat{x}(1) + \sum_{w \neq 1} L(w)^{-1} \sum_{m=1}^{L(w)} \hat{x}(\sigma^m(w))w + [\text{Pol}_2, \text{Pol}_2] \right).$$

By hypothesis,  $\hat{x}(1) = 0$  and

$$(2.5) \quad \sum_{m=1}^{L(w)} \hat{x}(\sigma^m(w)) = 0, \quad 1 \neq w \in W,$$

completing the proof.  $\square$

Next, we need some notation. Let  $k_1, k_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , and set

$$(2.6) \quad z_{k_1, k_2} = \begin{cases} 0, & k_1 = k_2 = 0, \\ k_1^{-1} A^{k_1}, & k_1 \in \mathbb{N}, k_2 = 0, \\ k_2^{-1} B^{k_2}, & k_1 = 0, k_2 \in \mathbb{N}, \\ \sum_{j=1}^{k_1+k_2} j^{-1} \sum_{\substack{\pi \in S_j, |\pi|=3 \\ |\pi_1|+|\pi_3|=k_1 \\ |\pi_2|+|\pi_3|=k_2}} (-1)^{|\pi_3|} z_\pi, & k_1, k_2 \in \mathbb{N}. \end{cases}$$

Here,  $S_j$  is the set of all partitions of the set  $\{1, \dots, j\}$ ,  $1 \leq j \leq k_1 + k_2$ . (The symbol  $|\cdot|$  abbreviating the cardinality of a subset of  $\mathbb{Z}$ .) The condition  $|\pi| = 3$  means that  $\pi$  breaks the set  $\{1, \dots, j\}$  into exactly 3 pieces denoted by  $\pi_1, \pi_2$ , and  $\pi_3$  (some of them can be empty). The element  $z_\pi$  denotes the product

$$(2.7) \quad z_\pi = \prod_{m=1}^j z_{m, \pi}, \quad z_{m, \pi} = \begin{cases} A, & m \in \pi_1, \\ B, & m \in \pi_2, \\ AB, & m \in \pi_3. \end{cases}$$

Finally, let  $W_{k_1, k_2}$  be the collection of all words with  $k_1$  letters  $A$  and  $k_2$  letters  $B$ .

Using this notation we now establish a combinatorial fact.

**Lemma 2.2.** *Let  $k_1, k_2 \in \mathbb{N}$ . Then*

$$(2.8) \quad z_{k_1, k_2} = \sum_{w \in W_{k_1, k_2}} \left( \sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(w)}{\ell} \right) w,$$

where

$$(2.9) \quad n(w) = |S(w)|, \quad S(w) = \{1 \leq \ell \leq L(w) - 1 \mid w_\ell = A, w_{\ell+1} = B\}.$$

*Proof.* For each  $j \in \{1, \dots, k_1 + k_2\}$ , let

$$(2.10) \quad \Pi_j = \{\pi \in S_j \mid |\pi| = 3, |\pi_1| + |\pi_3| = k_1, |\pi_2| + |\pi_3| = k_2\},$$

$$(2.11) \quad \Pi_{j,w} = \{\pi \in \Pi_j \mid z_\pi = w\}, \quad w \in W_{k_1, k_2}.$$

One observes that  $|\pi_3| \leq n(w) \leq \min\{k_1, k_2\}$  and that

$$(2.12) \quad j = |\pi_1| + |\pi_2| + |\pi_3| = k_1 + k_2 - |\pi_3|.$$

For any partition  $\pi \in \Pi_{j,w}$ , let  $I \subseteq S(w)$  indicate which subwords  $AB$  in  $w$  arise from elements in  $\pi_3$ . Then  $|I| = |\pi_3| = k_1 + k_2 - j$ . Therefore, each partition in  $\pi \in \Pi_{j,w}$  is determined by a unique choice of  $I$  and each such choice of  $I$  determines the choice of  $\pi$  uniquely. This implies that

$$(2.13) \quad |\Pi_{j,w}| = \binom{n(w)}{k_1 + k_2 - j}.$$

Thus,

$$(2.14) \quad \begin{aligned} z_{k_1, k_2} &= \sum_{w \in W_{k_1, k_2}} \sum_{j=1}^{k_1+k_2} j^{-1} \sum_{\pi \in \Pi_{j,w}} (-1)^{|\pi_3|} w \\ &= \sum_{w \in W_{k_1, k_2}} \sum_{j=1}^{k_1+k_2} (-1)^{k_1+k_2-j} j^{-1} |\Pi_{j,w}| w \\ &= \sum_{w \in W_{k_1, k_2}} \sum_{j=1}^{k_1+k_2} (-1)^{k_1+k_2-j} j^{-1} \binom{n(w)}{k_1 + k_2 - j} w. \end{aligned}$$

Taking into account that

$$(2.15) \quad \binom{n(w)}{k_1 + k_2 - j} = 0, \quad k_1 + k_2 - j \notin \{0, \dots, n(w)\},$$

it follows that

$$(2.16) \quad \begin{aligned} z_{k_1, k_2} &= \sum_{w \in W_{k_1, k_2}} \sum_{j=k_1+k_2-n(w)}^{k_1+k_2} (-1)^{k_1+k_2-j} j^{-1} \binom{n(w)}{k_1 + k_2 - j} w \\ &= \sum_{w \in W_{k_1, k_2}} \left( \sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(w)}{\ell} \right) w. \end{aligned}$$

□

We can now prove the main fact about the commutator subspace of  $\text{Pol}_2$  needed later on.

**Lemma 2.3.** *For every  $k_1, k_2 \in \mathbb{N}$ ,  $z_{k_1, k_2} \in [\text{Pol}_2, \text{Pol}_2]$ .*

*Proof.* Let  $w$  be any element in  $W_{k_1, k_2}$  and let  $m$  be any positive integer. If  $\sigma^m(w)$  starts with the subword  $AB$ , then  $\sigma^{m+1}(w)$  has the form  $B \cdots A$  and therefore has one fewer subwords  $AB$  than  $\sigma^m(w)$ ; that is,  $n(\sigma^{m+1}(w)) = n(\sigma^m(w)) - 1$ . If, however,  $\sigma^m(w)$  does not start with the subword  $AB$ , then the  $AB$  subwords of  $\sigma^{m+1}(w)$  are precisely the  $AB$  subwords of  $\sigma^m(w)$  each shifted once; hence,  $n(\sigma^{m+1}(w)) = n(\sigma^m(w))$ .

Now, to calculate  $\sum_{m=1}^{L(w)} \widehat{z_{k_1, k_2}}(\sigma^m(w))$ , one may assume, by applying cyclic shifts, that  $w$  starts with  $AB$ . Then there are  $n(w)$  shifted words  $\sigma^m(w)$  which start with the subword  $AB$ , and it follows that  $n(w)$  of the numbers  $\{n(\sigma^m(w)) : 1 \leq m \leq L(w)\}$  equal  $n(w) - 1$  and that the remaining  $L(w) - n(w) = k_1 + k_2 - n(w)$  numbers equal  $n(w)$ . Lemma 2.2 therefore implies that

$$\begin{aligned}
 \sum_{m=1}^{L(w)} \widehat{z_{k_1, k_2}}(\sigma^m(w)) &= \sum_{m=1}^{L(w)} \left( \sum_{\ell=0}^{n(\sigma^m(w))} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(\sigma^m(w))}{\ell} \right) \\
 &= n(w) \left( \sum_{\ell=0}^{n(w)-1} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(w)-1}{\ell} \right) \\
 (2.17) \quad &+ (k_1 + k_2 - n(w)) \left( \sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(w)}{\ell} \right).
 \end{aligned}$$

Since

$$(2.18) \quad \binom{n(w)-1}{n(w)} = 0,$$

it follows that

$$\begin{aligned}
 \sum_{m=1}^{L(w)} \widehat{z_{k_1, k_2}}(\sigma^m(w)) &= n(w) \left( \sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(w)-1}{\ell} \right) \\
 &+ (k_1 + k_2 - n(w)) \left( \sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \binom{n(w)}{\ell} \right) \\
 &= \sum_{\ell=0}^{n(w)} \frac{(-1)^\ell}{k_1 + k_2 - \ell} \left( n(w) \binom{n(w)-1}{\ell} \right. \\
 (2.19) \quad &\left. + (k_1 + k_2 - n(w)) \binom{n(w)}{\ell} \right).
 \end{aligned}$$

Clearly,

$$(2.20) \quad n(w) \binom{n(w)-1}{\ell} + (k_1 + k_2 - n(w)) \binom{n(w)}{\ell} = (k_1 + k_2 - \ell) \binom{n(w)}{\ell},$$

and thus

$$(2.21) \quad \sum_{m=1}^{L(w)} \widehat{z_{k_1, k_2}}(\sigma^m(w)) = \sum_{\ell=0}^{n(w)} (-1)^\ell \binom{n(w)}{\ell} = 0.$$

Hence, Lemma 2.1 completes the proof.  $\square$

Next, we introduce some further notation. Let  $k \in \mathbb{N}$  and set

$$(2.22) \quad \begin{aligned} x_1 &= 0, \\ x_k &= \sum_{j=1}^{k-1} j^{-1} \sum_{\substack{\mathcal{A} \subseteq \{1, \dots, j\} \\ j+|\mathcal{A}| \geq k}} (-1)^{|\mathcal{A}|} y_{\mathcal{A}}, \quad k \geq 2, \end{aligned}$$

$$(2.23) \quad \begin{aligned} y_1 &= 0, \\ y_k &= \sum_{j=1}^{k-1} j^{-1} \sum_{\substack{\mathcal{A} \subseteq \{1, \dots, j\} \\ j+|\mathcal{A}| \leq k-1}} (-1)^{|\mathcal{A}|} y_{\mathcal{A}}, \quad k \geq 2, \end{aligned}$$

$$(2.24) \quad y_{\mathcal{A}} = \prod_{m=1}^j y_{m, \mathcal{A}}, \quad y_{m, \mathcal{A}} = \begin{cases} A + B, & m \notin \mathcal{A}, \\ AB, & m \in \mathcal{A}. \end{cases}$$

In particular,

$$(2.25) \quad \sum_{j=1}^{k-1} j^{-1} (A + B - AB)^j = x_k + y_k,$$

and one notes that the length of the word  $y_{\mathcal{A}}$  subject to  $\mathcal{A} \subseteq \{1, \dots, j\}$ , equals

$$(2.26) \quad L(y_{\mathcal{A}}) = |\mathcal{A}^c| + 2|\mathcal{A}| = j + |\mathcal{A}|, \quad 1 \leq j \leq k-1, \quad k \geq 2$$

(with  $\mathcal{A}^c = \{1, \dots, j\} \setminus \mathcal{A}$  the complement of  $\mathcal{A}$  in  $\{1, \dots, j\}$ ).

Using this notation we can now state the following fact:

**Lemma 2.4.** *Let  $k \in \mathbb{N}$ ,  $k \geq 2$ , then*

$$(2.27) \quad y_k \in \left( \sum_{j=1}^{k-1} \frac{1}{j} (A^j + B^j) + [\text{Pol}_2, \text{Pol}_2] \right).$$

*Proof.* Employing

$$(2.28) \quad y_k = \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq k-1}} z_{k_1, k_2},$$

Lemma 2.3 yields

$$(2.29) \quad z_{k_1, k_2} \in [\text{Pol}_2, \text{Pol}_2], \quad k_1, k_2 \in \mathbb{N}.$$

Since by (2.6),

$$(2.30) \quad z_{0,0} = 0, \quad z_{k_1,0} = k_1^{-1} A^{k_1}, \quad k_1 \in \mathbb{N}, \quad z_{0,k_2} = k_2^{-1} B^{k_2}, \quad k_2 \in \mathbb{N},$$

combining (2.28)–(2.30) completes the proof.  $\square$

### 3. THE PRODUCT FORMULA FOR $k$ TH MODIFIED FREDHOLM DETERMINANTS

After these preparations we are ready to return to the product formula for regularized determinants and specialize the preceding algebraic considerations to the context of Theorem 1.1.

First we recall that by (2.22) and (2.26),

$$(3.1) \quad x_k = \sum_{j=1}^{k-1} j^{-1} \sum_{\substack{\mathcal{A} \subseteq \{1, \dots, j\} \\ j+|\mathcal{A}| \geq k}} (-1)^{|\mathcal{A}|} y_{\mathcal{A}} := X_k(A, B) \in \mathcal{B}_1(\mathcal{H}), \quad k \geq 2,$$

since for  $1 \leq j \leq k-1$ ,  $L(y_A) = j + |\mathcal{A}| \geq k$ , and hence one obtains the inequality

$$(3.2) \quad \|x_k\|_{\mathcal{B}_1(\mathcal{H})} \leq c_k \max_{\substack{0 \leq k_1, k_2 \leq k \\ k_1 + k_2 \geq k}} \|A\|_{\mathcal{B}_k(\mathcal{H})}^{k_1} \|B\|_{\mathcal{B}_k(\mathcal{H})}^{k_2}, \quad k \in \mathbb{N}, k \geq 2,$$

for some  $c_k > 0$ ,  $k \geq 2$ . We also set (cf. (2.22))  $X_1(A, B) = 0$ .

**Theorem 3.1.** *Let  $k \in \mathbb{N}$  and assume that  $A, B \in \mathcal{B}_k(\mathcal{H})$ . Then*

$$(3.3) \quad \begin{aligned} & \det_{\mathcal{H},k}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) \\ &= \det_{\mathcal{H},k}(I_{\mathcal{H}} - A) \det_{\mathcal{H},k}(I_{\mathcal{H}} - B) \exp(\operatorname{tr}_{\mathcal{H}}(X_k(A, B))). \end{aligned}$$

*Proof.* First, we suppose that  $A, B \in \mathcal{B}_1(\mathcal{H})$ . Then it is well-known that

$$(3.4) \quad \det_{\mathcal{H},1}(I_{\mathcal{H}} - A) \det_{\mathcal{H},1}(I_{\mathcal{H}} - B) = \det_{\mathcal{H},1}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)),$$

consistent with  $X_1(A, B) = 0$ . Without loss of generality we may assume that  $k \in \mathbb{N}$ ,  $k \geq 2$ , in the following. Employing

$$(3.5) \quad \det_{\mathcal{H},k}(I_{\mathcal{H}} - T) = \det_{\mathcal{H}}(I_{\mathcal{H}} - T) \exp\left(\operatorname{tr}_{\mathcal{H}}\left(\sum_{j=1}^{k-1} j^{-1} T^j\right)\right), \quad T \in \mathcal{B}_1(\mathcal{H}),$$

see, for instance, [2, Lemma XI.9.22 (e)], [10, Theorem 9.2 (d)], one infers that

$$(3.6) \quad \begin{aligned} & \det_{\mathcal{H},k}((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) \\ &= \det_{\mathcal{H},k}(I_{\mathcal{H}} - (A + B - AB)) \\ &= \det_{\mathcal{H}}(I_{\mathcal{H}} - (A + B - AB)) \exp\left(\operatorname{tr}_{\mathcal{H}}\left(\sum_{j=1}^{k-1} j^{-1} (A + B - AB)^j\right)\right) \\ &= \det_{\mathcal{H}}(I_{\mathcal{H}} - A) \det_{\mathcal{H}}(I_{\mathcal{H}} - B) \exp\left(\operatorname{tr}_{\mathcal{H}}\left(\sum_{j=1}^{k-1} j^{-1} (A + B - AB)^j\right)\right) \\ &= \det_{\mathcal{H},k}(I_{\mathcal{H}} - A) \det_{\mathcal{H},k}(I_{\mathcal{H}} - B) \\ &\quad \times \exp\left(\operatorname{tr}_{\mathcal{H}}\left(\sum_{j=1}^{k-1} j^{-1} [(A + B - AB)^j - A^j - B^j]\right)\right). \end{aligned}$$

By (2.25) one concludes that

$$(3.7) \quad \begin{aligned} & \operatorname{tr}_{\mathcal{H}}\left(\sum_{j=1}^{k-1} j^{-1} [(A + B - AB)^j - A^j - B^j]\right) \\ &= \operatorname{tr}_{\mathcal{H}}(x_k) + \operatorname{tr}_{\mathcal{H}}\left(y_k - \sum_{j=1}^{k-1} j^{-1} (A^j + B^j)\right). \end{aligned}$$

By Lemma 2.4,

$$(3.8) \quad y_k - \sum_{j=1}^{k-1} j^{-1} (A^j + B^j)$$

is a sum of commutators of polynomial expressions in  $A$  and  $B$ . Hence,

$$(3.9) \quad \left(y_k - \sum_{j=1}^{k-1} j^{-1} (A^j + B^j)\right) \in [\mathcal{B}_1(\mathcal{H}), \mathcal{B}_1(\mathcal{H})],$$



and thus,

$$(3.10) \quad \operatorname{tr}_{\mathcal{H}} \left( y_k - \sum_{j=1}^{k-1} j^{-1} (A^j + B^j) \right) = 0,$$

proving assertion (3.3) for  $A, B \in \mathcal{B}_1(\mathcal{H})$ .

Since both, the right and left-hand sides in (3.3) are continuous with respect to the norm in  $\mathcal{B}_k(\mathcal{H})$ , and  $\mathcal{B}_1(\mathcal{H})$  is dense in  $\mathcal{B}_k(\mathcal{H})$ , (3.3) holds for arbitrary  $A, B \in \mathcal{B}_k(\mathcal{H})$ .  $\square$

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SCHOOL OF MATHEMATICS AND STATISTICS, UNSW, KENSINGTON, NSW 2052, AUSTRALIA  
*Email address:* [britz@unsw.edu.au](mailto:britz@unsw.edu.au)  
*URL:* <https://research.unsw.edu.au/people/dr-thomas-britz>

MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, KINGSLEY STREET,  
CANBERRA, ACT 0200, AUSTRALIA; AND SCHOOL OF MATHEMATICS AND APPLIED STATISTICS,  
UNIVERSITY OF WOLLONGONG, NSW, AUSTRALIA, 2522  
*Email address:* [acarey@maths.anu.edu.au](mailto:acarey@maths.anu.edu.au)  
*URL:* <http://maths.anu.edu.au/~acarey/>

DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, SID RICHARDSON BUILDING., 1410 S. 4TH  
STREET, WACO, TEXAS 76706  
*Email address:* [Fritz.Gesztesy@baylor.edu](mailto:Fritz.Gesztesy@baylor.edu)  
*URL:* <http://www.baylor.edu/math/index.php?id=935340>

DEPARTMENT OF MATHEMATICS (DEPT. 6956), THE UNIVERSITY OF TENNESSEE AT CHAT-  
TANOOGA, 615 MCCALLIE AVENUE, CHATTANOOGA, TENNESSEE 37403  
*Email address:* [Roger-Nichols@utc.edu](mailto:Roger-Nichols@utc.edu)  
*URL:* <http://www.utc.edu/faculty/roger-nichols/index.php>

SCHOOL OF MATHEMATICS AND STATISTICS, UNSW, KENSINGTON, NSW 2052, AUSTRALIA  
*Email address:* [f.sukochev@unsw.edu.au](mailto:f.sukochev@unsw.edu.au)  
*URL:* <https://research.unsw.edu.au/people/scientia-professor-fedor-sukochev>

SCHOOL OF MATHEMATICS AND STATISTICS, UNSW, KENSINGTON, NSW 2052, AUSTRALIA  
*Email address:* [d.zanin@unsw.edu.au](mailto:d.zanin@unsw.edu.au)  
*URL:* <https://research.unsw.edu.au/people/dr-dmitriy-zanin>